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ON THE SYMMETRIES OF LORENTZIAN FOUR-DIMENSIONAL GENERALIZED SYMMETRIC SPACES OF TYPE C

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Abstract. We consider the four-dimensional generalized symmetric spaces of type C, equipped with a left-invariant Lorentzian metric. We completely describe its affine, homothetic and Killing vector fields. We also obtain a full classification of its Ricci, curvature and matter collineations.

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Key words. Generalized symmetric spaces, left-invariant metrics, Killing vector fields, affine vector fields, Lorentzian metrics, curvature and matter collineations.

1. INTRODUCTION

Let (M, g) be a pseudo-Riemannian manifold, a Killing vector field is a vector field on (M, g) that preserves the metric. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector will not distort distances on the object. Specifically, a vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes: $\mathcal{L}_X g = 0$. In terms of the Levi-Civita connection, this is equivalent to $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$ for all vector fields $Y, Z \in \mathfrak{X}(M)$. Therefore, it is sufficient to establish it in a preferred coordinate system in order to have it hold in the algebra systems. The Killing fields on a manifold M form a Lie subalgebra of vector fields on M. This is the Lie algebra of the isometry group of the manifold if M is complete. A typical use of the Killing field is to express a symmetry in general relativity (in which the geometry of spacetime as distorted by gravitational fields is viewed as a 4-dimensional pseudo-Riemannian manifold). In a static configuration, in which nothing changes with time, the time vector will be a Killing vector, and thus the Killing field will point in the direction of forward motion in time.

On the other hand, a vector field X tangent to (M, g) is said to be affine if it satisfies $\mathcal{L}_X \nabla = 0$, where ∇ is the Levi-Civita connection of (M, g) (or

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equivalently, if $[X, \nabla_Y Z] = \nabla_{[X,Y]}Z + \nabla_Y[X, Z]$ for all vector fields $Y, Z \in \mathfrak{X}(M)$) which means that the local fluxes of X are given by affine maps. Obviously, a Killing vector field is also affine. However, the converse does not holds in general. In particular, if (M, g) is a simply connected spacetime, the existence of a non Killing affine vector field implies the existence of a second-order covariantly constant symmetric tensor, nowhere vanishing, not proportional to g. As a consequence, the holonomy group of the manifold is reducible (see for example [19]).

A curvature (resp. Ricci) collineation is a vector field X which preserves the Riemann curvature tensor R (resp. the Ricci tensor Ric) in the sense that, $\mathcal{L}_X R = 0$ (resp. $\mathcal{L}_X Ric = 0$), where $\mathcal{L}$ denotes the Lie derivative. The set of all smooth curvature collineations forms a Lie algebra under the Lie bracket operation, which may be infinite-dimensional. Every affine vector field is a curvature collineation.

A matter collineation is a vector field X that satisfies the condition $\mathcal{L}_X T = 0$, where T is the energy-momentum tensor given by $T = Ric - \frac{1}{2}\tau g$ where τ denotes the scalar curvature. The relation between geometry and physics may be highlighted here, as the vector field X is regarded as preserving certain physical quantities along the flow lines of X, this being true for any two observers. In connection with this, it may be shown that every Killing vector field is a matter collineation (by the Einstein field equations, with or without cosmological constant). Thus, a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor. When the energy-momentum tensor represents a perfect fluid, every Killing vector field preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a Killing vector field does not necessarily preserve the electric and magnetic fields.

More general, a collineation or a symmetry of a tensor field S on a pseudo-Riemannian manifold (M, g) is a one-parameter group of diffeomorphisms of (M, g), which leaves S invariant. Therefore, each symmetry corresponds to a vector field X which satisfies $\mathcal{L}_X S = 0$. Symmetries of the metric tensor g which correspond to the Killing vector fields. Symmetries of the Levi-Civita connection ∇ which correspond to the affine vector fields. Since symmetries are more significant from physical aspects, they have been studied on several kinds of space-times (see [14, 15], [9, 12, 13, 10, 11],[16]).

The aim of this paper, is to study symmetries of the four-dimensional generalized symmetric spaces of type C, equipped with a left-invariant Lorentzian metric. The paper is organized in the following way. In Section 3, we shall report some basic information about four-dimensional generalized symmetric spaces of type C and its left-invariant metrics in global coordinates, we shall describe their Levi-Civita connection, the curvature and the Ricci tensor. In Section 4, affine, homothetic and Killing vector fields of four-dimensional generalized symmetric spaces of type C are characterized via a system of partial differential equations. Then, in Section 5 and 6, we shall respectively classify Ricci, curvature and matter collineations on the four-dimensional generalized symmetric spaces of type C equipped with Lorentzian left-invariant metric.

2. PRELIMINARIES

Let (M, g) be a connected pseudo-Riemannian manifold and x be a point of M. A symmetry at x is an isometry s_x of M, having x as an isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. This property was generalized by A.J. Ledger, who defined a regular $s-$structure as a family $\{s_x : x \in M\}$ of symmetries of (M, g) satisfying $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$, for all x, y of M. The order of an $s-$structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may happen that $k = \infty$). A generalized symmetric space is a connected pseudo-Riemannian manifold (M, g) admitting a regular $s-$structure. The order of a generalized symmetric space is the minimum of all integers $k \geq 2$ such that M admits a regular $s-$structure of order k. The classification of four-dimensional generalized symmetric spaces was obtained by J. Cerny and O. Kowalsky and is resumed in the following four types:

THEOREM 2.1. *All proper, simply connected generalized symmetric spaces (M, g) of dimension $n = 4$ are of order 3 or infinity. All this spaces are indecomposable, and belong (up to isometry) to the following four types:*

- ***Type A**. The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & x_3 \\ c & d & x_4 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad-bc = 1$. (M, g) is the space $\mathbb{R}^4(x_1, x_2, x_3, x_4)$ with the pseudo-Riemannian metric

$$g = \pm[(-x_1 + \sqrt{1 + x_1^2 + x_2^2})\mathrm{d}x_3^2 + (x_1 + \sqrt{1 + x_1^2 + x_2^2})\mathrm{d}x_4^2 - 2y^2\mathrm{d}x_3\mathrm{d}x_4]$$
$$+\lambda[(1 + x_2^2)\mathrm{d}x_1^2 + (1 + x_1^2)\mathrm{d}x_2^2 - 2x_1x_2\mathrm{d}x_1\mathrm{d}x_2]/(1 + x_1^2 + x_2^2),$$

where $\lambda \neq 0$ is a real constant. The order is $k = 3$ and possible signatures are $(4, 0), (0, 4), (2, 2)$. The typical symmetry of order 3 at the initial point $(0, 0, 0, 0)$ is the transformation

$$(x_1, x_2, x_3, x_4) \mapsto (-\frac{x_1}{2}x_1 + \frac{\sqrt{3}x_2}{2}, -\frac{\sqrt{3}x_1}{2} - \frac{x_2}{2}, -\frac{x_3}{2} - \frac{\sqrt{3}x_4}{2}, -\frac{\sqrt{3}x_3}{2} - \frac{x_4}{2}).$$

