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GENERALIZED SEMIDERIVATIONS IN PRIME RINGS WITH ALGEBRAIC IDENTITIES

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Abstract. Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$. Suppose that $\mathcal{R}$ admits a generalized semiderivation F with associated derivation $d \neq 0$. In the present paper we investigate the commutativity of a prime ring $\mathcal{R}$ satisfying any one of the identities: **(i)** $F([x,y]) \in Z(\mathcal{R})$, **(ii)** $F(x \circ y) \in Z(\mathcal{R})$, **(iii)** $F(xy) \pm xy \in Z(\mathcal{R})$, **(iv)** $F(xy) \pm yx \in Z(\mathcal{R})$, **(v)** $[F(x), F(y)] = 0$, **(vi)** $F(x) \circ F(y) = 0$ for all $x, y \in \mathcal{R}$.

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1. INTRODUCTION

Let $\mathcal{R}$ be an associative ring with center $Z(\mathcal{R})$. For any $x, y \in \mathcal{R}$; $[x, y]$ will denote the commutator $xy - yx$ while $x \circ y$ will represent the anti-commutator $xy + yx$. Recall that a ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is said to be 2-torsion free if $2a = 0$ (where $a \in \mathcal{R}$) implies $a = 0$. It is straight forward to see that a prime ring with characteristic different from two is 2-torsion free. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be centralizing on $\mathcal{R}$ if $[f(x), x] \in Z(\mathcal{R})$ holds for all $x \in \mathcal{R}$. In the special case if $[f(x), x] = 0$ for all $x \in \mathcal{R}$, f is said to be commuting on $\mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. A derivation d is said to be inner if there exists $a \in \mathcal{R}$ such that $d(x) = ax - xa$ for all $x \in \mathcal{R}$. Following Bresar [17], an additive mapping $F : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation if there exists a derivation $d : \mathcal{R} \to \mathcal{R}$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. The concept of generalized derivation includes both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $F : \mathcal{R} \to \mathcal{R}$ satisfying $F(xy) = F(x)y$ for all $x, y \in \mathcal{R}$). The notion of semiderivation was introduced by Bergen [16]. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a semiderivation on $\mathcal{R}$ if there exists a map $g : \mathcal{R} \to \mathcal{R}$ such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$, and (ii) $d(g(x)) = g(d(x))$ hold for all $x, y \in \mathcal{R}$. If g is the identity map on $\mathcal{R}$,

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then all semiderivations associated with g are merely ordinary derivations. Moreover, if g is any endomorphism of $\mathcal{R}$, then other example of semiderivations associated with g are of the form $d(x) = x - g(x)$. For an example of semiderivation which is not a derivation, let $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ where $\mathcal{R}_1$ and $\mathcal{R}_2$ are any rings. Let $\alpha_1 : \mathcal{R}_1 \to \mathcal{R}_1$ be any additive mapping and $\alpha_2 : \mathcal{R}_2 \to \mathcal{R}_2$ be a left and right $\mathcal{R}_2$-module map which is not a derivation. Define $d : \mathcal{R} \to \mathcal{R}$ such that $d(x_1, x_2) = \big(0, \alpha_2(x_2)\big)$ and $g : \mathcal{R} \to \mathcal{R}$ such that $g(x_1, x_2) = \big(\alpha_1(x_1), 0\big), x_1 \in \mathcal{R}_1, x_2 \in \mathcal{R}_2$. It can be easily verified that d is a semiderivation of $\mathcal{R}$ with associated map g which is not a derivation of $\mathcal{R}$. If the underlying ring is prime and the semiderivation $d \neq 0$, then in this case it was shown by Chang [19] that g must necessarily be a ring endomorphism. The notion of semiderivation can be generalized in terms of generalized semiderivation as follows: An additive mapping $F : \mathcal{R} \to \mathcal{R}$ is said to be a generalized semiderivation on $\mathcal{R}$ if there exists a semiderivation $d : \mathcal{R} \to \mathcal{R}$ associated with a map $g : \mathcal{R} \to \mathcal{R}$ such that (i) $F(xy) = F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$, and (ii) $F(g(x)) = g(F(x))$ hold for all $x, y \in \mathcal{R}$. Thus all semiderivations are generalized semiderivations. Further, if the associated mappings g is merely the identity mapping on $\mathcal{R}$, then all generalized semiderivations are generalized derivation on $\mathcal{R}$. This is straightforward that every generalized derivation is a generalized semiderivation but there exist generalized semiderivations which are not generalized derivations. For example, if $\mathcal{S}$ be a ring of characteristic different from 2 and

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}.$$

Define maps $F, d, g : \mathcal{R} \to \mathcal{R}$ by

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}; \ d = F, \ \ g \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be verified that $\mathcal{R}$ is a ring and F is a generalized semiderivation with associated semiderivation d and a map g associated with d. However F is not a generalized derivation on $\mathcal{R}$.

