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CLASSIFICATION OF RATIONAL HOMOTOPY TYPE FOR 10-COHOMOLOGICAL DIMENSION ELLIPTIC SPACES

SALOUA CHOUINGOU and MOHAMED RACHID HILALI

Abstract. The purpose of this paper is to give a classification of the rational homotopy type for any simply connected and elliptic space whose cohomological dimension is equal to 10. This classification treats two cases, according to whether the homotopic Euler characteristic is vanishing or not.

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1. INTRODUCTION

Rational homotopy theory started with the work of D. Quillen and D. Sullivan in the late 1960: it allows to describe what is called the rational homotopy type of a topological space lossless into algebra. This specific feature contributes to the strength and the elegance of the theory. In fact, the starting idea in the rational homotopy theory is to tensor the homotopy groups:

$$\Pi_k(X) \otimes \mathbb{Q} = \mathbb{Q}^{n_k}$$

and consider only the so called rational spaces generally denoted $X_{\mathbb{Q}}$ verifying the particular condition requiring that both $\Pi_*(X_{\mathbb{Q}})$ and $H^*(X_{\mathbb{Q}};\mathbb{Q})$ are $\mathbb{Q}$-vector spaces. One of the well-known results is that any simply connected space can be modeled up to homotopy equivalence by a rational CW-complex as follows:

$$\Pi_*(X) \otimes \mathbb{Q} \cong \Pi_*(X_{\mathbb{Q}}) \text{ as vector spaces}$$
$$H^*(X;\mathbb{Q}) \cong H^*(X_{\mathbb{Q}};\mathbb{Q}) \text{ as algebras.}$$

The rationalisation $X_{\mathbb{Q}}$ of X all have the same weak homotopy type, which depends only on the weak homotopy type of X, named the rational homotopy type of X.

For a detailed discussion we refer the reader to (see [5, 7, 9, 10, 14]), where the classification of the rational homotopy type for any simply connected elliptic space whose cohomological dimension varies from 1 to 9 is developed.

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Our goal is to extend this classification to the case of simply connected elliptic spaces whose cohomological dimension is equal to 10, which has never been studied before by means of particular tools involving the homotopic Euler characteristic.

Our main results are:

THEOREM 1.1. *If X is a simply connected elliptic space such that dim $H^*(X;\mathbb{Q})$ =10 with $\chi_\pi(X) = 0$, then its rational homotopy type is given by the following table:*

$H^*(X;\mathbb{Q}) \cong$	$(\Lambda V, d) \cong$	$X \simeq \mathbb{Q}$
$\mathbb{Q}[x]/\left(x^{10}\right)$	$(\Lambda(x,y), d)$ with $dx = 0,\ dy = x^{10}$	$\mathbb{S}^n_{(9)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1^2, x_2^5\right)$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$, $dy_1 = x_1^2,\ dy_2 = x_2^5$	$\mathbb{S}^n \times \mathbb{S}^m_{(4)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1x_2, \gamma_1x_1^5 + \gamma_2x_2^5\right)$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$, $dy_1 = x_1x_2,\ dy_2 = \gamma_1x_1^5 + \gamma_2x_2^5$	$\mathbb{S}^n_{(5)}\#\mathbb{S}^n_{(5)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1x_2, x_2^4 + \lambda x_1^6\right)$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$ $dy_1 = x_1x_2,\ dy_2 = x_2^4 + \lambda x_1^6$	$\mathbb{S}^n_{(6)}\#\mathbb{S}^m_{(4)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1x_2, x_2^3 + \lambda x_1^7\right)$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$ $dy_1 = x_1x_2,\ dy_2 = x_2^3 + \lambda x_1^7$	$\mathbb{S}^n_{(7)}\#\mathbb{S}^n_{(3)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1^2x_2, x_2^3 + x_1^4\right)$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$ $dy_1 = x_1^2x_2,\ dy_2 = x_2^3 + x_1^4$	
$\mathbb{Q}[x_1,x_2]/_{(x_2^2+\gamma x_1^2x_2, x_1^5+\lambda_1x_1x_2^2+\lambda_2x_1^3x_2)}$	$(\Lambda(x_1,x_2,y_1,y_2), d)$ with $dx_1 = dx_2 = 0$ $dy_1 = x_2^2 + \lambda_1x_1^2x_2$ $dy_2 = x_1^5 + \lambda_1x_1x_2^2 + \lambda_2x_1^3x_2$	