- ***Type B**. The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} \mathrm{e}^{-(x_1+x_2)} & 0 & 0 & a \\ 0 & \mathrm{e}^{x_1} & 0 & b \\ 0 & 0 & \mathrm{e}^{x_2} & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & -\omega \\ 0 & 1 & 0 & -2\omega \\ 0 & 0 & 1 & 2\omega \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) is the space $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ with the pseudo-Riemannian metric

$$g=\lambda(\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_1\mathrm{d}x_2)+\mathrm{e}^{-x_2}(2\mathrm{d}x_1+\mathrm{d}x_2)\mathrm{d}x_4+\mathrm{e}^{-x_1}(\mathrm{d}x_1+2\mathrm{d}x_2)\mathrm{d}x_3,$$

where λ is a real constant. The order is $k=3$ and the signature is $(2,2)$. The typical symmetry of order 3 at the initial point $(0,0,0,0)$ is the transformation

$$(x_1,x_2,x_3,x_4))\mapsto\left(x_2,-x_1-x_2,-x_3\mathrm{e}^{-x_1+x_2}-x_4,x_3\mathrm{e}^{-2x_1-x_2}\right).$$

- ***Type C***. *The underlying homogeneous space is the matrix group*

$$G=\begin{pmatrix}\mathrm{e}^{-x_4} & 0 & 0 & x_1\\ 0 & \mathrm{e}^{x_4} & 0 & x_2\\ 0 & 0 & 1 & x_3\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

(M,g) is the space $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ with the pseudo-Riemannian metric

$$g=\varepsilon(\mathrm{e}^{-2x_4}\mathrm{d}x_1^2+\mathrm{e}^{2x_4}\mathrm{d}x_2^2)+\mathrm{d}x_3\mathrm{d}x_4, \tag{1}$$

where $\varepsilon\in\{-1,1\}$. The possible signatures of g are $(3,1),(1,3)$.

- ***Type D***. *The underlying homogeneous space is G/H, where*

$$G=\begin{pmatrix}a & b & x_1\\ c & d & x_2\\ 0 & 0 & 1\end{pmatrix},\quad H=\begin{pmatrix}\mathrm{e}^{t} & 0 & 0\\ 0 & \mathrm{e}^{-t} & 0\\ 0 & 0 & 1\end{pmatrix},$$

with $ad-bc=1$. (M,g) is the space $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ with the pseudo-Riemannian metric

$$\begin{aligned}g=&\,(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\,\mathrm{d}x_1^2+(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\,\mathrm{d}x_2^2\\ &-2\cosh(2x_3)\cos(2x_4)\mathrm{d}x_1\mathrm{d}x_2+\lambda\left(\mathrm{d}x_3^2-\cosh(2x_3)^2\mathrm{d}x_4^2\right),\end{aligned}$$

where λ is a non-zero real constant. The signature of g is $(2,2)$. The space is of order infinity.

3. CONNECTION AND CURVATURE OF FOUR-DIMENSIONAL GENERALIZED SYMMETRIC SPACE OF TYPE C

Let (M,g) be a four-dimensional generalized symmetric spaces of type C which is the space $\mathbb{R}(x_1,x_2,x_3,x_4)$, and denote by ∇, R and Ric the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor of $M,g)$ respectively. We will denote the coordinate basis $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\right\}$ by $\{\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{x_4}\}$.

The non-vanishing components of the Levi-Civita connection ∇ of (M,g) are given by

$$\begin{cases}\nabla_{\partial_{x_1}}\partial_{x_1}=2\varepsilon\mathrm{e}^{-2x_4}\partial_{x_3},\ \nabla_{\partial_{x_1}}\partial_{x_4}=-\partial_{x_1},\\ \nabla_{\partial_{x_2}}\partial_{x_2}=-2\varepsilon\mathrm{e}^{2x_4}\partial_{x_3},\ \nabla_{\partial_{x_2}}\partial_{x_4}=\partial_{x_2},\\ \nabla_{\partial_{x_4}}\partial_{x_1}=-\partial_{x_1},\qquad \nabla_{\partial_{x_4}}\partial_{x_2}=\partial_{x_2}.\end{cases} \tag{2}$$

The curvature tensor R is taken with the sign convention

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

The non-vanishing curvature tensor R components are computed as

$$(3) \qquad \begin{cases} R(\partial_{x_1}, \partial_{x_4})\partial_{x_1} = -2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}, & R(\partial_{x_1}, \partial_{x_4})\partial_{x_4} = \partial_{x_1}, \\ R(\partial_{x_2}, \partial_{x_4})\partial_{x_2} = -2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}, & R(\partial_{x_2}, \partial_{x_4})\partial_{x_4} = \partial_{x_2}. \end{cases}$$

The Ricci curvature Ric is defined by

$$(4) \qquad Ric(X,Y) = trace\{Z \to R(Z,X)Y\}.$$

The components $\{Ric_{ij}\}$ of the Ricci tensor are defined by

$$(5) \qquad Ric(\partial_{x_i}, \partial_{x_j}) = Ric_{ij} = \sum_{k=1}^{4} g(\partial_{x_k}, \partial_{x_k}) g(R(\partial_{x_k}, \partial_{x_i})\partial_{x_j}, \partial_{x_k}).$$

The non-vanishing components $\{Ric_{ij}\}$ are computed as

$$(6) \qquad Ric_{44} = -2.$$

The scalar curvature τ of (M,g) is constant and we have

$$(7) \qquad \tau = \mathrm{tr} Ric = \sum_{i=1}^{3} g(\partial_{x_i}, \partial_{x_i}) Ric(\partial_{x_i}, \partial_{x_i}) = -2.$$

4. AFFINE, HOMOTHETIC AND KILLING VECTOR FIELDS

We firstly classify affine, homothetic and Killing vector fields of the four-dimensional generalized symmetric spaces of type C. The classifications we obtain are summarized in the following theorem.