The study of centralizing and commuting derivations was initiated by E. C. Posner who proved that the existence of nonzero centralizing derivation on a prime ring $\mathcal{R}$ forces $\mathcal{R}$ to be commutative. This result has been very influential and after its inception there has been a great deal of work concerning commutativity of prime and semiprime rings satisfying certain differential identities (see for reference [1, 2, 3, 5, 6, 7, 9, 11] etc. where further references can be found). Many results in the recent past have also appeared concerning commutativity of rings satisfying certain differential identities involving generalized derivations, for reference see [4], [13] etc. In this paper, we shall consider similar problems when the ring $\mathcal{R}$ is equipped with generalized semiderivation d. More precisely, we obtain commutativity of $\mathcal{R}$ admitting generalized semiderivation F satisfying any one of the identities:

(**i**) $F([x, y] \in Z(\mathcal{R})$, (**ii**) $F(x \circ y) \in Z(\mathcal{R})$, (**iii**) $F(xy) + xy \in Z(\mathcal{R})$, (**iv**) $F(xy) + yx \in Z(\mathcal{R})$, (**v**) $[F(x), F(y)] = 0$, (**vi**) $F(x) \circ F(y) = 0$ for all $x, y \in \mathcal{R}$.

In this paper, we suppose that F is a generalized semiderivation with associated semiderivation d, while it is also assumed that the associated maps g is onto.

2. PRELIMINARY LEMMAS

This section, includes some well known results which will be used for developing the proof of our main results. The proof of the first lemma is straightforward while the second lemma is an easy consequence of [11, Theorem 2.9]

LEMMA 2.1. *Let $\mathcal{R}$ be a prime ring which admits a semiderivation d whose associated map g is onto. Then $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$.*

LEMMA 2.2. *Let $\mathcal{R}$ be a prime ring. If I is a nonzero ideal of $\mathcal{R}$ such that $[x, y] \in Z(\mathcal{R})$ for all $x, y \in I$, then $\mathcal{R}$ is a commutative.*

LEMMA 2.3. *Let $\mathcal{R}$ be a prime ring, and let F be a generalized semiderivation with associated semiderivation $d \neq 0$. If $d(F(\mathcal{R})) = \{0\}$, then $F(d(\mathcal{R})) = \{0\}$.*

Proof. Since $d(F(x)) = 0$ for all $x \in \mathcal{R}$, it follows that $0 = d(F(xy)) = d(F(x)y) + d(g(x)d(y))$ for all $x, y \in \mathcal{R}$ which implies that

$$F(x)d(y) + d(g(x))d(y) + g^2(x)d^2(y) = 0 \quad \text{for all } x, y \in \mathcal{R}. \tag{2.1}$$

Applying d again, we get

$$F(x)d^2(y)+d^2(g(x))g(d(y))+d(g(x))d^2(y)+d(g^2(x))g(d^2(y))+g^2(x)d^3(y) = 0. \tag{2.2}$$

Taking $d(y)$ in place of y in (2.1), then (2.2) becomes

$$d^2(g(x))g(d(y)) + d(g^2(x))g(d^2(y)) = 0 \quad \text{for all } x, y \in \mathcal{R}.$$

Using the fact that $d(g(x)) = g(d(x))$ for all $x \in\in \mathcal{R}$, we find that

$$d^2(g(x))d(g(y)) + d(g^2(x))d^2(g(y)) = 0 \quad \text{for all } x, y \in \mathcal{R}.$$

Since g is onto, we obtain

$$d^2(g(x))d(y) + d(g^2(x))d^2(y) = 0 \quad \text{for all } x, y \in \mathcal{R}. \tag{2.3}$$

Replacing x by $d(x)$ in (2.1) and using (2.3), we conclude that $F(d(x))d(y) = 0$ for all $x, y \in \mathcal{R}$. Letting ry in place of y in the above equation and using it again, we arrive at $F(d(x))\mathcal{R}d(y) = \{0\}$ for all $x, y \in \mathcal{R}$. By primeness of $\mathcal{R}$ with $d \neq 0$, we conclude that $F(d(\mathcal{R})) = \{0\}$. $\square$

3. SOME RESULTS FOR PRIME RINGS

THEOREM 3.1. *Let $\mathcal{R}$ be a prime ring with $char(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a generalized semiderivation F associated with a nonzero semiderivation, then the following assertions are equivalent:*

(i) $F([x,y]) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$;

(ii) $F(x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is clear that (iii) implies (i) and (ii). So we need to prove that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.
$(i) \Rightarrow (iii)$ Suppose that $\mathcal{R}$ satisfies (i), i.e.;

$$F([x,y]) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.1}$$

Then

$$F([[u,v],y]) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, y \in \mathcal{R}. \tag{3.2}$$

Replacing y by $y[u,v]$ in (3.2), we get

$$F([[u,v],y])g([u,v]) + [[u,v],y]d([u,v]) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, y \in \mathcal{R}. \tag{3.3}$$

