MATHEMATICA, 62 (85), N° 1, 2020, pp. 47–60

COMMON FIXED POINTS FOR GENERALIZED CONTRACTIONS IN UNIFORM SPACES ENDOWED WITH A GRAPH

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Abstract. In this paper, we will define a new kind of generalized contractions to establish some common fixed point theorems for self-mappings on Hausdorff uniform spaces endowed with a graph. This new notion enables us to extend some known results in the literature. We also show that our results can be applied to a homotopy theorem. Related examples are also given to support our main results.

MSC 2010. Primary: 47H10; Secondary: 47H9,54E15, 05C40, 54H25.

Key words. Connectivity, fixed points, graph, uniform spaces.

1. INTRODUCTION

In 2004, Ran and Reurings [17] established an extension of Banach's fixed point theorem for continuous self-mappings on complete metric spaces endowed with a partial ordering.

Jachymski [12] noticed that every partially ordered metric space $(X, d, \preceq)$ can be considered as the metric space (X, d) endowed with the graph

$$V(G) = X \quad \text{and} \quad E(G) = \{(x, y) \in X \times X : x \preceq y\}.$$

This observation led to some generalizations of some fixed point theorems in partial ordered spaces (see e.g. [4, 8, 14, 15, 16]).

In 1965, Knill [13] generalized the notion of contractive mappings for uniform spaces and established some fixed point theorems in uniform spaces. This motivated some mathematicians to study different kinds of contractions in uniform spaces (see e.g. [2, 3, 5, 7, 11, 18]).

In this paper, we define a new generalized contraction for two self-mappings on a Hausdorff uniform space endowed with a graph. We will show that, under certain circumstances, these functions have a unique common fixed point. Our results lead to genuine generalization of some old results such as Aamri's theorem [1], Jachymski's fixed point theorem [12] and Edelstein's fixed point theorem [10]. We also provide some examples to support our main results.

The authors thank the referee for his helpful comments and suggestions.

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DOI: 10.24193/mathcluj.2020.1.06

Moreover, we will show that under certain conditions common fixed points of self-mappings in uniform spaces remain invariant under a homotopy.

2. PRELIMINARIES

Following [6], a uniform space (X, υ) is a nonempty set X endowed with a uniformity υ that is a special kind of filter on $X \times X$ such that

υ_1) for each $U \in \upsilon$, $\Delta = \{(x, x) : x \in X\} \subseteq U$,

υ_2) $U \in \upsilon$ and $U \subseteq W \subseteq X \times X$ imply $W \in \upsilon$,

υ_3) $U \in \upsilon$ and $W \in \upsilon$ imply $U \cap W \in \upsilon$,

υ_4) $U \in \upsilon$ implies $U^{-1} \in \upsilon$,

υ_5) if $U \in \upsilon$, then there exists $V \in \upsilon$ with $V \circ V \subseteq U$ (the composition of two subsets V and U of $X \times X$ is defined by $V \circ U = \{(x, z) : \exists y \in X : (x, y) \in V,\ (y, z) \in U\}$).

Every element of υ is called an *entourage*. $U \in \upsilon$ is called symmetric if $U = U^{-1} = \{(y, x); (x, y) \in U\}$. Since each entourage U contains a symmetric entourage that is $U \cap U^{-1}$, we can assume that all the members of υ are symmetric. A uniformity υ induces a unique topology $\tau(\upsilon)$ on X in which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$, where $V \in \upsilon$.

A uniform space (X, υ) is said to be Hausdorff if and only if the intersection of all members of υ reduces to the diagonal Δ of X. This guarantees the uniqueness of the limits of sequences.

Intuition about uniformities is provided by the following example of metric space: if (X, d) is a metric space, the sets

$$U_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\} \quad \text{where} \quad \varepsilon > 0,$$

form a fundamental system of entourages for the standard uniform structure of X. Then x and y are U_ε-close when the distance between x and y is at most ε.

There are other kinds of distances which can be attached to a uniformity. The following definitions introduce two kinds of these distances.

Definition 2.1 ([1]). Let (X, υ) be a uniform space. We call a function $\rho : X \times X \to \mathbb{R}^{\geq 0}$ an A-distance, if for any $U \in \upsilon$ there exists $\delta > 0$ such that if $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in U$. The function ρ is said to be an E-distance, if

1) ρ is an A-distance,
2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for each $x, y, z \in X$.

Example 2.2. Let (X, d) be a metric space, then the metric d is an E-distance for the uniformity generated by the metric.

Example 2.3. Let $X = [0, 1]$ be endowed with usual uniformity. For each $x, y \in X$ define $\rho : X \times X \to \mathbb{R}^{\geq 0}$ by $\rho(x, y) = y$, where $\mathbb{R}^{\geq 0} = [0, +\infty)$. Then ρ is an E-distance on X which is not a metric.

We also need the following notions.

Definition 2.4 ([1]). Let (X, υ) be a uniform space endowed with an A-distance ρ.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called ρ-Cauchy if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ρ-convergent to a point $x \in X$ if
$$\lim_{n\to\infty} \rho(x_n, x) = 0.$$
(3) X is called S-complete provided that every ρ-Cauchy sequence in X is ρ-convergent.
(4) $f : X \to X$ is called ρ-continuous if
$$\lim_{n\to\infty} \rho(x_n, x) = 0 \text{ implies } \lim_{n\to\infty} \rho(fx_n, fx) = 0.$$

The following lemma implies the uniqueness of the limit of ρ-convergent sequences in Hausdorff uniform spaces.