THEOREM 4.1. *Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ be an arbitrary vector field on the four-dimensional generalized symmetric spaces of type C.*

(1) *X is a affine vector field if and only if*

$$\begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \alpha_4 x_1 + \alpha_5, \\ f_2 = (-\alpha_2 x_1 + \alpha_7)\mathrm{e}^{-2x_4} + (\alpha_4 - 2\alpha_1)x_2 + \alpha_9, \quad \alpha_i \in \mathbb{R}, \\ f_3 = 2(\alpha_4 - \alpha_1)x_3 + \alpha_{11}x_4 - 4\varepsilon\alpha_3 x_1 + 4\varepsilon(-\alpha_2 x_1 + \alpha_7)x_2 + \alpha_{12}, \\ f_4 = \alpha_1, \ \alpha_i \in \mathbb{R}. \end{cases}$$

(2) *X is a homothetic vector field if and only if*

$$\begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + (\alpha_1 + \eta)x_1 + \alpha_4, \\ f_2 = (-\alpha_2 x_1 + \alpha_5)\mathrm{e}^{-2x_4} - \alpha_1 x_2 + \alpha_6, \\ f_3 = 4\varepsilon(-\alpha_2 x_1 + \alpha_5)x_2 - 4\varepsilon\alpha_3 x_1 + \eta x_3 + \alpha_7, \\ f_4 = \alpha_1 + \frac{\eta}{2}, \quad \alpha_i \in \mathbb{R}. \end{cases}$$

(3) *X is a Killing vector field if and only if*

$$\begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \alpha_1 x_1 + \alpha_4, \\ f_2 = (-\alpha_2 x_1 + \alpha_5)\mathrm{e}^{-2x_4} - \alpha_1 x_2 + \alpha_6, \\ f_3 = 4\varepsilon(-\alpha_2 x_1 + \alpha_5)x_2 - 4\varepsilon\alpha_3 x_1 + \alpha_7, \\ f_4 = \alpha_1, \quad \alpha_i \in \mathbb{R}. \end{cases}$$

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ denote an arbitrary vector field on the four-dimensional generalized symmetric spaces of type C, for some arbitrary smooth functions f_1, f_2, f_3, f_4 on M. Starting from (1), a direct calculation yields the following description of the Lie derivative of the metric tensor g :

$$\begin{cases} (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_1}) = 2\varepsilon\mathrm{e}^{-2x_4}(\partial_{x_1} f_1 - f_4), \\ (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_2}) = \varepsilon(\mathrm{e}^{-2x_4}\partial_{x_2} f_1 + \mathrm{e}^{2x_4}\partial_{x_1} f_2), \\ (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_3}) = \frac{1}{2}\partial_{x_1} f_4 + \varepsilon\mathrm{e}^{-2x_4}\partial_{x_3} f_1, \\ (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_4}) = \frac{1}{2}\partial_{x_1} f_3 + \varepsilon\mathrm{e}^{-2x_4}\partial_{x_4} f_1, \\ (\mathcal{L}_X g)(\partial_{x_2}, \partial_{x_2}) = 2\varepsilon\mathrm{e}^{2x_4}(f_4 + \partial_{x_2} f_2), \\ (\mathcal{L}_X g)(\partial_{x_2}, \partial_{x_3}) = \frac{1}{2}\partial_{x_2} f_4 + \varepsilon\mathrm{e}^{2x_4}\partial_{x_3} f_2, \\ (\mathcal{L}_X g)(\partial_{x_2}, \partial_{x_4}) = \frac{1}{2}\partial_{x_2} f_3 + \varepsilon\mathrm{e}^{2x_4}\partial_{x_4} f_2, \\ (\mathcal{L}_X g)(\partial_{x_3}, \partial_{x_3}) = \partial_{x_3} f_4, \\ (\mathcal{L}_X g)(\partial_{x_3}, \partial_{x_4}) = \frac{1}{2}(\partial_{x_3} f_3 + \partial_{x_4} f_4), \\ (\mathcal{L}_X g)(\partial_{x_4}, \partial_{x_4}) = 2\partial_{x_4} f_3. \end{cases} \tag{8}$$

In order to determine the Killing vector fields, we then must solve the system of PDEs obtained, requiring that all the coefficients in the above Lie derivative are equal to zero.

A straightforward calculations lead to prove that

$$\begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \alpha_1 x_1 + \alpha_4, \\ f_2 = (-\alpha_2 x_1 + \alpha_5)\mathrm{e}^{-2x_4} - \alpha_1 x_2 + \alpha_6, \\ f_3 = 4\varepsilon(-\alpha_2 x_1 + \alpha_5)x_2 - 4\varepsilon\alpha_3 x_1 + \alpha_7, \\ f_4 = \alpha_1, \quad \alpha_i \in \mathbb{R}. \end{cases}$$

Then, we make again use the above formula $\mathcal{L}_X g$ and now require that $\mathcal{L}_X g = \eta g$, for some real constant $\eta \neq 0$. The solutions of the corresponding system of PDEs give us the homothetic vector fields of the four-dimensional generalized symmetric spaces of type C, proving part (2) of the statement of the Theorem 4.1 .

To determine the affine Killing vector fields, we need to calculate the Lie derivative of the Levi-Civita connection ∇. Staring from (2), we find the following possibly non-vanishing components:

$$
(9)\quad \begin{cases}
(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_1}) = (\partial^2_{x_1}f_1 - 2\partial_{x_1}f_4 - 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}f_1)\partial_{x_1} \\
\qquad +(\partial^2_{x_1}f_2 - 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}f_2)\partial_{x_2} \\
\qquad +(\partial^2_{x_1}f_3 - 4\varepsilon \mathrm{e}^{-2x_4}f_4 + 4\varepsilon \mathrm{e}^{-2x_4}\partial_{x_1}f_1 - 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}f_3)\partial_{x_3} \\
\qquad +(\partial^2_{x_1}f_4 - 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}f_4)\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_2}) = (\partial_{x_1}\partial_{x_2}f_1 - \partial_{x_2}f_4)\partial_{x_1} + (\partial_{x_1}\partial_{x_2}f_2 + \partial_{x_1}f_4)\partial_{x_2} \\
\qquad +(\partial_{x_1}\partial_{x_2}f_3 - 2\varepsilon \mathrm{e}^{2x_4}\partial_{x_1}f_2 + 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_2}f_1)\partial_{x_3} + \partial_{x_1}\partial_{x_2}f_4\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_3}) = (\partial_{x_1}\partial_{x_3}f_1 - \partial_{x_3}f_4)\partial_{x_1} + \partial_{x_1}\partial_{x_3}f_2\partial_{x_2} \\
\qquad +(\partial_{x_1}\partial_{x_3}f_3 + 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_3}f_1)\partial_{x_3} + \partial_{x_1}\partial_{x_3}f_4\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_4}) = (\partial_{x_1}\partial_{x_4}f_1 - \partial_{x_4}f_4)\partial_{x_1} + (\partial_{x_1}\partial_{x_4}f_2 + 2\partial_{x_1}f_2)\partial_{x_2} \\
\qquad +(\partial_{x_1}\partial_{x_4}f_3 + \partial_{x_1}f_3 + 2\varepsilon \mathrm{e}^{-2x_4}\partial_{x_4}f_1)\partial_{x_3} + (\partial_{x_1}\partial_{x_4}f_4 + \partial_{x_1}f_4)\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_2}) = (\partial^2_{x_2}f_1 + 2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}f_1)\partial_{x_1} + (\partial^2_{x_2}f_2 + 2\partial_{x_2}f_4 \\
\qquad +2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}f_2)\partial_{x_2} + (\partial^2_{x_2}f_3 - 4\varepsilon \mathrm{e}^{2x_4}f_4 - 4\varepsilon \mathrm{e}^{2x_4}\partial_{x_2}f_2 \\
\qquad +2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}f_3)\partial_{x_3} + (\partial^2_{x_2}f_4 + 2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}f_4)\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_3}) = \partial_{x_2}\partial_{x_3}f_1\partial_{x_1} + (\partial_{x_2}\partial_{x_3}f_2 + \partial_{x_3}f_4)\partial_{x_2} \\
\qquad +(\partial_{x_2}\partial_{x_3}f_3 + 2\varepsilon \mathrm{e}^{2x_4}\partial_{x_3}f_2)\partial_{x_3} + \partial_{x_2}\partial_{x_3}f_4\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_4}) = (\partial_{x_2}\partial_{x_4}f_1 - 2\partial_{x_2}f_1)\partial_{x_1} + (\partial_{x_2}\partial_{x_4}f_2 + \partial_{x_4}f_4)\partial_{x_2} \\
\qquad +(\partial_{x_2}\partial_{x_4}f_3 - \partial_{x_2}f_3 + 2\varepsilon \mathrm{e}^{2x_4}\partial_{x_4}f_2)\partial_{x_3} + (\partial_{x_2}\partial_{x_4}f_4 - \partial_{x_2}f_4)\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_3},\partial_{x_3}) = \partial^2_{x_3}f_1\partial_{x_1} + \partial^2_{x_3}f_2\partial_{x_2} + \partial^2_{x_3}f_3\partial_{x_3} + \partial^2_{x_3}f_4\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_3},\partial_{x_4}) = (\partial_{x_3}\partial_{x_4}f_1 - \partial_{x_3}f_1)\partial_{x_1} + (\partial_{x_3}\partial_{x_4}f_2 + \partial_{x_3}f_2)\partial_{x_2} \\
\qquad +\partial_{x_3}\partial_{x_4}f_3\partial_{x_3} + \partial_{x_3}\partial_{x_4}f_4\partial_{x_4}, \\
(\mathcal{L}_X\nabla)(\partial_{x_4},\partial_{x_4}) = (\partial^2_{x_4}f_1 - 2\partial_{x_4}f_1)\partial_{x_1} + (\partial^2_{x_4}f_2 + 2\partial_{x_4}f_2)\partial_{x_2} \\
\qquad +\partial^2_{x_4}f_3\partial_{x_3} + \partial^2_{x_4}f_4\partial_{x_4}.
\end{cases}
$$

In order to determine the affine vector fields, we then must solve the system of PDEs obtained, requiring that all the coefficients in the above Lie derivative are equal to zero.

Deriving equations $(\mathcal{L}_X\nabla)^4(\partial_{x_1},\partial_{x_4}) = 0$ and $(\mathcal{L}_X\nabla)^4(\partial_{x_2},\partial_{x_4}) = 0$ with respect to x_4, and using equation $(\mathcal{L}_X\nabla)^4(\partial_{x_4},\partial_{x_4}) = 0$, we obtain $\partial_{x_1}f_4 = \partial_{x_2}f_4 = 0$. And from equations $(\mathcal{L}_X\nabla)^4(\partial_{x_1},\partial_{x_1}) = 0$ and $(\mathcal{L}_X\nabla)^1(\partial_{x_1},\partial_{x_4}) = 0$, since $\partial_{x_1}f_4 = 0$, we get $\partial_{x_3}f_4 = \partial_{x_4}f_4 = 0$. Thus $f_4 = \alpha_1, \alpha_1 \in \mathbb{R}$.

Integrating equations $(\mathcal{L}_X\nabla)^1(\partial_{x_4},\partial_{x_4}) = 0$ and $(\mathcal{L}_X\nabla)^2(\partial_{x_4},\partial_{x_4}) = 0$ with respect to x_4, we get

$$
(10)\quad \begin{cases} f_1 = \mathrm{e}^{2x_4}h(x_1,x_2,x_3) + H(x_1,x_2,x_3), \\ f_2 = \mathrm{e}^{-2x_4}k(x_1,x_2,x_3) + K(x_1,x_2,x_3). \end{cases}
$$

where h, H, k and K are smooth functions depending on x_1, x_2, x_3.

Then, from equations $(\mathcal{L}_X\nabla)^1(\partial_{x_1},\partial_{x_4})=0$ and $(\mathcal{L}_X\nabla)^2(\partial_{x_2},\partial_{x_4})=0$, we have

$$\partial_{x_1}\partial_{x_4}f_1+\partial_{x_2}\partial_{x_4}f_2=0.$$

Then we replace f_1 and f_2 to obtain

$$\mathrm{e}^{2x_4}\partial_{x_1}h-\mathrm{e}^{-2x_4}\partial_{x_2}k=0,$$

which, since x_4 is arbitrary, gives

$$\partial_{x_1}h=\partial_{x_2}k=0. \tag{11}$$

Deriving equations

$$(\mathcal{L}_X\nabla)^1(\partial_{x_3},\partial_{x_4})=0 \ \text{ and } \ (\mathcal{L}_X\nabla)^2(\partial_{x_3},\partial_{x_4})=0$$

with respect to x_4, and using equations

$$(\mathcal{L}_X\nabla)^1(\partial_{x_4},\partial_{x_4})=0 \ \text{ and } \ (\mathcal{L}_X\nabla)^2(\partial_{x_4},\partial_{x_4})=0,$$

we obtain $\partial_{x_3}f_1=\partial_{x_3}f_2=0$. And we replace f_1 and f_2 to find that

$$h=h(x_2),\ \ H=H(x_1,x_2),\ \ k=k(x_1),\ \ K=K(x_1,x_2). \tag{12}$$

Replacing f_1 and f_2 in equations

$$(\mathcal{L}_X\nabla)^1(\partial_{x_2},\partial_{x_4})=0 \ \text{ and } \ (\mathcal{L}_X\nabla)^2(\partial_{x_1},\partial_{x_4})=0$$

respectively, we get

$$\partial_{x_2}H=\partial_{x_1}K=0. \tag{13}$$

Thus $H=H(x_1)$ and $K=K(x_2)$.