Using the definition of F and the fact that $F([u,v]) \in Z(\mathcal{R})$, we get

$$\begin{aligned} F([x,y])F([u,v]) + g([x,y])d(F([u,v])) &= F([x,y]F([u,v])) \\ &= F([x,yF([u,v])]), \end{aligned}$$

for all $u, v, x, y \in \mathcal{R}$. By (3.1), we find that

$$F([x,y])F([u,v]) + g([x,y])d(F([u,v])) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, x, y \in \mathcal{R}. \tag{3.4}$$

Using again equation (3.1), we can write the last expression in the form

$$g([x,y])d(F([u,v])) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, x, y \in \mathcal{R}.$$

Sine $F([u,v]) \in Z(\mathcal{R})$, $d(F([u,v])) \in Z(\mathcal{R})$ by Lemma 2.1 and primeness of $\mathcal{R}$ gives

$$d(F([u,v])) = 0 \;\text{ or }\; g([x,y]) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, x, y \in \mathcal{R}. \tag{3.5}$$

Suppose that $g([x,y]) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Then by (3.3) we get

$$[[u,v],y]d([u,v]) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, y \in \mathcal{R}. \tag{3.6}$$

Replacing y by $[u,v]y$, we find that

$$[u,v][[u,v],y]d([u,v]) \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, y \in \mathcal{R}. \tag{3.7}$$

Combining (3.6), (3.7) and using the primeness of $\mathcal{R}$, we get

$$[[u,v],y]d([u,v]) = 0 \;\text{ or }\; [u,v] \in Z(\mathcal{R}) \quad \text{for all} \;\; u, v, y \in \mathcal{R}. \tag{3.8}$$

Suppose there exist two elements u_0, v_0 of $\mathcal{R}$ such that $[[u_0,v_0],y]d([u_0,v_0]) = 0$ for all $y \in \mathcal{R}$. Taking ty instead of y in the last expression and using it again, we can easily find the relation $[[u_0,v_0],t]\mathcal{R}d([u_0,v_0]) = \{0\}$ for all $t \in \mathcal{R}$. Using the primeness of $\mathcal{R}$, the last expression gives that either $[u_0, v_0] \in Z(\mathcal{R})$ or $d([u_0, v_0]) = 0$. In this case, (3.8) becomes

$$d([u,v]) = 0 \text{ or } [u,v] \in Z(\mathcal{R}) \quad \text{for all } u, v \in \mathcal{R}. \tag{3.9}$$

Suppose there are two elements $u_0, v_0 \in \mathcal{R}$ such that $d([u_0, v_0]) = 0$. Returning to the definition of F, we have $F(x)y + g(x)d(y) = F(x)g(y) + xd(y)$ for all $x, y \in \mathcal{R}$ and replacing x by $[r, s]$ and y by $[u_0, v_0]$, we arrive at $F([r, s])\mathcal{R}\big(g([u_0, v_0]) - [u_0, v_0]\big) = \{0\}$. Since $\mathcal{R}$ is prime, $F([r, s]) = 0$ or $g([u_0, v_0]) = [u_0, v_0] \in Z(\mathcal{R})$ and hence by (3.9), we obtain $F([r, s]) = 0$ or $[u, v] \in Z(\mathcal{R})$ for all $r, s, u, v \in \mathcal{R}$. By Lemma 2.2 we then conclude that $F([r, s]) = 0$ for all $r, s \in \mathcal{R}$ or $\mathcal{R}$ is commutative. Since $F([r, s]) = 0$ for all $r, s \in \mathcal{R}$ implies $d(F([r, s])) = 0$ for all $r, s \in \mathcal{R}$, the equality (3.5) becomes

$$d(F([r,s])) = 0 \quad \text{for all } r, s \in \mathcal{R} \quad \text{or} \quad \mathcal{R} \text{ is commutative.} \tag{3.10}$$

Suppose that $d(F([r, s])) = 0$ for all $r, s \in \mathcal{R}$. Using the definition of F and the choice of $x = [u, v]$, $y = F([r, s])$, we can easily arrive at $F([u, v])\mathcal{R}\big(g(F([r, s])) - [r, s]\big) = \{0\}$ for all $r, s, u, v \in \mathcal{R}$. Since $\mathcal{R}$ is prime, $g(F([r, s])) = [r, s]$ for all $r, s \in \mathcal{R}$ or $F([u, v]) = 0$ for all $u, v \in \mathcal{R}$.
Suppose that $g(F([r, s])) = [r, s]$ for all $r, s \in \mathcal{R}$. By definition of F, it is clear that $F(F([u, v])x) = F^2([u, v])x + F([u, v])d(x)$ and $F(F([u, v])x) = F^2([u, v])g(x) + F([u, v])d(x)$ for all $u, v, x \in \mathcal{R}$. By comparing the last two equalities, we can easily arrive at

$$F^2([u,v])x = F^2([u,v])g(x) \quad \text{for all } u, v, x \in \mathcal{R}. \tag{3.11}$$