Lemma 2.5 ([1]). *Let (X, υ) be a Hausdorff uniform space and ρ be an A-distance on X. Let $\{x_n\}$ be an arbitrary sequence in X. Then for each $x, y, z \in X$ the following hold:*

(a) *If $\lim_{n\to\infty} \rho(x_n, y) = 0$ and $\lim_{n\to\infty} \rho(x_n, z) = 0$ then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then y=z.*
(b) *If $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, υ).*

Consider a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G by the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G. That is $V(G^{-1}) = V(G)$ and
$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$
The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of the edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

A graph G is called connected if there is a path between any two of its vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in a path beginning at x is called the component of G containing x. In this case, $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:
$$yRx \text{ if there is a path in } G \text{ from } x \text{ to } y.$$
Clearly, G_x is connected.

3. FIXED POINTS IN UNIFORM SPACES ENDOWED WITH E−DISTANCES

Throughout this section, we will assume that $\mathbb{R}^{\geq 0} = [0, +\infty)$ and (X, υ) is a Hausdorff uniform space endowed with an E-distance ρ and with a directed graph G, i.e. $V(G) = X$ and $E(G) \supseteq \Delta$. We denote by Ψ the set of all non-decreasing functions $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $\psi(0) = 0$, $\psi(r) > 0$ and $\Sigma_{n=1}^{\infty} \psi^n(r) < \infty$ for each $r > 0$. It follows from the definition that $\psi(r) < r$ for all $\psi \in \Psi$ and $r > 0$.

Definition 3.1. Let (X, υ) be a Hausdorff uniform space endowed with a graph G and let $\rho : X \times X \to \mathbb{R}^{\geq 0}$ be an E-distance. Assume that $\psi \in \Psi$ and $f, g : X \to X$. We say that f is a (ρ, ψ, G)-contraction with respect to g if the following conditions hold:

(1) If $x \in X$ is such that $fx, gx \in [x]_{\widetilde{G}}$, then there exists $y \in [x]_{\widetilde{G}}$ such that $fx = gy$.
(2) f and g are G-invariant, that is $(x_0, y_0) \in E(G)$ implies that (fx_0, fy_0), $(gx_0, gy_0) \in E(G)$. Moreover, if $fx_0 = gx_1$ and $fy_0 = gy_1$, then $(x_1, y_1) \in E(G)$.
(3) If $(x_0, y_0) \in E(G)$, then $\rho(fx_0, fy_0) \leq \psi(\rho(gx_0, gy_0))$.

We denote by $\text{fix}\{f, g\}$ the set of common fixed points of f and g.

Example 3.2. Let (X, d) be a metric space and let $f : X \to X$ be a mapping such that for some $0 \leq \alpha < 1$ satisfies

$$d(fx, fy) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

Define graph G_0 by $V(G_0) = X$ and $E(G_0) = X \times X$. Then f is a (d, ψ, G_0)-contraction with respect to the identity function, where $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is defined by $\psi(r) = \alpha r$.

Definition 3.3. Two sequences $\{x_n\}$ and $\{y_n\}$ are said to be ρ-Cauchy equivalent, if each of them is a ρ-Cauchy and $\lim_{n\to\infty} \rho(x_n, y_n) = 0$.

In order to state our main results we need the following result.

Lemma 3.4. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and E-distance ρ. Assume that $\psi \in \Psi$ and $f, g : X \to X$ where f is a $(\rho, \psi, G)-$contraction with respect to g. Let $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ for an $x_0 \in X$. Then $[x_0]_{\widetilde{G}}$ is both f and g invariant and $f\mid_{[x_0]_{\widetilde{G}}}$ is a $(\rho, \psi, G_{x_0})-$contraction with respect to $g\mid_{[x_0]_{\widetilde{G}}}$, where G_{x_0} is the directed subgraph of G containing all the edges and vertices contained in a path beginning at x_0.*

Proof. Let $x \in [x_0]_{\widetilde{G}}$. There exists a path $\{r_i\}_{i=0}^N$ in $\widetilde{G}$ from x_0 to x, i.e., $r_0 = x_0$, $r_N = x$ and $(r_{i-1}, r_i) \in E(\widetilde{G})$ for all $1 \leq i \leq N$. By Definition 3.1 (2), $(fr_{i-1}, fr_i) \in E(\widetilde{G})$ for all $1 \leq i \leq N$. It means that $\{fr_i\}_{i=0}^N$ is a path in $\widetilde{G}$ from fx_0 to fx. It follows that $fx \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Similarly, one can show that $gx \in [x_0]_{\widetilde{G}}$.

Suppose that $(x_0, y_0) \in E(G_{x_0})$, then $(x_0, y_0) \in E(G)$. Since f is (ρ, ψ, G)-contraction with respect to g, $(fx_0, fy_0), (gx_0, gy_0), (x_1, y_1) \in E(G)$.

Since $[x_0]_{\widetilde{G}}$ is f and g invariant $(fx_0, fy_0), (gx_0, gy_0) \in E(G_{x_0})$. By the definition, $x_1, y_1 \in [x_0]_{\widetilde{G}}$. Thus, $(x_1, y_1) \in E(G_{x_0})$. Moreover, if $(x_0, y_0) \in E(G_{x_0})$, then $(x_0, y_0) \in E(G)$. It follows that $\rho(fx_0, fy_0) \leq \psi(\rho(gx_0, gy_0))$. □

REMARK 3.5. Let x_0 be such that $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$. By Lemma 3.4, $[x_0]_{\widetilde{G}}$ is both f and g invariant. Definition 3.1 (1) implies that there exists $x_1 \in [x_0]_{\widetilde{G}}$ such that $fx_0 = gx_1$. Similarly, since $fx_1, gx_1 \in [x_0]_{\widetilde{G}} = [x_1]_{\widetilde{G}}$, there exists $x_2 \in [x_0]_{\widetilde{G}}$ such that $fx_1 = gx_2$. By continuing this procedure, we can obtain a sequence $\{fx_n\}$ such that $x_n \in [x_0]_{\widetilde{G}}$ and $fx_{n-1} = gx_n$. Moreover, since $[x_0]_{\widetilde{G}}$ is both f and g invariant, for each $y_0 \in [x_0]_{\widetilde{G}}$, we have $fy_0, gy_0 \in [y_0]_{\widetilde{G}}$. Hence we may construct a sequence $\{fy_n\}$ with $y_n \in [y_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$ and $fy_{n-1} = gy_n$, for each $n \geq 1$.