Then, replacing f_1 and f_2 in equations

$$\begin{aligned}(\mathcal{L}_X\nabla)^1(\partial_{x_1},\partial_{x_1})&=0, \qquad (\mathcal{L}_X\nabla)^2(\partial_{x_1},\partial_{x_1})=0,\\ (\mathcal{L}_X\nabla)^1(\partial_{x_2},\partial_{x_2})&=0, \qquad (\mathcal{L}_X\nabla)^2(\partial_{x_2},\partial_{x_2})=0,\end{aligned}$$

since $\partial_{x_3}f_1=\partial_{x_3}f_2=0$, we find,

$$H''=h''=K''=k''=0.$$

Thus

$$\begin{cases} f_1=(\alpha_2x_2+\alpha_3)\mathrm{e}^{2x_4}+\alpha_4x_1+\alpha_5,\\ f_2=(\alpha_6x_1+\alpha_7)\mathrm{e}^{-2x_4}+\alpha_8x_2+\alpha_9,\ \ \alpha_i\in\mathbb{R}.\end{cases} \tag{14}$$

From equations

$$\begin{aligned}(\mathcal{L}_X\nabla)^3(\partial_{x_1},\partial_{x_3})&=0, \qquad (\mathcal{L}_X\nabla)^3(\partial_{x_2},\partial_{x_3})=0,\\ (\mathcal{L}_X\nabla)^3(\partial_{x_3},\partial_{x_3})&=0, \qquad (\mathcal{L}_X\nabla)^3(\partial_{x_3},\partial_{x_4})=0,\end{aligned}$$

since $\partial_{x_3}f_1=\partial_{x_3}f_2=0$, we get that

$$f_3=\alpha_{10}x_3+F(x_1,x_2,x_4), \tag{15}$$

where F is a smooth function.

Replacing f_3 in equation $(\mathcal{L}_X\nabla)^3(\partial_{x_4},\partial_{x_4})=0$, gives $\partial^2_{x_4}F=0$. Thus

$$F=x_4\overline{F}(x_1,x_2)+\overline{\overline{F}}(x_1,x_2),$$

where $\overline{F}$ and $\overline{\overline{F}}$ are smooth functions.

Then, replacing f_3 in equations

$$(\mathcal{L}_X\nabla)^3(\partial_{x_1}, \partial_{x_4}) = 0 \ \text{ and } \ (\mathcal{L}_X\nabla)^3(\partial_{x_2}, \partial_{x_4}) = 0,$$

we get

$$\text{(16)} \qquad \begin{cases} x_4\partial_{x_1}\overline{F} + \partial_{x_1}\overline{\overline{F}} + 4\varepsilon(\alpha_1 x_2 + \alpha_2) = 0, \\ -x_4\partial_{x_2}\overline{F} - \partial_{x_2}\overline{\overline{F}} - \partial_{x_2}\overline{F} + 4\varepsilon(\alpha_6 x_2 + \alpha_7) = 0. \end{cases}$$

Thus $\partial_{x_1}\overline{F} = \partial_{x_2}\overline{F} = 0$. Then $\overline{F} = \alpha_{11},\ \alpha_{11} \in \mathbb{R}$, and so

$$\text{(17)} \qquad \begin{cases} \partial_{x_1}\overline{\overline{F}} = -4\varepsilon(\alpha_2 x_2 + \alpha_3), \\ \partial_{x_2}\overline{\overline{F}} = 4\varepsilon(\alpha_6 x_2 + \alpha_7). \end{cases}$$

Integrating the first equation of (17) with respect to x_1, we get

$$\overline{\overline{F}} = -4\varepsilon(\alpha_2 x_2 + \alpha_3)x_1 + \widetilde{F}(x_2),$$

where $\widetilde{F}$ is a smooth function. By replacing $\overline{\overline{F}}$ in the second equation of (17), to obtain

$$\widetilde{F} = 4\varepsilon((\alpha_2 + \alpha_6)x_1 + \alpha_7)x_2 + \alpha_{12},\ \ \alpha_{12} \in \mathbb{R}.$$

Thus $f_3 = \alpha_{10}x_3 + \alpha_{11}x_4 - 4\varepsilon\alpha_3 x_1 + 4\varepsilon(\alpha_6 x_1 + \alpha_7)x_2 + \alpha_{12}$. We replace f_1, f_2, f_3 and f_4 in equations $(\mathcal{L}_X\nabla)^3(\partial_{x_1}, \partial_{x_1}) = 0$, $(\mathcal{L}_X\nabla)^3(\partial_{x_1}, \partial_{x_2}) = 0$ and $(\mathcal{L}_X\nabla)^3(\partial_{x_2}, \partial_{x_2}) = 0$, we obtain $\alpha_6 = -\alpha_2$, $\alpha_{10} = 2(\alpha_4 - \alpha_1)$ and $\alpha_8 = \alpha_4 - 2\alpha_1$.

Thus the final solution of PDEs system obtained requiring that all the coefficients in the above Lie derivative of the Levi-Civita connection ∇ are equal to zero are given by

$$\text{(18)} \qquad \begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \alpha_4 x_1 + \alpha_5, \\ f_2 = (-\alpha_2 x_1 + \alpha_7)\mathrm{e}^{-2x_4} + (\alpha_4 - 2\alpha_1)x_2 + \alpha_9,\ \ \alpha_i \in \mathbb{R}, \\ f_3 = 2(\alpha_4 - \alpha_1)x_3 + \alpha_{11}x_4 - 4\varepsilon\alpha_3 x_1 + 4\varepsilon(-\alpha_2 x_1 + \alpha_7)x_2 + \alpha_{12}, \\ f_4 = \alpha_1,\ \ \alpha_i \in \mathbb{R}. \end{cases}$$

□

5. RICCI AND CURVATURE COLLINEATIONS

In this section we give a full classification of Ricci and curvature collineations vector fields of the four-dimensional generalized symmetric spaces of type C. The classifications we obtain are summarized in the following theorem.

THEOREM 5.1. *Let* $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ *be an arbitrary vector field on the four-dimensional generalized symmetric spaces of type* C.

(1) *X is a Ricci collineation if and only if*

$$X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + \alpha\partial_{x_4},$$

where $\alpha \in \mathbb{R}$, and f_1, f_2, f_3 are any smooth functions on the four-dimensional generalized symmetric spaces of type C.

(2) *X is a curvature collineation vector field if and only if*

$$\begin{cases} f_1 = (\frac{1}{2}\varphi(x_4) + \alpha_1)x_1 + (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \overline{G}(x_4), \\ f_2 = (\frac{1}{2}\varphi(x_4) - \alpha_1)x_2 + (-\alpha_2 x_1 + \alpha_4)\mathrm{e}^{-2x_4} + \overline{F}(x_4), \\ f_3 = \varphi(x_4)x_3 - 4\varepsilon(\alpha_2 x_2 + \alpha_3)x_1 + 4\varepsilon\alpha_4 x_2 - \varepsilon\mathrm{e}^{-2x_4}(\frac{x_1^2}{2}\varphi'(x_4) + 2x_1\overline{G}'(x_4)) \\ -\varepsilon\mathrm{e}^{2x_4}(\frac{x_2^2}{2}\varphi'(x_4) + 2x_2\overline{F}'(x_4)) + \varphi_2(x_4), \\ f_4 = \alpha_1, \end{cases}$$

where $\alpha_i \in \mathbb{R}$, and $\varphi, \varphi_2, \overline{G}$ and $\overline{F}$ are smooth functions of variable x_4.