Replacing x by xt in (3.11), we get

$$F^2([u,v])xt = F^2([u,v])g(xt) = F^2([u,v])g(x)g(t) = F^2([u,v])xg(t)$$

for all $u, v, x, t \in \mathcal{R}$. This implies that

$$F^2([u,v])\mathcal{R}(g(t) - t) = \{0\} \quad \text{for all } u, v, t \in \mathcal{R}.$$

Now the primeness of $\mathcal{R}$ yields that $F^2([u, v]) = 0$ for all $u, v \in \mathcal{R}$ or $g = id_{\mathcal{R}}$. In all cases, we can conclude

$$F^2([u,v]) = 0 \quad \text{for all } u, v \in \mathcal{R} \quad \text{or} \quad g = id_{\mathcal{R}}. \tag{3.12}$$

Suppose that $F^2([u, v]) = 0$. Now further expansion of

$$F(xF([u,v])) = F(F([u,v])x)$$

for all $x \in \mathcal{R}$ yields $F([u, v])F(x) = F([u, v])d(x)$ for all $x \in \mathcal{R}$. This implies that

$$F([u,v])\mathcal{R}(F(x) - d(x)) = \{0\} \quad \text{for all } x \in \mathcal{R},$$

and by primeness of $\mathcal{R}$, we get either $F([u, v]) = 0$ for all $u, v \in \mathcal{R}$ or $F = d$.

If $F = d$, by Theorem 3.1 of [8] we conclude that $\mathcal{R}$ is commutative.

If $F([u, v]) = 0$ for all $u, v \in \mathcal{R}$, taking vu in place of v we get

$$0 = F([u,vu]) = F([u,v])g(u) + [u,v]d(u) = [u,v]d(u) \quad \text{for all } u, v \in \mathcal{R},$$

which implies that $uvd(u) = vud(u)$ for all $u, v \in \mathcal{R}$. Replacing v by tv in the last equation and using it, we arrive at $[u, t]\mathcal{R}d(u) = \{0\}$ for all $u, t \in \mathcal{R}$ and the primeness of $\mathcal{R}$ assures that either $u \in Z(\mathcal{R})$ or $d(u) = 0$ for all $u \in \mathcal{R}$. It follows from Lemma 2.1 that $d(\mathcal{R}) \subseteq Z(\mathcal{R})$ which implies that $d(xy)t = td(xy)$ for all $x, y, t \in \mathcal{R}$. By developing the last expression, we arrive at

$$d(x)yt + g(x)d(y)t = td(x)y + tg(x)d(y) \quad \text{for all} \quad x, y, t \in \mathcal{R}. \tag{3.13}$$

Replacing t by $g(x)$, we arrive at $d(x)\mathcal{R}[g(x), y] = \{0\}$ for all $x, y \in \mathcal{R}$. Since $\mathcal{R}$ is prime, either $d(x) = 0$ or $g(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. If there is $x_0 \in \mathcal{R}$ such that $d(x_0) = 0$, replace x by x_0 in (3.13), we get $d(y)\mathcal{R}[g(x_0), t] = \{0\}$ for all $x, t \in \mathcal{R}$. By primeness of $\mathcal{R}$ with $d \neq 0$, we obtain $g(x_0) \in Z(\mathcal{R})$. In all case, $g(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and the fact g is onto forces that $\mathcal{R}$ is commutative.

If $g = id_{\mathcal{R}}$, then F becomes a generalized derivation and [18, Lemma 1] forces that $\mathcal{R}$ is commutative.

$(ii) \Rightarrow (iii)$. Using similar techniques with necessary variations, we get the desired result. We skip the details of the proof just to avoid repetition. □

The following example shows that we cannot removed the primeness of $\mathcal{R}$ in Theorem 3.1.

EXAMPLE 3.1. *Let*

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$

Define maps $F, d, g : \mathcal{R} \to \mathcal{R}$ by

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ g \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } d = g.$$

It is clear that $\mathcal{R}$ is not prime and F is a generalized semiderivation on $\mathcal{R}$ with associated semiderivation d of $\mathcal{R}$ which satisfies the following:

(i) $F([A, B]) \in Z(\mathcal{R})$ **(ii)** $F(A \circ B) \in Z(\mathcal{R})$

for all $A, B \in \mathcal{R}$. However, $\mathcal{R}$ is not commutative.

THEOREM 3.2. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a generalized semiderivation F associated with a semiderivation d. If $F \neq id_{\mathcal{R}}$, then the following assertions are equivalent:*

(i) $F(xy) - xy \in Z(\mathcal{R})$ *for all $x, y \in \mathcal{R}$;*
(ii) $F(xy) - yx \in Z(\mathcal{R})$ *for all $x, y \in \mathcal{R}$;*
(iii) *$\mathcal{R}$ is commutative.*

Proof. $(i) \Rightarrow (iii)$. We start the proof by treating the case $Z(\mathcal{R}) = \{0\}$. In this case, we have $F(xy) = xy$ for all $x, y \in \mathcal{R}$.