In what follows, whenever $x_0 \in X$ with $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and $y_0 \in [x_0]_{\widetilde{G}}$, $\{fx_n\}$ and $\{fy_n\}$ will be the sequences described above.

The following result shows that under certain circumstances, for each $y_0 \in [x_0]_{\widetilde{G}}$, the corresponding sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ-Cauchy equivalent.

LEMMA 3.6. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and E-distance ρ. Assume that $\psi \in \Psi$ and $f, g : X \to X$ where f is a $(\rho, \psi, G)-$ contraction with respect to g. Let $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ for an $x_0 \in X$. Then, for each $y_0 \in [x_0]_{\widetilde{G}}$, the sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ-Cauchy equivalent, where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$ for each $n \in \mathbb{N}$.*

Proof. First, we will show that, for all $y_0 \in [x_0]_{\widetilde{G}}$, the sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ-equivalent. By the definition, we can find a path $\{t_0^i\}_{i=0}^M$ in $\widetilde{G}$ from x_0 to y_0, i.e. $t_0^0 = x_0$, $t_0^M = y_0$ and $(t_0^{i-1}, t_0^i) \in E(\widetilde{G})$ for each $1 \leq i \leq M$.

For each $1 \leq i \leq M$, construct a sequence $\{t_n^i\}_{n=0}^{\infty}$ such that $ft_n^i = gt_{n+1}^i$. By Definition 3.1 (2), $(t_n^{i-1}, t_n^i) \in E(\widetilde{G})$ for each $n \geq 0$ and for all $1 \leq i \leq M$. Thus, for each $n \geq 0$, $\{t_n^i\}_{i=0}^M$ is a path in $\widetilde{G}$ from x_n to y_n. Since ρ is an E-distance and by Definition 3.1 (1), for each $n \geq 0$ we get

$$\begin{aligned}
\rho(fx_n, fy_n) &\leq \rho(fx_n, ft_n^1) + \rho(ft_n^1, ft_n^2) + \cdots + \rho(ft_n^{M-1}, fy_n) \\
&\leq \psi\rho(fx_{n-1}, ft_{n-1}^1) + \psi\rho(ft_{n-1}^1, ft_{n-1}^2) + \cdots + \\
&\quad \psi\rho(ft_{n-1}^{M-1}, fy_{n-1}) \\
&\vdots \\
&\leq \psi^n\rho(fx_0, ft_0^1) + \psi^n\rho(ft_0^1, ft_0^2) + \cdots + \psi^n\rho(ft_0^{M-1}, fy_0).
\end{aligned}$$

Letting $n \to \infty$, we have $\rho(fx_n, fy_n) \to 0$. It follows that the sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ-equivalent.

Since $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and f is (ρ, ψ, G)-contraction with respect to g, by Definition 3.1 (1), $x_1 \in [x_0]_{\widetilde{G}}$. Let $\{t_i\}_{i=0}^{M'}$ be a path from x_0 to x_1, i.e. $x_0 = t_0$,

$x_1 = t_{M'}$ and $(t_{i-1}, t_i) \in E(\widetilde{G})$ for each $1 \le i \le M'$. By the above argument, for each $m > n$ we have

$$\begin{aligned}
\rho(fx_n, fx_{n+m}) &\le \rho(fx_n, fx_{n+1}) + \cdots + \rho(fx_{n+m-1}, fx_{n+m}) \\
&\le [\psi^n \rho(fx_0, ft_1) + \psi^n \rho(ft_1, ft_2) + \cdots + \\
&\quad \psi^n \rho(ft_{M'-1}, ft_{M'})] + \cdots + \\
&\quad [\psi^{n+m-1} \rho(fx_0, ft_1) + \psi^{n+m-1} \rho(ft_1, ft_2) + \cdots + \\
&\quad \psi^{n+m-1} \rho(ft_{M'-1}, ft_{M'})] \\
&= \Sigma_{k=n}^{n+m-1} \psi^k \rho(fx_0, ft_1) + \Sigma_{k=n}^{n+m-1} \psi^k \rho(ft_1, ft_2) + \cdots + \\
&\quad \Sigma_{k=n}^{n+m-1} \psi^k \rho(ft_{M'-1}, ft_{M'}).
\end{aligned}$$

Hence $\lim_{n,m\to\infty} \rho(fx_n, fx_{n+m}) = 0$. This means that $\{fx_n\}$ is ρ-Cauchy. □

Now, we are ready to state one of the main results of this section.

THEOREM 3.7. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and an E-distance ρ. Assume that X is S-complete and $\psi \in \Psi$. Let $f, g : X \to X$ be commuting ρ-continuous functions such that f is a $(\rho, \psi, G)-$ contraction with respect to g and the triple (X, ρ, G) has the following property: $(*)$ For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with*

$$\lim_{n\to\infty} \rho(x_n, x) = 0 \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for each } n \in \mathbb{N},$$

there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.