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ denote an arbitrary vector field on the four-dimensional generalized symmetric spaces of type C, for some arbitrary smooth functions f_1, f_2, f_3, f_4 on M. Starting from (6), a direct calculation yields the following description of the Lie derivative of the Ricci tensor Ric in the direction of X given by:

$$\begin{cases} (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_1}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_2}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_3}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_4}) = -2\partial_{x_1} f_4, \\ (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_2}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_3}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_4}) = -2\partial_{x_2} f_4, \\ (\mathcal{L}_X Ric)(\partial_{x_3}, \partial_{x_3}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_3}, \partial_{x_4}) = -2\partial_{x_3} f_4, \\ (\mathcal{L}_X Ric)(\partial_{x_4}, \partial_{x_4}) = -4\partial_{x_4} f_4. \end{cases} \tag{19}$$

Ricci collineations are then calculated by solving the system of PDEs obtained by requiring that all the above coefficients of $\mathcal{L}_X Ric$ vanish.

From equations given by

$$(\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_4}) = 0, \qquad (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_4}) = 0,$$
$$(\mathcal{L}_X Ric)(\partial_{x_3}, \partial_{x_4}) = 0, \qquad (\mathcal{L}_X Ric)(\partial_{x_4}, \partial_{x_4}) = 0,$$

we get that $f_4 = \alpha$, where $\alpha \in \mathbb{R}$ and f_1, f_2, f_3 are any smooth functions on M.

To determine the curvature collineations, we need to calculate the Lie derivative of the curvature tensor R in the direction of X. Staring from (3),we find the following possibly non-vanishing components:

$$
(20)\quad \begin{cases}
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_1}) = -2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_2} f_4 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}) = 2\varepsilon \mathrm{e}^{2x_4} \partial_{x_1} f_4 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_4}) = \partial_{x_2} f_4 \partial_{x_1} - \partial_{x_1} f_4 \partial_{x_2}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_1}) = -2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_4 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_4}) = \partial_{x_3} f_4 \partial_{x_1}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_4}, \partial_{x_1}) = (2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_1 + \partial_{x_1} f_4) \partial_{x_1} \\
\qquad + 2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_2 \partial_{x_2} + 2\varepsilon \mathrm{e}^{-2x_4} (\partial_{x_3} f_3 + 2 f_4 - 2\partial_{x_1} f_1 - \partial_{x_4} f_4) \partial_{x_3} \\
\qquad\qquad + 2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_4 \partial_{x_4}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_4}, \partial_{x_2}) = \partial_{x_2} f_4 \partial_{x_1} - 2\varepsilon (\mathrm{e}^{2x_4} \partial_{x_1} f_2 + \mathrm{e}^{-2x_4} \partial_{x_2} f_1) \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_4}, \partial_{x_3}) = \partial_{x_3} f_4 \partial_{x_1} - 2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_1 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_4}, \partial_{x_4}) = 2\partial_{x_4} f_4 \partial_{x_1} - (2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_4} f_1 + \partial_{x_1} f_3) \partial_{x_3} \\
\qquad\qquad - \partial_{x_1} f_4 \partial_{x_4}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_2}) = -2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_4 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_4}) = \partial_{x_3} f_4 \partial_{x_2}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_4}, \partial_{x_1}) = \partial_{x_1} f_4 \partial_{x_2} - 2\varepsilon (\mathrm{e}^{2x_4} \partial_{x_1} f_2 + \mathrm{e}^{-2x_4} \partial_{x_2} f_1) \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_4}, \partial_{x_2}) = 2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_1 \partial_{x_1} + (2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_2 + \partial_{x_2} f_4) \partial_{x_2} \\
\qquad + 2\varepsilon \mathrm{e}^{2x_4} (\partial_{x_3} f_3 - 2 f_4 - 2\partial_{x_2} f_2 - \partial_{x_4} f_4) \partial_{x_3} + 2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_4 \partial_{x_4}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_4}, \partial_{x_3}) = \partial_{x_3} f_4 \partial_{x_2} - 2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_2 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_4}, \partial_{x_4}) = 2\partial_{x_4} f_4 \partial_{x_2} - (2\varepsilon \mathrm{e}^{2x_4} \partial_{x_4} f_2 + \partial_{x_2} f_3) \partial_{x_3} \\
\qquad\qquad - \partial_{x_2} f_4 \partial_{x_4}, \\
(\mathcal{L}_X R)(\partial_{x_3}, \partial_{x_4}, \partial_{x_1}) = -2\varepsilon \mathrm{e}^{-2x_4} \partial_{x_3} f_1 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_3}, \partial_{x_4}, \partial_{x_2}) = -2\varepsilon \mathrm{e}^{2x_4} \partial_{x_3} f_2 \partial_{x_3}, \\
(\mathcal{L}_X R)(\partial_{x_3}, \partial_{x_4}, \partial_{x_4}) = \partial_{x_3} f_1 \partial_{x_1} + \partial_{x_3} f_2 \partial_{x_2}.
\end{cases}
$$

In order to determine the curvature collineation vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero.

Which together with equations

$$
\begin{aligned}
&(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}) = 0, \qquad (\mathcal{L}_X R)^1(\partial_{x_1}, \partial_{x_2}, \partial_{x_4}) = 0, \\
&(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_3}, \partial_{x_1}) = 0, \qquad (\mathcal{L}_X R)^1(\partial_{x_1}, \partial_{x_4}, \partial_{x_4}) = 0,
\end{aligned}
$$

gives

$$f_4 = \alpha_1, \ \alpha_1 \in \mathbb{R}.$$

From the equations given by

$$
(21)\quad \begin{aligned}
&(\mathcal{L}_X R)^1(\partial_{x_3}, \partial_{x_4}, \partial_{x_4}) = 0, \\
&(\mathcal{L}_X R)^2(\partial_{x_3}, \partial_{x_4}, \partial_{x_4}) = 0,
\end{aligned}
$$

we get

$$\begin{cases} f_1 = f_1(x_1, x_2, x_4), \\ f_2 = f_2(x_1, x_2, x_4). \end{cases} \tag{22}$$

Deriving equations $(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_1}) = 0$, $(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_4}) = 0$ and $(\mathcal{L}_X R)^3(\partial_{x_2}, \partial_{x_4}, \partial_{x_4}) = 0$, with respect to x_3, we get that $\partial^2_{x_3} f_3 = 0, \partial_{x_1}\partial_{x_3} f_3 = 0$ and $\partial_{x_2}\partial_{x_3} f_3 = 0$.