If $F = 0$, then $xy = 0$ for all $x, y \in \mathcal{R}$ which implies that $x\mathcal{R}y = \{0\}$ for all $x, y \in \mathcal{R}$. The primeness of $\mathcal{R}$ gives a contradiction.

Now suppose that $F \neq 0$, then

$$\begin{aligned} xyt &= F(xyt) \\ &= F(xy)t + g(xy)d(t) \\ &= xyt + g(xy)d(t) \quad \text{for all } \ x, y, t \in \mathcal{R}. \end{aligned}$$

This implies that $g(xy)d(t) = 0$ for all $x, y, t \in \mathcal{R}$ and replacing t by rt in the last expression, we arrive at $g(xy)\mathcal{R}d(t) = \{0\}$ for all $x, y, t \in \mathcal{R}$. By primeness of $\mathcal{R}$, we get

$$g(xy) = 0 \quad \text{for all } \ x, y \in \mathcal{R} \quad \text{or} \quad d = 0. \tag{3.14}$$

If $g(xy) = 0$ for all $x, y \in \mathcal{R}$, then by definition of F, we have

$$\begin{aligned} F(xyz) &= F(x)g(yz) + xd(yz) \\ &= F(x)yz + g(x)d(yz) \quad \text{for all } \ x, y, z \in \mathcal{R} \end{aligned}$$

which means that $xd(yz) = F(x)yz + g(x)d(yz)$ for all $x, y, z \in \mathcal{R}$. Taking xt in the place of x, we can easily arrive at $x\mathcal{R}(d(yz) - yz) = \{0\}$ for all $x, y, z \in \mathcal{R}$. Since $\mathcal{R}$ is prime, we get $d(yz) = yz$ for all $y, z \in \mathcal{R}$. Using the definition of d, we have $d(xy)z + g(xy)d(z) = d(xy)g(z) + xyd(z)$ for all $x, y, z \in \mathcal{R}$ which easily gives us the equation $x\mathcal{R}(d(z) + g(z) - z) = \{0\}$ for all $x, z \in \mathcal{R}$. By primeness of $\mathcal{R}$, we get $d = id_{\mathcal{R}} - g$ and using again the definition of F, we have $F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$ for all $x, y \in \mathcal{R}$, then $F(x)(g(y) - y) + (x - g(x))d(y) = 0$ for all $x, y \in \mathcal{R}$. This implies that $(F(x) - d(x))d(y) = 0$ for all $x, y \in \mathcal{R}$. Replacing y by yt in last equation and using it again, we arrive at $(F(x) - d(x))\mathcal{R}d(t) = \{0\}$ for all $x, t \in \mathcal{R}$. Using the primeness of $\mathcal{R}$ and $d \neq 0$, we arrive at $F = d$.

Now, replacing x by $g(x)$ in equation $d(xy) = xy$, we obtain $d(g(x))g(y) + g(x)d(y) = g(x)y$ for all $x, y \in \mathcal{R}$. Using $d(g(x)) = g(d(x))$ for all $x \in \mathcal{R}$ and the fact that $g(xy) = 0$ for all $x, y \in \mathcal{R}$, we arrive at $g(x)(d(y) - y) = 0$ for all $x, y \in \mathcal{R}$. Since g is onto, $x(d(y) - y) = 0$ for all $x, y \in \mathcal{R}$ and by the primeness of $\mathcal{R}$, we obtain $d = id_{\mathcal{R}} = F$ and this expression contradicts with $F \neq id_{\mathcal{R}}$.

If $d = 0$, then $F(x)g(y) = F(x)y$ for all $x, y \in \mathcal{R}$ which implies that $F(x)(g(y) - y) = 0$ for all $x, y \in \mathcal{R}$. Putting xr instead of x we get $xr(g(y) - y) = 0$ for all $x, y, r \in \mathcal{R}$ which means $x\mathcal{R}(g(y) - y) = \{0\}$ for all $x, y \in \mathcal{R}$ and by the primeness of $\mathcal{R}$ again, we conclude that $g = id_{\mathcal{R}}$, in this case $F(xy) = F(x)y = xy$ for all $x, y \in \mathcal{R}$. Taking ry in place of y we get $(F(x) - x)\mathcal{R}y = \{0\}$ for all $x, y \in \mathcal{R}$ which gives a contradiction with $F \neq id_{\mathcal{R}}$.