Let $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and $(gx_n, fx_n) \in E(G)$ for all $n \in \mathbb{N}\}$. Then, for any $x_0 \in X_{f,g}$, $f \mid_{[x_0]_{\widetilde{G}}}$ and $g \mid_{[x_0]_{\widetilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X_{f,g}$. Then $(fx_{n-1}, fx_n), (gx_n, gx_{n+1}) \in E(G)$, where $fx_{n-1} = gx_n$, for each $n \in \mathbb{N}$. Since $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$, by Lemma 3.4, $[x_0]_{\widetilde{G}}$ is both f and g invariant. Therefore, $fx_n, gx_n \in [x_0]_{\widetilde{G}}$ for each $n \ge 0$. Lemma 3.6 implies that, for each $y_0 \in [x_0]_{\widetilde{G}}$, the sequences $\{fy_n\}$ and $\{fx_n\}$ are ρ-Cauchy equivalent, where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$ for each natural number n. Since X is S-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} \rho(fx_n, u) = \lim_{n\to\infty} \rho(gx_n, u) = 0.$$

Thanks to ρ-continuity of f and g, we get

$$\lim_{n\to\infty} \rho(gfx_n, gu) = \lim_{n\to\infty} \rho(fgx_n, fu) = 0.$$

Since f and g also commute, $\lim_{n\to\infty} \rho(gfx_n, gu) = \lim_{n\to\infty} \rho(gfx_n, fu) = 0$. Therefore, by Lemma 2.5 (a), $fu = gu$. By property $(*)$, there exists a subsequence $\{fx_{k_n}\}$ of $\{fx_n\}$ such that $(fx_{k_n}, u) \in E(G)$ for each $n \ge 1$ Therefore, there exists a path $fx_0, fx_1, \cdots, fx_{k_1}, u$ from fx_0 to u in G. Thus,

$u \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Since $fu = gu$, we get a sequence $u = u_0 = u_1 = \cdots$, for each $n \geq 0$. By Lemma 3.6, $\lim_{n\to\infty} \rho(fx_n, fu_n) = \lim_{n\to\infty} \rho(fx_n, fu) = 0$.

Since $\lim_{n\to\infty} \rho(fx_n, u) = 0$, Lemma 2.5 (a) implies that $fu = u$. Therefore, $gu = fu = u$ is a common fixed point of f and g.

Next, we will show that u is a unique common fixed point of $f\mid_{[x_0]_{\widetilde{G}}}$ and $g\mid_{[x_0]_{\widetilde{G}}}$. Indeed, if $a_0, b_0 \in fix\{f, g\} \cap [x_0]_{\widetilde{G}}$, then $fa_0 = ga_0 = a_0$ and $fb_0 = gb_0 = b_0$. We get $a_0 = a_1 = \cdots$ and $b_0 = b_1 = \cdots$. Lemma 3.6 implies that the sequences $\{fa_n\}_{n\geq 1}$ and $\{fb_n\}_{n\geq 1}$ are ρ-Cauchy equivalent. Hence $\rho(a_0, b_0) = \lim_{n\to\infty} \rho(fa_n, fb_n) = 0$ and $\rho(b_0, a_0) = \lim_{n\to\infty} \rho(fb_n, fa_n) = 0$. Since ρ is an E-distance we have $\rho(a_0, a_0) \leq \rho(a_0, b_0) + \rho(b_0, a_0)$. Therefore, $\rho(a_0, a_0) = 0$. By Lemma 2.5 (a), $a_0 = b_0$.

Let $x_0 \in X_{f,g} \neq \emptyset$ and G be weakly connected. Then $[x_0]_{\widetilde{G}} = X$. By the above argument, f and g have a unique common fixed point. $\square$

The following result shows that one can replace continuity of f by continuity of the E-distance ρ in Theorem 3.7, provided that $\rho(x, x) = 0$ for all $x \in X$.

THEOREM 3.8. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and a continuous E-distance ρ such that $\rho(x, x) = 0$ for each $x \in X$. Assume that X is S-complete and $\psi \in \Psi$. Let $f, g : X \to X$ be commuting functions such that f is a (ρ, ψ, G)- contraction with respect to g and let g be ρ-continuous. Assume that the triple (X, ρ, G) has the following property:*
$(*)$ For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if

$$\lim_{n\to\infty} \rho(x_n, x) = 0 \text{ and } (x_n, x_{n+1}) \in E(G)$$

for each $n \in \mathbb{N}$, then there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.

Let $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and $(gx_n, fx_n) \in E(G)$ for all $n \in \mathbb{N}\}$. Then, for any $x_0 \in X_{f,g}$, $f\mid_{[x_0]_{\widetilde{G}}}$ and $g\mid_{[x_0]_{\widetilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X_{f,g}$. Then $(fx_{n-1}, fx_n), (gx_n, gx_{n+1}) \in E(G)$, where $fx_{n-1} = gx_n$, for each $n \in \mathbb{N}$.

Since $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$, by Lemma 3.4, $[x_0]_{\widetilde{G}}$ is both f and g invariant. Thus, $fx_n, gx_n \in [x_0]_{\widetilde{G}}$ for each $n \geq 0$ and, for each $y_0 \in [x_0]_{\widetilde{G}}$, the sequences $\{fy_n\}$ and $\{fx_n\}$ are ρ-Cauchy equivalent, where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$ for each $n \in \mathbb{N}$. Since X is S-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} \rho(fx_n, u) = \lim_{n\to\infty} \rho(gx_n, u) = 0.$$

By $(*)$ there exists a subsequence $\{gx_{k_n}\}$ of $\{gx_n\}$ such that $(gx_{k_n}, u) \in E(G)$ for each $n \geq 1$. By continuity of ρ and by ρ-continuity of g for each $n \geq 1$, we get

$$\begin{aligned}\lim_{n\to\infty}\rho(fgx_{k_n},fu) &\leq \lim_{n\to\infty}\psi(\rho(ggx_{k_n},gu)) < \lim_{n\to\infty}\rho(ggx_{k_n},gu)\\ &= \rho(g\lim_{n\to\infty}gx_{k_n},gu)=\rho(gu,gu)=0.\end{aligned}$$