We deduce that

$$f_3 = \varphi(x_4)x_3 + \psi(x_1, x_2, x_4),$$

where φ is a smooth function depending only on x_4, and ψ is a smooth function depending on x_1, x_2, x_4.

Next, replace f_3 in equations

$$(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_1}) = 0,$$
$$(\mathcal{L}_X R)^3(\partial_{x_2}, \partial_{x_4}, \partial_{x_2}) = 0,$$

we find

$$\begin{cases} \partial_{x_1} f_1 = \frac{1}{2}\varphi(x_4) + \alpha_1, \\ \partial_{x_2} f_2 = \frac{1}{2}\varphi(x_4) - \alpha_1. \end{cases} \tag{23}$$

Integrating first and second equations of (23) with respect to x_1 and x_2 respectively, we get

$$\begin{cases} f_1 = (\frac{1}{2}\varphi(x_4) + \alpha_1)x_1 + G(x_2, x_4), \\ f_2 = (\frac{1}{2}\varphi(x_4) - \alpha_1)x_2 + F(x_1, x_4), \end{cases} \tag{24}$$

where F and G are smooth functions.

Replacing f_1 and f_2 in equation $(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_2}) = 0$, we find

$$\partial_{x_2} G \mathrm{e}^{-2x_4} + \partial_{x_1} F \mathrm{e}^{2x_4} = 0. \tag{25}$$

Deriving equations $(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_4}) = 0$ and $(\mathcal{L}_X R)^3(\partial_{x_2}, \partial_{x_4}, \partial_{x_4}) = 0$, we find that

$$\partial_{x_2}\partial_{x_4} f_1 \mathrm{e}^{-2x_4} = \partial_{x_1}\partial_{x_4} f_2 \mathrm{e}^{2x_4} = 0. \tag{26}$$

And replacing f_1 and f_2 in (26), since $\partial_{x_2} G\mathrm{e}^{-2x_4} = -\partial_{x_1} F\mathrm{e}^{2x_4}$, gives

$$\partial_{x_2}\partial_{x_4} G \mathrm{e}^{-2x_4} = \partial_{x_1}\partial_{x_4} F \mathrm{e}^{2x_4}. \tag{27}$$

Deriving equations (25) with respect to x_4, and using equation (27), we get

$$\begin{cases} \partial_{x_2}\partial_{x_4} G - 2\partial_{x_2} G = 0, \\ \partial_{x_1}\partial_{x_4} F + 2\partial_{x_1} F = 0. \end{cases} \tag{28}$$

Integrating equations (28) with respect to x_4, we find that

$$\begin{cases} \partial_{x_2} G = A(x_2)\mathrm{e}^{2x_4}, \\ \partial_{x_1} F = B(x_1)\mathrm{e}^{-2x_4}, \end{cases} \tag{29}$$

where A and B are smooth functions.

Next, deriving equation (25) with respect to x_2 and x_1 we prove that $\partial^2_{x_2} G = \partial^2_x F = 0$. Thus

$$\begin{cases} A(x_2) = \alpha_2 x_2 + \alpha_3, \\ B(x_1) = \alpha_4 x_1 + \alpha_5, \ \alpha_i \in \mathbb{R}. \end{cases} \tag{30}$$

Hence,

$$\begin{cases} G = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \overline{G}(x_4), \\ F = (\alpha_4 x_1 + \alpha_5)\mathrm{e}^{-2x_4} + \overline{F}(x_4), \ \alpha_i \in \mathbb{R}, \end{cases} \tag{31}$$

where $\overline{G}$ and $\overline{F}$ are smooth functions. By replacing G and F in equation (25), we get $\alpha_2 = \alpha_4$.

Together with replacing f_1, f_2 and f_3 in equations

$$(\mathcal{L}_X R)^3(\partial_{x_1}, \partial_{x_4}, \partial_{x_4}) = 0,$$
$$(\mathcal{L}_X R)^3(\partial_{x_2}, \partial_{x_4}, \partial_{x_4}) = 0,$$

we find

$$\begin{aligned} \psi = {} & -4\varepsilon(\alpha_2 x_2 + \alpha_3)x_1 + 4\varepsilon\alpha_4 x_2 - \varepsilon \mathrm{e}^{-2x_4}\Big(\frac{x_1^2}{2}\varphi'(x_4) + 2x_1\overline{G}'(x_4)\Big) \\ & - \varepsilon \mathrm{e}^{2x_4}\Big(\frac{x_2^2}{2}\varphi'(x_4) + 2x_2\overline{F}'(x_4)\Big) + \varphi_2(x_4), \end{aligned}$$

where φ_2 is a smooth function.

The final solution of the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero are given by

$$\begin{cases} f_1 = (\frac{1}{2}\varphi(x_4) + \alpha_1)x_1 + (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \overline{G}(x_4), \\ f_2 = (\frac{1}{2}\varphi(x_4) - \alpha_1)x_2 + (-\alpha_2 x_1 + \alpha_4)\mathrm{e}^{-2x_4} + \overline{F}(x_4), \\ f_3 = \varphi(x_4)x_3 - 4\varepsilon(\alpha_2 x_2 + \alpha_3)x_1 + 4\varepsilon\alpha_4 x_2 - \varepsilon \mathrm{e}^{-2x_4}(\frac{x_1^2}{2}\varphi'(x_4) + 2x_1\overline{G}'(x_4)) \\ -\varepsilon \mathrm{e}^{2x_4}(\frac{x_2^2}{2}\varphi'(x_4) + 2x_2\overline{F}'(x_4)) + \varphi_2(x_4), \\ f_4 = \alpha_1, \end{cases}$$

where $\alpha_i \in \mathbb{R}$, and $\varphi, \varphi_2, \overline{G}$ and $\overline{F}$ are smooth functions of variable x_4. □

6. MATTER COLLINEATIONS

In this section we classify matter collineation vector fields of the four-dimensional generalized symmetric spaces of type C. The classifications we obtain are summarized in the following theorem.

THEOREM 6.1. *Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ be an arbitrary vector field on the four-dimensional generalized symmetric spaces of type C.*

X is a matter collineation if and only if

$$\begin{cases} f_1 = (\alpha_2 x_2 + \alpha_3)\mathrm{e}^{2x_4} + \alpha_1 x_1 + \alpha_4, \\ f_2 = (-\alpha_2 x_1 + \alpha_5)\mathrm{e}^{-2x_4} - \alpha_1 x_2 + \alpha_6, \\ f_3 = 4\varepsilon(-\alpha_2 x_1 + \alpha_5)x_2 - 4\varepsilon\alpha_3 x_1 + \alpha_7, \\ f_4 = \alpha_1, \quad \alpha_i \in \mathbb{R}, \end{cases}$$

where $\alpha_i \in \mathbb{R}$.