Now suppose that $Z(\mathcal{R}) \neq \{0\}$ and

$$F(xy) - xy \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.15}$$

It follows that $F(x)g(y) + xd(y) - xy \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}$. This can be rewritten as

$$(F(xt) - xt)g(y) + xt(d(y) + g(y) - y) \in Z(\mathcal{R}) \quad \text{for all } \ x, y, t \in \mathcal{R}. \tag{3.16}$$

This implies that

(3.17) $g(y)xt(d(y)+g(y)-y) = xt(d(y)+g(y)-y)g(y)$ for all $x, y, t \in \mathcal{R}$.

Putting xu in place of x in the above equality and using it, then we get,

$$[x, g(y)]\mathcal{R}t(d(y)+g(y)-y) = \{0\} \quad \text{for all } x, y, t \in \mathcal{R}.$$

By primeness of $\mathcal{R}$, we get

(3.18) $g(y) \in Z(\mathcal{R})$ or $d(y)+g(y)-y = 0$ for all $y \in \mathcal{R}$.

If there exists $y_0 \in \mathcal{R}$ such that $g(y_0) \in Z(\mathcal{R})$, by (3.16) we arrive at

$$xt(d(y_0)+g(y_0)-y_0) \in Z(\mathcal{R}) \quad \text{for all } x, t \in \mathcal{R}.$$

Replacing x by ux in the last expression with primeness of $\mathcal{R}$, we can easily show that $d(y_0)+g(y_0)-y_0 = 0$ or $\mathcal{R}$ is commutative, and in this case, (3.18) becomes

(3.19) $d(y)+g(y)-y = 0$ for all $y \in \mathcal{R}$ or $\mathcal{R}$ is commutative.

Suppose that $d(y)+g(y)-y = 0$ for all $y \in \mathcal{R}$. By (3.15), (3.16) and the primeness of $\mathcal{R}$, we get

(3.20) $F(xt) = xt$ for all $x, t \in \mathcal{R}$ or $g(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$.

The first case is already treated previously and for the second case, the fact that g is onto forces that $\mathcal{R}$ is commutative.

(ii) $\Rightarrow$ *(iii)*. Assume that $Z(\mathcal{R}) = \{0\}$, then $F(xy) = yx$ for all $x, y \in \mathcal{R}$. If $F = 0$, then $yx = 0$ for all $x, y \in \mathcal{R}$. The primeness of $\mathcal{R}$ gives a contradiction.

If $F \neq 0$, using the definition of F we obtain $g(x)d(y) = yx - F(x)y$ for all $x, y \in \mathcal{R}$. Replacing x by tx in the last equation, we get $g(t)g(x)d(y) = ytx - xty$ for all $x, y, t \in \mathcal{R}$. For $y = x$, we have $g(t)g(x)d(x) = 0$ for all $x, t \in \mathcal{R}$. Using the fact that g is onto and the primeness of $\mathcal{R}$, we find that $g(x)d(x) = 0$ for all $x \in \mathcal{R}$. Using the definition of F, we have $F(x)g(y) + xd(y) = yx$ and right multiplying by $d(y)$, we arrive at $xd(y)d(y) = yxd(y)$ for all $x, y \in \mathcal{R}$. Replacing x by xt in last equation and using it again, we arrive at $[x, y]\mathcal{R}d(y) = \{0\}$ for all $x, y \in \mathcal{R}$. By primeness of $\mathcal{R}$ we get either $y \in Z(\mathcal{R})$ or $d(y) = 0$ for all $y \in \mathcal{R}$. By Lemma 2.1, we get $d(\mathcal{R}) \subseteq Z(\mathcal{R})$ and this forces that $\mathcal{R}$ is commutative.

Assume that

(3.21) $F(xy) - yx \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.

It follows that

(3.22) $(F(x)-x)y + g(x)d(y) + [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.

Putting yx in place of y, we get

$$((F(x)-x)y + g(x)d(y) + [x, y])x + g(x)g(y)d(x) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}.$$

Using (3.22), we can rewrite the last equality as

(3.23) $g(x)g(y)d(x) \in Z(\mathcal{R})$ for all $x \in Z(\mathcal{R}), y \in \mathcal{R}$.

Since g is onto, (3.23) implies that $g(x)yd(x) \in Z(\mathcal{R})$ for all $x \in Z(\mathcal{R})$, $y \in \mathcal{R}$. Since $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$, we get $g(x)y \in Z(\mathcal{R})$ or $d(x) = 0$ for all $x \in Z(\mathcal{R})$, $y \in \mathcal{R}$. Replacing y by yr in the last equation and using it with the primeness of $\mathcal{R}$, we obtain either $g(x)y = 0$ or $d(x) = 0$ or $r \in Z(\mathcal{R})$ for all $x \in Z(\mathcal{R})$, $y, r \in \mathcal{R}$ which implies that $g(x)\mathcal{R}y = \{0\}$ or $d(x) = 0$ for all $x \in Z(\mathcal{R})$, $y \in \mathcal{R}$ or $\mathcal{R}$ is commutative. By primeness of $\mathcal{R}$, we arrive at $g(x) = 0$ or $d(x) = 0$ for all $x \in Z(\mathcal{R})$ or $\mathcal{R}$ is commutative. Using the fact that g is onto, we obtain $d(Z(\mathcal{R}) = \{0\}$ or $\mathcal{R}$ is commutative. Assume that $d(Z(\mathcal{R}) = \{0\}$. By (3.22), we obtain $(F(x) - x)y \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, $y \in Z(\mathcal{R})$. By primeness of $\mathcal{R}$, we get $F(x) - x \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ or $Z(\mathcal{R}) = \{0\}$. The second case is already treated previously and for the first case, we have $F(xy) - xy \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Combining this expression with (3.21), we conclude that $[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ which forces that $\mathcal{R}$ is commutative by Lemma 2.2 . □