Also, for each $n\in\mathbb{N}$,

$$\lim_{n\to\infty}\rho(fgx_{k_n},gu)=\lim_{n\to\infty}\rho(gfx_{k_n},gu))=\rho(g\lim_{n\to\infty}fx_{k_n},gu)=\rho(gu,gu)=0.$$

By Lemma 2.5 (a), $fu=gu$. Now, we will show that u is a common fixed point of f and g. By $(*)$ there exists a subsequence $\{fx_{k_n}\}$ of $\{fx_n\}$ such that for each $n\in\mathbb{N}$, $(fx_{k_n},u)\in E(G)$. Thus, we can find a path $fx_0,fx_1,\cdots,fx_{k_1},u$ from fx_0 to u in G. Hence $u\in[x_0]_{\widetilde{G}}$. Since $fu=gu$, we get a sequence $u=u_0=u_1=\cdots$, for each $n\geq 0$. By Lemma 3.6, $\lim_{n\to\infty}\rho(fu,fx_n)=0$. So by continuity of ρ we have

$$\rho(fu,u)=\rho(fu,\lim_{n\to\infty}fx_n)=\lim_{n\to\infty}\rho(fu,fx_n)=0.$$

Lemma 2.5 (a) implies that $fu=u$. Thus u is a common fixed point of f and g. Moreover, u is a unique common fixed point of $f\mid_{[x_0]_{\widetilde{G}}}$ and $g\mid_{[x_0]_{\widetilde{G}}}$. Indeed, if $a_0,b_0\in fix\{f,g\}\cap[x_0]_{\widetilde{G}}$, then $fa_0=ga_0=a_0$ and $fb_0=gb_0=b_0$. We get $a_0=a_1=\cdots$ and $b_0=b_1=\cdots$. Lemma 3.6 implies that the sequences $\{fa_n\}_{n\geq 1}$ and $\{fb_n\}_{n\geq 1}$ are ρ-Cauchy equivalent. Therefore,

$$\rho(a_0,b_0)=\lim_{n\to\infty}\rho(fa_n,fb_n)=0 \text{ and } \rho(b_0,a_0)=\lim_{n\to\infty}\rho(fb_n,fa_n)=0.$$

Also, we have $\rho(b_0,b_0)=\rho(a_0,a_0)=0$. By Lemma 2.5 (a), $a_0=b_0$.

Finally, if $x_0\in X_{f,g}\neq\emptyset$ and G is weakly connected, $[x_0]_{\widetilde{G}}=X$. The argument that was used shows that f and g have a unique fixed point. □

Next, we will show that the following extension of Jachymski's Theorem [12, Theorem 3.2] follows from our main result.

Corollary 3.9. *Let (X,d) be a complete metric space endowed with a graph G and let the triple (X,d,G) have the following property:*
$(*)$ *For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if*

$$\lim_{n\to\infty}d(x_n,x)=0 \textit{ and } (x_n,x_{n+1})\in E(G),\ n\in\mathbb{N},$$

then there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ with $(x_{k_n},x)\in E(G)$ for each $n\in\mathbb{N}$.

Assume that $f:X\to X$ satisfies the following conditions:

1. *For all $x,y\in X,(x,y)\in E(G)$ implies that $(fx,fy)\in E(G)$.*
2. *There is $\psi\in\Psi$ such that $(x,y)\in E(G)$ implies that*

$$d(fx,fy)\leq\psi(d(x,y)).$$

If $X_f=\{x\in X:(x,fx)\in E(G)\}$, then, for any $x\in X_f$, $f\mid_{[x]_{\widehat{G}}}$ has a unique fixed point. Moreover, if $X_f\neq\emptyset$ and G is weakly connected, then f has a unique fixed point.

Proof. As in Example 2.2, the metric d generates a Hausdorff uniformity on X and at the same time is an E-distance on X. By our assumption, X is S-complete. Clearly, d is continuous, the identity map I is d-continuous and f and I are commuting. Moreover, we have the following:

(1) Let $fx \in [x]_{\widetilde{G}}$. Put $y = fx$. Then there exists $y \in [x]_{\widetilde{G}}$ such that $fx = gy$, where $g = I$.

(2) $(x_0, y_0) \in E(G)$ implies that $(fx_0, fy_0), (Ix_0, Iy_0) = (x_0, y_0), (x_1, y_1) = (fx_0, fy_0) \in E(G)$.

(3) If $(x_0, y_0) \in E(G)$, then $d(fx_0, fy_0) \leq \psi(d(x_0, y_0))$.

It follows that f is (d, ψ, G)-contraction with respect to the identity map on X. Moreover, by our assumption, the triple (X, d, G) satisfies property $(*)$ of Theorem 3.8 and

$$\begin{aligned} X_f &= \{x_0; (x_0, fx_0) \in E(G)\} \\ &\subseteq X_{f,I} = \{x_0 \in X : fx_0, Ix_0 \in [x_0]_{\widetilde{G}} \text{ and } (Ix_n, fx_n) \in E(G),\ n \in \mathbb{N}\}. \end{aligned}$$

Let $x_0 \in X_f \subseteq X_{f,I}$. Then, by Theorem 3.8, the functions $f\mid_{[x_0]_{\widetilde{G}}}$ and $I\mid_{[x_0]_{\widetilde{G}}}$ have a unique common fixed point. This means that $f\mid_{[x_0]_{\widetilde{G}}}$ have a unique fixed point. Let $x_0 \in X_f \neq \emptyset$ and G be weakly connected. Then $[x_0]_{\widetilde{G}} = X$. By the above argument, f has a unique fixed point. □

In 2004, Aamri and El Moutawakil [1] obtained the following result.