THEOREM 1.2. *If X is a simply connected elliptic space such that $dimH^*(X;\mathbb{Q}) = 10$ with $\chi_\pi(X) \neq 0$, then its rational homotopy type is given by the following table:*

$H^*(X;\mathbb{Q}) \cong$	$(\Lambda V, d) \cong$	$X \simeq \mathbb{Q}$
$\mathbb{Q}[x_1]/\left(x_1^5\right) \otimes H^*(\Lambda y, 0)$	$(\Lambda(x_1,y_1,y), d)$ with $dx_1 = 0$ $dy_1 = x_1^5,\ dy = 0$	$\mathbb{S}^{2k+1} \times \mathbb{S}^{2n}_{(4)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1x_2, x_2^2 + x_1^3\right) \otimes H^*(\Lambda y, 0)$	$(\Lambda(x_1,x_2,y_1,y_2,y), d)$ with $dx_1 = 0$ $dx_2 = 0,\ dy_1 = x_1x_2$ $dy_2 = x_2^2 + x_1^3, dy = 0$	$\mathbb{S}^{2p+1} \times \mathbb{S}^n_{(3)}\#\mathbb{S}^m_{(2)}$
$\mathbb{Q}[x_1,x_2]/\left(x_1^3, x_1x_2, x_2^3\right)$	$(\Lambda(x_1,x_2,y_1,y_2,y_3), d)$ with $dx_1 = 0$ $dx_2 = 0,\ dy_1 = x_1^3$ $dy_2 = x_2^3,\ dy_3 = x_2x_1$	E with E is the total space of this fibration $\mathbb{S}^p \to E \to \mathbb{S}^n \times \mathbb{S}^m$

COROLLARY 1.3. *The conjecture (H) is true for every space X simply connected elliptic such that $dimH^*(X;\mathbb{Q}) \leq 10$.*

COROLLARY 1.4. *Let $(\Lambda V, d)$ be a Sullivan minimal model of a simply connected elliptic space X. If $dimH^*(X;\mathbb{Q}) \leq 10$, then $(\Lambda V, d)$ is pure.*

We have organized the content of this paper in the following way. In Section 2 we recall the necessary definitions and preliminaries concerning the Sullivan minimal model, elliptic spaces and some of their properties. In Section 3 we establish our main results. The proof will be split into two parts, according to whether the homotopic Euler characteristic is equal to zero or not.

2. PRELIMINARIES

A minimal model is a particularly tractable kind of commutative differential graded algebra "cdga" that can be associated to any nice cdga or to any nice space. The word minimal emphasizes that, at least in many cases of interest, the model is calculable. The amazing feature of minimal models of spaces is their ability to algebraically encode all rational homotopy information about a space. This is, of course, why minimal models are important. Further details can be found in the reference [3]. We use the Sullivan minimal model of simply connected CW-complex X of finite type. It is a free graded commutative algebra ΛV, for some finite type graded vector space V, together with a differential d of degree +1 that is decomposable, i.e., d: $V^i \to \left(\Lambda^{\geq 2} V\right)^{i+1}$. We assume that the minimal algebra is simply connected, i.e., that the vector space V has no generators for degree lower than 2. If $\{v_1, \ldots, v_n\}$ is a graded basis for V, then we write ΛV as $\Lambda\left(v_1, \ldots, v_n\right)$. A basis can always be chosen so that $\mathrm{d}v_1 = 0$ and $\mathrm{d}v_i \in \Lambda\left(v_1, \ldots, v_{i-1}\right)$ for $i \geq 2$. In particular, if $(\Lambda V, \mathrm{d})$ is the Sullivan minimal model of X, there are isomorphisms:

$$V \cong \Pi_*(X) \otimes \mathbb{Q} \quad \text{and} \quad H^*(\Lambda V; \mathbb{Q}) \cong H^*(X; \mathbb{Q}).$$

Example 2.1. The spheres $\mathbb{S}^k$.
- The minimal model of an odd sphere is $(\Lambda\{a\}, 0)$.
- The minimal model of an even sphere is $(\Lambda\{a, x\}, \mathrm{d})$ with $\mathrm{d}a = 0$, $\mathrm{d}x = a^2$.

Definition 2.2. A simply connected topological space X is called rationally elliptic if it satisfies the two conditions:

$$\dim H_*(X; \mathbb{Q}) < \infty \quad \text{and} \dim \Pi_*(X) \otimes \mathbb{Q} < \infty.$$

By analogy, a minimal Sullivan algebra $(\Lambda V,\mathrm{d})$ is elliptic if both $H(\Lambda V,\mathrm{d})$ and V are finite dimensional. There is a remarkable sub-class of elliptic spaces called pure spaces.