Using similar techniques with necessary variations, one can easily prove the following theorem.

THEOREM 3.3. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a generalized semiderivation F associated with a semiderivation d. If $F \neq -id_{\mathcal{R}}$, then the following assertions are equivalent:*

(i) *$F(xy) + xy \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,*
(ii) *$F(xy) + yx \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,*
(iii) *$\mathcal{R}$ is commutative.*

REMARK 3.1. *For $F = \pm id_{\mathcal{R}}$, is easy to demonstrate (ii) $\Leftrightarrow$ (iii) in Theorems 3.2 and 3.3, just by apply Lemma 2.2.*

The following example shows that we cannot removed the primeness of $\mathcal{R}$ in Theorems 3.2 (i) and 3.4 (i).

EXAMPLE 3.2. *Let*

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z}_2 \right\}.$$

Define maps $F, d, g : \mathcal{R} \to \mathcal{R}$ by

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, \ d = F, \ g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that $\mathcal{R}$ is not prime and F is a generalized semiderivation on $\mathcal{R}$ with associated semiderivation d of $\mathcal{R}$ which satisfies the following:

(i) *$F(AB) \pm AB \in Z(\mathcal{R})$ for all $A, B \in \mathcal{R}$.*

However, $\mathcal{R}$ is not commutative.

In [12], H. E. Bell and N.-Ur Rehman showed that the prime ring $\mathcal{R}$ with 1 and char$(\mathcal{R}) \neq 2$ is commutative if the condition $[F(x), F(y)] = 0$ for all

$x, y \in \mathcal{R}$ holds in $\mathcal{R}$. Several authors investigated this result for prime ring admitting derivation and generalized derivation. Motivated by these works, we will prove this result in prime ring with 1 involving generalized semiderivation.

THEOREM 3.4. *Let $\mathcal{R}$ be a prime ring with* 1 *such that char($\mathcal{R}$) $\neq 2$. If $\mathcal{R}$ admits a generalized semiderivation F associated with a nonzero semiderivations d, then the following assertions are equivalent:*

(i) $[F(x), F(y)] = 0$ *for all* $x, y \in \mathcal{R}$;

(ii) $\mathcal{R}$ *is commutative.*

Proof. Obviously $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Suppose that $[F(x), F(y)] = 0$ for all $x, y \in \mathcal{R}$, This means that $F(x)F(y) = F(y)F(x)$ for all $x, y \in \mathcal{R}$. Using the definition of F after putting y by $yF(z)$, we arrive at $F(x)yd(F(z)) = y(F(z))F(x)$ for all $x, y, z \in \mathcal{R}$. By replacing y by ys for arbitrary $s \in \mathcal{R}$ it is easy to see that $[F(x), y]\mathcal{R}d(F(z)) = \{0\}$ for all $x, y, z \in \mathcal{R}$. Since $\mathcal{R}$ is prime, either $d(F(\mathcal{R})) = \{0\}$ or $F(\mathcal{R}) \subseteq Z(\mathcal{R})$.

Now, suppose the second case. By Lemma 2.3 we obtain $F(d(x)) = 0$ for all $x \in \mathcal{R}$. Using the definition of F, we have

$$\begin{aligned} F(d(x)y) &= F(d(x))y + g(d(x))d(y) \\ &= F(d(x))g(y) + d(x)d(y) \quad \text{for all } x, y \in \mathcal{R}. \end{aligned}$$