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3} + f_4\partial_{x_4}$ denote an arbitrary vector field on the four-dimensional generalized symmetric spaces of type C, for some arbitrary smooth functions f_1, f_2, f_3, f_4 on M. Starting from equations (1),(6) and (7), a direct calculation yields the the four-dimensional generalized symmetric spaces of type C, with respect to the basis $\{\partial_{x_i}\}_{i\in\{1,2,3,4\}}$ the tensor $T = Ric - \frac{\tau}{2}g$ is described by:

$$T = \begin{pmatrix} \varepsilon\mathrm{e}^{-2x_4} & 0 & 0 & 0 \\ 0 & \varepsilon\mathrm{e}^{2x_4} & 0 & \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -2 \end{pmatrix}. \tag{32}$$

Then we compute the Lie derivative of T with respect to X and we find:

$$\begin{cases} (\mathcal{L}_X T)(\partial_{x_1}, \partial_{x_1}) = 2\varepsilon\mathrm{e}^{-2x_4}(\partial_{x_1} f_1 - f_4), \\ (\mathcal{L}_X T)(\partial_{x_1}, \partial_{x_2}) = \varepsilon(\mathrm{e}^{-2x_4}\partial_{x_2} f_1 + \mathrm{e}^{2x_4}\partial_{x_1} f_2), \\ (\mathcal{L}_X T)(\partial_{x_1}, \partial_{x_3}) = \frac{1}{2}\partial_{x_1} f_4 + \varepsilon\mathrm{e}^{-2x_4}\partial_{x_3} f_1, \\ (\mathcal{L}_X T)(\partial_{x_1}, \partial_{x_4}) = \frac{1}{2}\partial_{x_1} f_3 + \varepsilon\mathrm{e}^{-2x_4}\partial_{x_4} f_1 - 2\partial_{x_1} f_4, \\ (\mathcal{L}_X T)(\partial_{x_2}, \partial_{x_2}) = 2\varepsilon\mathrm{e}^{2x_4}(f_4 + \partial_{x_2} f_2), \\ (\mathcal{L}_X T)(\partial_{x_2}, \partial_{x_3}) = \frac{1}{2}\partial_{x_2} f_4 + \varepsilon\mathrm{e}^{2x_4}\partial_{x_3} f_2, \\ (\mathcal{L}_X T)(\partial_{x_2}, \partial_{x_4}) = \frac{1}{2}\partial_{x_2} f_3 + \varepsilon\mathrm{e}^{2x_4}\partial_{x_4} f_2 - 2\partial_{x_2} f_4, \\ (\mathcal{L}_X T)(\partial_{x_3}, \partial_{x_3}) = \partial_{x_3} f_4, \\ (\mathcal{L}_X T)(\partial_{x_3}, \partial_{x_4}) = \frac{1}{2}(\partial_{x_3} f_3 + \partial_{x_4} f_4) - 2\partial_{x_3} f_4, \\ (\mathcal{L}_X T)(\partial_{x_4}, \partial_{x_4}) = 2\partial_{x_4} f_3 - 2\partial_{x_4} f_4. \end{cases} \tag{33}$$

To determine matter collineation we solve the system of PDEs obtained, requiring that all the coefficients in the above Lie derivative of the tensor field T in the direction of X are equal to zero (i.e. $\mathcal{L}_X T = 0$), we get that all solutions coincide with Killing vector fields of the four-dimensional generalized symmetric spaces of type C.

□

REFERENCES

[1] W. Batat, M. Brozos-Vázquez, E. García-Río, and S. Gavino-Fernández, *Ricci solitons on Lorentzian manifolds with large isometry groups*, Bull. Lond. Math. Soc., **43** (2011), 1219–1227.

[2] W. Batat and K. Onda, *Four-dimensional pseudo-Riemannian generalized symmetric spaces which are algebraic Ricci solitons*, Results Math., **64** (2013), 254–267.

[3] W. Batat and K. Onda, *Ricci and Yamabe solitons on second-order symmetric, and plane wave 4-dimensional Lorentzian manifolds*, J. Geom., **105** (2014), 561–575.

[4] L. Belarbi, *On the symmetries of the Sol_3 Lie group*, accepted in J. Korean Math. Soc., DOI: 10.4134/JKMS.j190198, 1–15.
[5] A. Bouharis and B. Djebbar, *Ricci solitons on Lorentzian four-dimensional generalized symmetric spaces*, Zh. Mat. Fiz. Anal. Geom., **14** (2018), 132–140.
[6] M. Božek, *Existence of generalized symmetric Riemannian spaces with solvable isometry group*, Časopis Pro Pěstování Matematiky, **105** (1980), *4*, 368–384.
[7] M. Brozos-Vázquez, G. Calvaruso, E. García-Río, and S. Gavino-Fernández, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math., **188** (2012), 385–403.
[8] G. Calvaruso, O. Kowalski and A. Marinosci, *Homogeneous geodesics in solvable Lie groups*, Acta. Math. Hungar., **101** (2003), *4*, 313–322.
[9] G. Calvaruso and A. Zaeim, *Invariant symmetries on non-reductive homogeneous pseudo-Riemannian fourmanifolds*, Rev. Mat. Complut., **28** (2015), 599–622.
[10] G. Calvaruso and A. Zaeim, *On the symmetries of the Lorentzian oscillator group*, Collect. Math., **68** (2017), 51–67.
[11] G. Calvaruso, A. Zaeim, *Symmetries of Lorentzian three-manifolds with recurrent curvature*, SIGMA Symmetry Integrability Geom. Methods Appl., **12** (2016), 1–12.
[12] E. Calvino-Louzao, J. Seoane-Bascoy, M.E. Vazquez-Abal and R. Vazquez-Lorenzo, *Invariant Ricci collineations on three-dimensional Lie groups*, J. Geom. Phys., **96** (2015), 59–71.
[13] U. Camci, I. Hussain and Y. Kucukakca, *Curvature and Weyl collineations of Bianchi type V spacetimes*, J. Geom. Phys., **59** (2009), 1476–1484.
[14] U. Camci and M. Sharif, *Matter collineations of spacetime homogeneous Gödel-type metrics*, Classical Quantum Gravity, **20** (2003), 2169–2179.
[15] J. Carot, J. da Costa and E.G.L.R. Vaz, *Matter collineations: the inverse "symmetry inheritance" problem*, J. Math. Phys., **35** (1994), 4832–4838.
[16] G. Hall, *Symmetries of curvature structure in general relativity*, World Science Lecture Notes in Physics, 2004.
[17] O. Kowalski, *Generalized symmetric spaces*, Lectures Notes in Math., Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[18] A. Mostefaoui, L. Belarbi and W. Batat, *Ricci solitons of five-dimensional Solvable Lie group*, PanAmerican Math. J., **29** (2019), *1*, 1–16.
[19] G. Shabbir, *Proper affine vector fields in spherically symmetic static space-times*, Differ. Geom. Dyn. Syst., **8** (2006), 244–252.

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