Definition 2.3 (Pure space/pure Sullivan minimal model). An elliptic Sullivan minimal model $(\Lambda V, \mathrm{d})$ is called pure, if $\mathrm{d}V^{\text{even}} = 0$ and $\mathrm{d}V^{\text{odd}} \subset \Lambda V^{\text{even}}$. Also, a simply connected elliptic space X is pure if its Sullivan minimal model is pure.

Definition 2.4. An elliptic Sullivan minimal model $(\Lambda V, \mathrm{d})$ is called hyperelliptic if $\mathrm{d}V^{\text{even}} = 0$ and $\mathrm{d}V^{\text{odd}} \subset \Lambda^+ V^{\text{even}} \otimes \Lambda V^{\text{odd}}$.

The class of elliptic spaces has a variety of very nice properties. Let us briefly sum them up:

Formal dimension, $\mathrm{fd}(X) := \max\left\{k \in \mathbb{N} \ / \ H^k(X; \mathbb{Q}) \neq 0\right\}$;

Homotopical Euler characteristic, $\chi_\Pi(X) := \sum_k (-1)^k \dim \Pi_k(X) \otimes \mathbb{Q}$;

Cohomological Euler characteristic, $\chi_c(X) := \sum_k (-1)^k \dim H^k(X;\mathbb{Q})$.

It is well known that:

THEOREM 2.5 ([4, Theorem 32.10]). *If X is a simply connected elliptic space, then $\chi_\Pi \leq 0$ and $\chi_c \geq 0$. Moreover, the following conditions are equivalent:*

(i) $\chi_\Pi(X) = 0$; (ii) $\chi_c(X) > 0$; (iii) $H^*(X;\mathbb{Q}) = H^{even}(X;\mathbb{Q})$.

Regarding Theorem 2.5, we have:

REMARK 2.6. $\dim V^{\text{odd}} \geq \dim V^{\text{even}}$ and the inequality is strict if and only if $\chi_c(X) = 0$, hence $\dim H^*(X;\mathbb{Q}) = 2 \dim H^{\text{even}}(X;\mathbb{Q})$. In particular, $\dim V^{\text{odd}} = \dim V^{\text{even}}$, when $\dim H^*(X;\mathbb{Q})$ is odd.

PROPOSITION 2.7. *If X is a rationally elliptic space, then the following conditions are equivalent:*
(i) $\chi_c(X) > 0$*;*
(ii) $H^*(X;\mathbb{Q})$ *is the quotient of a polynomial algebra in r variables of even degree by an ideal truncated by a Borel ideal, more precisely:*

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$$

where $\{f_1, \ldots, f_r\}$ is a regular sequence of graded elements in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_r]$;
(iii) *$\dim \Pi_{even}(X) \otimes \mathbb{Q} = \dim \Pi_{odd}(X) \otimes \mathbb{Q}$.*

If these conditions hold, then:

$$\dim H^*(X;\mathbb{Q}) = |f_1| \ldots |f_r| / |x_1| \ldots |x_r|.$$

Moreover, (see [14]), $|f_i| \geq 2|x_i|$ and $\dim H^*(X;\mathbb{Q}) \geq 2^r$.
Our proofs are essentially based on these theorems:

THEOREM 2.8 ([2] and [4]). *If X is a simply connected elliptic space, $(\Lambda V, d)$ its minimal model and $(a_i)_i$ is a homogeneous basis of V, then:*
- $\sum_{|a_i|\ even} |a_i| \leq fd(X)$;
- $\sum_{|a_i|\ odd} |a_i| \leq 2fd(X) - 1$;
- $fd(X) = \sum_{|a_i|\ odd} |a_i| - \sum_{|a_i|\ even} (|a_i| - 1)$.