Theorem 3.10 ([1, Theorems 3.1 and 3.2]). *Let (X, υ) be a Hausdorff uniform spaces and ρ be an A-distance on X. Suppose X is ρ-bounded and S-complete. Suppose that $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfies*

$$\psi(t) > 0 \text{ and } \lim_{n\to\infty} \psi^n(t) = 0 \text{ for each } t > 0.$$

Let f and g be commuting ρ-continuous or $\tau(\upsilon)$-continuous self-mappings of X such that

(1) $f(X) \subseteq g(X)$,

(2) $\rho(f(x), f(y)) \leq \psi(\rho(g(x), g(y)))$, *for all $x, y \in X$.*

Then f and g have a common fixed point. Moreover, if ρ is an E-distance, then f and g have a unique common fixed point.

Theorem 3.7 enables us to omit the condition of ρ-boundedness of X in Theorem 3.10, provided that $\psi \in \Psi$. We present this extension in the following corollary.

Corollary 3.11. *Let (X, υ) be a Hausdorff uniform spaces and ρ be an E-distance on X. Suppose that X is S-complete. Let f and g be ρ-continuous commuting mappings on X and let $\psi \in \Psi$. Assume that the followings conditions hold:*

(1) $f(X) \subseteq g(X)$,

(2) $\rho(f(x), f(y)) \leq \psi(\rho(g(x), g(y)))$, *for all $x, y \in X$.*

Then f and g have a unique common fixed point.

Proof. Assume that G is a graph such that $V(G) = X$ and $E(G) = X \times X$. Then G is weakly connected and the following conditions hold:

(1) By our assumption $f(X) \subseteq g(X)$, so for each $x \in X$, with $fx, gx \in [x]_{\widetilde{G}} = X$, there exists $y \in [x]_{\widetilde{G}} = X$ such that $fx = gy$.
(2) Let $(x_0, y_0) \in E(G) = \{(x, y) : x, y \in X\}$. Then both f and g are G-invariant and $(x_1, y_1) \in E(G) = \{(x, y) : x, y \in X\}$, where $fx_0 = gx_1$ and $fy_0 = gy_1$.
(3) If $(x_0, y_0) \in E(G) = \{(x, y) : x, y \in X\}$, then

$$\rho(fx_0, fy_0) \leq \psi(\rho(gx_0, gy_0)).$$

Hence, we may consider f as a (ρ, ψ, G)- contraction with respect to g. By our assumption, f and g are ρ-continuous commuting mappings. Moreover, $X_{f,g} = X \neq \emptyset$. By Theorem 3.7, f and g have a unique common fixed point. □

Definition 3.12. Let (X, d) be a metric space and $\varepsilon > 0$. Then X is called ε-chainable if for each $x, y \in X$, there exist $N \in \mathbb{N}$ and $x_0 = x, x_1, \cdots, x_N = y \in X$ such that $d(x_{i-1}, x_i) < \varepsilon$ for all $1 \leq i \leq N$.

In 1961, Edelstein [10] proved the following extension of the Banach fixed point theorem.

Theorem 3.13 (Edelstein's Theorem). *Let (X, d) be a complete ε-chainable metric space for an $\varepsilon > 0$. Assume that $f : X \to X$ for some $0 \leq \alpha < 1$ satisfies*

$$d(x, y) < \varepsilon \Rightarrow d(fx, fy) < \alpha d(x, y).$$

Then f has a unique fixed point

The following result, which is an extension of Edelstein's Theorem for two self-mappings, is a consequence of Theorem 3.8.

Corollary 3.14. *Let (X, d) be a complete ε-chainable metric space for an $\varepsilon > 0$ and $\psi \in \Psi$. Assume that f and g are commuting self mappings on X such that g is continuous and the following conditions hold:*

1. $f(X) \subseteq g(X)$;
2. $d(x_0, y_0) < \varepsilon \Rightarrow d(fx_0, fy_0), d(gx_0, gy_0), d(x_1, y_1) < \varepsilon$, *for some* x_0, $y_0 \in X$ *where* $fx_0 = gx_1$ *and* $fy_0 = gy_1$;
3. $d(x_0, y_0) < \varepsilon \Rightarrow d(fx_0, fy_0) < \psi(d(gx_0, gy_0))$.

Then f and g have a unique common fixed point.

Proof. By Example 2.2, the metric d generates a Hausdorff uniformity on X and at the same time is an E-distance on X. Define a graph G by

$$V(G) = X \text{ and } E(G) = \{(x, y) \in X \times X : \rho(x, y) < \varepsilon\}.$$

Since X is ε-chainable, G is weakly connected. Therefore, f is a (d, ψ, G)-contraction with respect to g.

Let $\{x_n\}_{n\geq 0}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x^*) = 0$, for an $x^* \in X$ and $d(x_n, x_{n+1}) < \varepsilon$ for each $n \in \mathbb{N}$. Since $\{x_n\}$ is convergent to x^*, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x^*) < \varepsilon$ for each $n \geq n_0$. Therefore, for the subsequence $\{x_{n_0+k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, we have $(x_{n_0+k}, x) \in E(G)$ for each $k \geq 0$. This means that property $(*)$ of Theorem 3.8 holds.