This implies that $(g(d(x)) - d(x))d(y) = 0$ for all $x, y \in \mathcal{R}$. Letting ry in place of y, we arrive at $(g(d(x)) - d(x))\mathcal{R}d(y) = \{0\}$ for all $x, y \in \mathcal{R}$ and by the primeness of $\mathcal{R}$ and the fact that $d \neq 0$, we get $g(d(x)) = d(x) = d(g(x))$ for all $x \in \mathcal{R}$. Invoking this in the relation of $F(d(xy)) = 0$, using two different ways, we arrive at $F(x)d(y) + d(x)d(y) + xd^2(y) = 0$ and $F(x)d(y) + d(x)d(y) + g(x)d^2(y) = 0$ for all $x, y \in \mathcal{R}$. By comparing the latter expressions, we can easily arrive at $g(x)d^2(y) = xd^2(y)$ for all $x, y \in \mathcal{R}$. Replacing x by rx in the latter equation we find that $(g(x) - x)\mathcal{R}d^2(y) = \{0\}$ for all $x, y \in \mathcal{R}$. By the primeness of $\mathcal{R}$, we arrive at either $g = id_{\mathcal{R}}$ or $d^2(y) = 0$ for all $y \in \mathcal{R}$. Suppose we have the second condition. Then

$$\begin{aligned} 0 &= d^2(yz) = d^2(y)g(z) + g(d(y))d(z) + d(g(y))d(z) + g(y)d^2(z) \\ &= 2d(y)d(z) \text{ for all } y, z \in \mathcal{R}. \end{aligned}$$

Since $char(\mathcal{R}) \neq 2$, the last expression becomes $d(x)d(y) = 0$ for all $x, y \in \mathcal{R}$. Letting $y = rt$ for arbitrary $r \in \mathcal{R}$ and using the definition of d, we can conclude that $d(x)\mathcal{R}d(t) = \{0\}$ for all $x, t \in \mathcal{R}$ and primeness of $\mathcal{R}$ forces that $d = 0$, a contradiction.

Now assume that $g = id_{\mathcal{R}}$. Then F becomes a generalized derivation and by Theorem 3.4 of [12], we find that $\mathcal{R}$ is commutative.

Now suppose that $F(\mathcal{R}) \subseteq Z(\mathcal{R})$. Then $F(x)y = yF(x)$ for all $x, y \in \mathcal{R}$. Replacing x by xt in the latter expression and using it again, we get $xd(y)g(y) = g(y)xd(y)$ for all $x, y \in \mathcal{R}$. Putting xr in place of x, we arrive at

$[g(y), x]\mathcal{R}d(y) = \{0\}$ for all $x, y \in \mathcal{R}$. Since $\mathcal{R}$ is prime, we obtain

$$d(y) = 0 \quad \text{or} \quad g(y) \in Z(\mathcal{R}) \quad \text{for all} \quad y \in \mathcal{R}. \tag{3.24}$$

Suppose there is $y_0 \in \mathcal{R}$ such that $d(y_0) = 0$. Using the definition of F and the fact that $F(\mathcal{R}) \subseteq Z(\mathcal{R})$, we get $F(x)g(y_0)t + xd(y_0)t = tF(x)g(y_0) + txd(y_0)$ for all $x, t \in \mathcal{R}$ which forces that $F(x)\mathcal{R}[g(y_0), t] = 0$ for all $x, t \in \mathcal{R}$. Using the primeness of $\mathcal{R}$ and $F \neq 0$, we find that $g(y_0) \in Z(\mathcal{R})$. In all case, (3.24) forces that $g(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Since g is onto, we obtain that $\mathcal{R} \subseteq Z(\mathcal{R})$ which forces that $\mathcal{R}$ is commutative. □

If the commutator is replaced by the anti-commutator in Theorem 3.4, then the conclusions of this theorem is not valid.

THEOREM 3.5. *Let $\mathcal{R}$ be a prime ring with 1 such that $char(\mathcal{R}) \neq 2$. Then there exists no generalized semiderivation F associated with a nonzero semiderivation d satisfying $F(x) \circ F(y) = 0$ for all $x, y \in \mathcal{R}$.*

Proof. By the assumption, for all $x, y, z \in \mathcal{R}$, we get $F(x)F(yF(z)) + F(yF(z))F(x) = 0$. This implies that $F(x)yd(F(z)) = -yd(F(z))F(x)$. Replacing y by yt, we get

$$F(x)ytd(F(z)) = -ytd(F(z))F(x) = yF(x)td(F(z)) \text{ for all } x, y, z, t \in \mathcal{R}.$$

This yields that $[F(x), y]\mathcal{R}d(F(z)) = \{0\}$ for all $x, y, z \in \mathcal{R}$. Since $\mathcal{R}$ is prime, $F(\mathcal{R}) \subseteq Z(\mathcal{R})$ or $d(F(\mathcal{R})) = \{0\}$.

Suppose the first case. Then $F(x) \circ F(y) = 0$ for all $x, y \in \mathcal{R}$ becomes $2F(x)\mathcal{R}F(y) = \{0\}$ for all $x, y \in \mathcal{R}$. Using primeness of $\mathcal{R}$ and $char(\mathcal{R}) \neq 2$, we conclude that $F = 0$; a contradiction.

Now, suppose that $d(F(\mathcal{R})) = \{0\}$. Proceeding in the similar manner as above, we obtain $g = id_{\mathcal{R}}$, then F becomes a generalized derivation and by Theorem 4.1 of [12], we get $F = 0$. □

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