THEOREM 2.9 ([4]). *If X is a simply connected elliptic space, then $H^*(X;\mathbb{Q})$ satisfies the Poincaré duality, which means that:*
- *$\dim H^n(X;\mathbb{Q}) = 1$, where $fd(X) = n$, i.e., $H^n(X;\mathbb{Q}) = \mathbb{Q}\mu$ (μ is called fundamental class of $H^*(X;\mathbb{Q})$);*
- *for any $0 \leq k \leq n$, the cup-product $H^k(X;\mathbb{Q}) \times H^{n-k}(X;\mathbb{Q}) \to H^n(X;\mathbb{Q}) \cong \mathbb{Q}$ is a non-degenerate bilinear form.*

In [11], James introduced the construction of *reduced product of pointed spaces.* If X is a topological based space, we set $X_{(1)} = X$ and

$$X_{(p)} = X \times \ldots \times X \ / \ (\ldots, *, \ldots) \sim (*, \ldots).$$

Applying this for even spheres, we construct $\mathbb{S}^n_{(p)}$ verifying:

$$H^*(\mathbb{S}^n_{(p)};\mathbb{Q}) \cong \mathbb{Q}[a]/(a^{p+1}).$$

The notation $\mathbb{S}^n_{(p)}$ will be used in the rest of the text only if n is even.

We conclude this part by a conjecture given by M.R.Hilali (see [6]), which is based on the size of the rationally elliptic spaces.

Conjecture 2.10. *Let X be a simply connected rationally elliptic space. Then it holds that*

$$\text{(H)} \qquad dim\ H^*(X;\mathbb{Q}) \geq\ dim\ \Pi_*(X) \otimes \mathbb{Q}.$$

We remark that, in terms of Sullivan minimal models $(\Lambda V, d)$, this conjecture can be stated equivalently as

$$dim\ H^*(\Lambda V, d) \geq\ dim\ V.$$

In the remainder of the paper, X is an elliptic rational and simply connected finite cell complex, $(\Lambda V$,d) its minimal model and μ its fundamental class. We also denote by $|v|$ the degree of v. The main tool we shall use is the Sullivan minimal model.

3. CLASSIFICATION

Consider $\dim H^*(X;\mathbb{Q}) = 10$ and let $B = \{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \mu\}$ be a basis of $H^*(X;\mathbb{Q})$ ordered in an increasing degree. We will divide the proof in two parts; in the first one, we will discuss the case when $\chi_\Pi(X) = 0$, then we will suppose $\chi_\pi(X) \neq 0$.

We are now ready to proceed with the proof of Theorem 1.1.

3.1. PROOF OF THEOREM 1.1

We discuss this case according to the number of generators n and the ideal is generated by n polynomials f_i for $1 \leq i \leq n$. Since $\dim H^*(X;\mathbb{Q}) = \prod_{i=1}^{i=n} |f_i| \diagup \prod_{i=1}^{i=n} |x_i| \geq 2^n$ and $\dim H^*(X;\mathbb{Q}) = 10$, $n \in \{1,2,3\}$.

• If $\boldsymbol{n = 1}$, then $H^*(X;\mathbb{Q}) = \mathbb{Q}[x] \diagup (x^{10})$ which implies

$$X \sim_{\mathbb{Q}} \mathbb{S}^m_{(9)} \text{ with } \mathsf{fd}(X) = 9m.$$

• If $\boldsymbol{n = 2}$, then $H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1,x_2] \diagup (f_1,f_2)$, where $(f_1,f_2) \in \mathbb{Q}[x_1,x_2]$, and we consider firstly the case $|x_1| < |x_2|$.

i) Assume that $|f_1|$ is an integer multiple of $|x_2|$, so $|f_1| = k|x_2|$ for some integer $k \geq 1$. From the relation $\frac{|f_1||f_2|}{|x_1||x_2|} = 10$, we have $2|x_2| \leq |f_2| = \frac{10}{k}|x_1| < \frac{10}{k}|x_2|$ and we automatically get $k \leq 5$.

Let us start by supposing $k = 1$; then $|f_1| = |x_2|$ and $|f_2| = 10|x_1|$, thus $f_1 = x_1^m$ for $m \geq 2$; by the dimension formula, we get $2|x_2| \leq |f_2| = \frac{10}{m}|x_2|$, so $m \in \{2,3,4,5\}$. If $m = 2$, then $|f_1| = 2|x_1| = |x_2|$ and $|f_2| = 10|x_1| = 5|x_2|$, thus $(f_1,f_2) = \left(x_1^2, \sum \lambda_{ij} x_1^i x_2^j\right)$ with $i|x_1| + j|x_2| = 10|x_1|$, then $i + 2j = 10$, which gives $f_2 = x_2^5 + \lambda_1 x_1^{10} + \lambda_2 x_1^8 x_2 + \lambda_3 x_1^6 x_2^2 + \lambda_4 x_1^4 x_2^3 + \lambda_5 x_1^2 x_2^4$ for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{Q}$. Since $f_2 - x_2^5 = f_3 \in \langle f_1 \rangle$, by this change of variable, we get $(f_1, f_2 - f_3) = (x_1^2, x_2^5)$. Therefore,