Since G is weakly connected, $fx, gx \in [x]_{\widetilde{G}}$ for each $x \in X$. Fix $x_0 \in X$. By Lemma 3.6, the sequence $\{fx_n\}$ is Cauchy, where $fx_{n-1} = gx_n$, for each $n \geq 0$. Therefore, $\lim_{n,m\to\infty} d(fx_n, fx_m) = 0$. Let $M \in \mathbb{N}$ be such that $d(fx_n, fx_m) < \varepsilon$ for all $n, m \geq M$. Put $z_0 = x_M$. One can see that $z_0 \in X_{f,g}$. Since $[z_0]_{\widetilde{G}} = X$, by Theorem 3.8, $f\,|_{[z_0]_{\widetilde{G}}} = f$ and $g\,|_{[z_0]_{\widetilde{G}}} = g$ have a unique common fixed point on $[z]_{\widetilde{G}} = X$. □

The next example shows that our results are different from Theorem 3.10.

Example 3.15. Let $X = \{\frac{1}{n} : n \geq 1\} \cup \{0, \frac{2}{3}\}$ be endowed with usual uniformity and graph G, where $V(G) = X$ and

$$E(G) = \Delta(X) \cup \{(\frac{1}{n}, \frac{1}{n+1}) : n \geq 2\} \cup \{(\frac{1}{n}, 0) : n \geq 2\} \cup \{(\frac{2}{3}, 0), (0, \frac{2}{3})\}.$$

Define $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by $\psi(r) = \frac{r}{2}$ and define $\rho : X \times X \to \mathbb{R}^{\geq 0}$ by

$$\rho(x,y) = \begin{cases} 1 & x = \frac{2}{3}, y \neq x \\ 0 & x = y = \frac{2}{3} \\ y & \text{otherwise} \end{cases}. \tag{1}$$

Then ρ is an E-distance on X. One can easily verify that X is S-complete. Define $f, g : X \to X$ by

$$fx = \begin{cases} \frac{2}{3} & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases} \tag{2}$$

$$gx = \begin{cases} 0 & \text{if } x = 0, \frac{2}{3}, \\ \frac{2}{3} & \text{if } x = 1, \\ \frac{1}{1+n} & \text{if } x = \frac{1}{n}, n \geq 2. \end{cases} \tag{3}$$

Then $fgx = gfx$, for all $x \in X$, and $f(X) = \{0, \frac{2}{3}\} \subseteq g(X) = \{0, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}, \cdots\}$.

We show that f and g are ρ-continuous. Assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} \rho(x_n, x) = 0$ for an $x \in X$. By the definition of ρ, $x = 0$ or $x = \frac{2}{3}$.

If $x = \frac{2}{3}$, there exists $N_0 \in \mathbb{N}$ such that $x_n = \frac{2}{3}$ for each $n \geq N_0$. Thus, $\lim_{n\to\infty} \rho(fx_n, fx) = \rho(f\frac{2}{3}, f\frac{2}{3}) = \rho(0,0) = 0$ and $\lim_{n\to\infty} \rho(gx_n, gx) = \rho(g\frac{2}{3}, g\frac{2}{3}) = \rho(0,0) = 0$.

If $x = 0$, then $\lim_{n\to\infty} \rho(fx_n, fx) = \lim_{n\to\infty} \rho(fx_n, f0) = \lim_{n\to\infty} \rho(fx_n, 0) = 0$ and $\lim_{n\to\infty} \rho(gx_n, gx) = \lim_{n\to\infty} \rho(gx_n, g0) = \lim_{n\to\infty} \rho(gx_n, 0) = 0$. Therefore, f and g are ρ-continuous. Moreover, the following conditions hold:

(1) For each $x \in X$ with $fx, gx \in [x]_{\widetilde{G}}$, there exists $y \in [x]_{\widetilde{G}} = X$ such that $fx = gy$.

(2) If $(x_0, y_0) \in E(G)$, then $(fx_0, fy_0), (gx_0, gy_0), (x_1, y_1) \in E(G)$, where $fx_0 = gx_1$ and $fy_0 = gy_1$.
(3) If $(x_0, y_0) \in E(G)$, then $\rho(fx_0, fy_0) \leq \psi(\rho(x_0, y_0))$.

Now, we show that the triple (X, ρ, G) satisfies property $(*)$ of Theorem 3.7.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} \rho(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Then $x = 0$ or $x = \frac{2}{3}$.

If $x = \frac{2}{3}$, there exists $n_0 \in \mathbb{N}$ such that $x_n = \frac{2}{3}$ for $n \geq n_0$. In this case, $(x_n, x) \in E(G)$ for all $n \geq n_0$. If $x = 0$, then there is $n_1 \in \mathbb{N}$ such that $x_n \neq \frac{2}{3}$ for all $n \geq n_1$. Hence, for each $n \geq n_1$, $x_n \in \{\frac{1}{k}\}_{k\geq 1} \cup \{0\}$. Therefore, $(x_n, x) \in E(G)$ for each $n \geq n_1$.

Also, $0 \in X_{f,g} \neq \emptyset$. By Theorem 3.7, f and g have a unique common fixed point on $[0]_{\widetilde{G}} = \{\frac{1}{n}; n \geq 2\} \cup \{0, \frac{2}{3}\}$, that is $x = 0$. However, Aamri's Theorem (Theorem 3.10) can't be used, since

$$\rho(f(\frac{1}{2}), f(1)) = \rho(0, \frac{2}{3}) = \frac{2}{3} \nleq \psi(\rho(g(\frac{1}{2}), g(1))) = \psi(\rho(\frac{1}{3}, \frac{2}{3})) = \psi(\frac{2}{3}) = \frac{1}{3}.$$

4. HOMOTOPY RESULTS IN HAUSDORFF UNIFORM SPACES

Homotopy of continuous functions plays an important role in topology, since some known topological properties are homotopy invariant. In this section we are going to apply our results to get a homotopy theorem. We start by recalling some definitions.

DEFINITION 4.1 ([9]). Let X and Y be two topological spaces and let $f, g : X \to Y$ be two continuous mappings. A homotopy from f to g is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = fx$ and $H(x, 1) = gx$ for all $x \in X$. In this case, f and g are called homotopic mappings.

DEFINITION 4.2. Let (X, υ) be a uniform space and let $\rho : X \times X \to \mathbb{R}^{\geq 0}$ be an E-distance. A function $H : X \times [0, 1] \to X$ is called ρ-continuous at $(x, t) \in X \times [0, 1]$ if $\rho(x_n, x) \to 0$ and $| t_n - t | \to 0$ imply that

$$\rho(H(x_n, t_n), H(x, t)) \to 0,$$

where $\{x_n\} \subseteq X$ and $\{t_n\} \subseteq [0, 1]$.

THEOREM 4.3. *Let (X, υ) be a uniform space endowed with a graph G and E-distance $\rho : X \times X \to \mathbb{R}^{\geq 0}$. Assume that X is S-complete and the triple (X, ρ, G) satisfies property $(*)$ in Theorem 3.7. Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an element of Ψ and let $f : X \to X$ and $H : X \times [0, 1] \to X$ be commuting ρ-continuous mappings. Assume that $[fx]_{\widetilde{G}} = [H(x, 1)]_{\widetilde{G}}$ implies that $H(x, 0) \in [fx]_{\widetilde{G}}$ for each $x \in X$. Assume that H satisfies the following property: For each $t_1, t_2 \in [0, 1]$, if $t_1 < t_2$ then $H(., t_1)$ is a $(\rho, \psi, G)-$contraction with respect to $H(., t_2)$. Let f is $(\rho, \psi, G)-$contraction with respect to $H(., 0)$. Then f is also a $(\rho, \psi, G)-$contraction with respect to $H(., 1)$. Moreover, if u is a common fixed point of f and $H(., 0)$ such that there is $x_0 \in X_{H(.,0),H(.,1)}$ such* *that* $\{H(x_n,0)\}$ *is* ρ*-convergent to* u*, where* $H(x_{n-1},0) = H(x_n,1)$ *for each natural* n*, then* u *is also a unique common fixed point of* f *and* $H(.,1)$ *on* $[u]_{\widetilde{G}}$.

Proof. Suppose that f is a (ρ,ψ,G)-contraction with respect to $H(.,0)$. Let $x \in X$ be such that $f(x), H(x,1) \in [x]_{\widetilde{G}}$.

By our hypothesis, $H(x,0) \in [x]_{\widetilde{G}}$. f is (ρ,ψ,G)-contraction with respect to $H(.,0)$ and $H(.,0)$ is (ρ,ψ,G)-contraction with respect to $H(.,1)$. By Lemma 3.4, $[x]_{\widetilde{G}}$ is f and $H(.,0)$ and $H(.,1)$ invariant. Moreover, there exists $y \in [x]_{\widetilde{G}}$ such that $fx = H(y,0)$. So $H(y,0), H(y,1) \in [y]_{\widetilde{G}}$. Thus, there exists $z \in [y]_{\widetilde{G}} = [x]_{\widetilde{G}}$ such that $H(y,0) = H(z,1)$. Therefore, there is $z \in [x]_{\widetilde{G}}$ such that $fx = H(z,1)$.

Suppose that $(x_0,y_0) \in E(G)$. f is (ρ,ψ,G)-contraction with respect to $H(.,0)$ and $H(.,0)$ is (ρ,ψ,G)-contraction with respect to $H(.,1)$, hence

$$(fx_0, fy_0), (H(x_0,0), H(y_0,0)), (H(x_0,1), H(y_0,1)) \in E(G).$$

Moreover, $(z,w) \in E(G)$ where $fx_0 = H(z,0)$ and $fy_0 = H(w,0)$. There exist $z', w' \in X$ such that $H(z,0) = H(z',1)$ and $H(w,0) = H(w',1)$ and, since $H(.,0)$ is (ρ,ψ,G)-contraction with respect to $H(.,1)$, $(z',w') \in E(G)$. Put $x_1 = z'$ and $y_1 = w'$. Thus, $(x_1,y_1) \in E(G)$.

Let $(x,y) \in E(G)$. We have

$$\begin{aligned} \rho(fx,fy) &\le \psi(\rho(H(x,0),H(y,0))) \le \rho(H(x,0),H(y,0)) \\ &\le \psi(\rho(H(x,1),H(y,1))). \end{aligned}$$

Therefore, f is a (ρ,ψ,G)-contraction with respect to $H(.,1)$.

Let u be a common fixed point of f and $H(.,0)$ and let $x_0 \in X_{H(.,0),H(.,1)}$ be such that $\{H(x_n,0)\}$ is ρ-convergent to u, where $H(x_{n-1},0) = H(x_n,1)$ for each $n \ge 1$. Property $(*)$ in Theorem 3.7 implies that $u \in [x_0]_{\widetilde{G}}$. Since $H(.,0)$ is a $(\rho,\psi,G)-$contraction with respect to $H(.,1)$ and since $x_0 \in X_{H(.,0),H(.,1)} \neq \emptyset$, Theorem 3.7 implies that $H(.,0) \mid_{[x_0]_{\widetilde{G}}=[u]_{\widetilde{G}}}$ and $H(.,1) \mid_{[x_0]_{\widetilde{G}}=[u]_{\widetilde{G}}}$ have a unique common fixed point that is u. Thus, u is a unique common fixed point of $f \mid_{[u]_{\widetilde{G}}}$ and $H(.,1) \mid_{[u]_{\widetilde{G}}}$. □

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Received October 18, 2018
Accepted March 14, 2019

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