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1^2, x_2^5\right), \text{ in particular}, X \sim_{\mathbb{Q}} \mathbb{S}^k \times \mathbb{S}^m_{(4)}.$$

If $m = 3$ or 4, we choose for example $m = 3$, we obtain $|f_1| = 3|x_1| = |x_2|$ and $|f_2| = 10|x_1| = \frac{10}{3}|x_2|$, which implies $|f_1|$ is an integer multiple of $|x_2|$, but $|f_1| = |x_2|$, then $\dim \mathbb{Q}[x_1, x_2] \diagup (f_1, f_2) = \infty$ (because $[x_2^m] \neq 0\ \forall m$), so it is impossible. The same thing goes for $m = 4$. If $m = 5$, we get $|f_2| = 10|x_1| = 2|x_2|$, so $(f_1, f_2) = \left(x_1^5, \sum \lambda_{ij} x_1^i x_2^j\right)$ with $i + 5j = 10$, hence $f_2 = x_2^2 + \lambda_1 x_1^5 x_2 + \lambda_2 x_1^{10}$ for some $\lambda_1, \lambda_2 \in \mathbb{Q}$. Since $f_2 - x_2^2 = f_3 \in \langle f_1 \rangle$, by the variable change, we get $(f_1, f_2 - f_3) = (x_1^5, x_2^2)$. So,

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1^5, x_2^2\right).$$

If $k = 2$, we have $|f_1| = 2|x_2|$ and $|f_2| = 5|x_1|$, since $|f_1| \leq |f_2|$ so $|x_1| + |x_2| < 2|x_2| \leq 5|x_1|$, then $f_1 = x_2^2 + \sum \lambda_{ij} x_1^i x_2^j$ with $i|x_1| + j|x_2| = 2|x_2|$, it is obvious that $j = 1$ and $i|x_1| = |x_2|$ for $i \geq 2$. Supposing that $i \geq 3$ leads us to a contradiction, because $5|x_1| \geq 2|x_2| \geq 6|x_1|$, hence $i = 2$, i.e., $f_1 = x_2^2 + \lambda_1 x_1^2 x_2$. Similarly, we have $f_2 = x_1^5 + \sum \lambda_{ij} x_1^i x_2^j$ with $i|x_1| + j|x_2| = 5|x_1|$ so $i + 2j = 5$, then $f_2 = x_1^5 + \gamma_1 x_1 x_2^2 + \gamma_2 x_1^3 x_2$ for some $\lambda_1, \gamma_1, \gamma_2 \in \mathbb{Q}$. We conclude

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_2^2 + \lambda_1 x_1^2 x_2, x_1^5 + \gamma_1 x_1 x_2^2 + \gamma_2 x_1^3 x_2\right).$$

If $k = 3$, then $|f_1| = 3|x_2|$ and $|f_2| = \frac{10}{3}|x_1|$, so $f_2 \in (x_2)$ and $|f_1|$ is an integer multiple of $|x_1|$, but $2|x_1| \leq |f_1| = 3|x_2| \leq \frac{10}{3}|x_1|$, thus, if $|f_1| = 2|x_1|$ and $|f_1| = 3|x_2|$, we get $|x_1| > |x_2|$, and, if $|f_1| = 3|x_1|$, we get $|x_1| = |x_2|$, which contradicts the hypothesis $|x_1| < |x_2|$. The same justification applies if we suppose $k = 4$.

If $k = 5$, then $|f_1| = 5|x_2|$ and $|f_2| = 2|x_1|$, which implies $|f_1| > |f_2|$, because $|x_1| < |x_2|$, so it is impossible.

ii) Assume that $|f_1|$ is an integer multiple of $|x_1|$ and not of $|x_2|$, i.e., $|f_1| = k|x_1|$ for $k \geq 2$, we will obtain $|f_2| = \frac{10}{k}|x_2| \geq 2|x_2|$ so $k \in \{2, 3, 4, 5\}$.

If $k = 2$, then $f_1 = x_1^2$ and $|f_2| = 5|x_2|$, thus $f_2 = x_2^5 + \sum \lambda_{ij} x_1^i x_2^j = x_2^5 + \lambda_1 x_1^{k_1} x_2 + \lambda_2 x_1^{k_2} x_2^2 + \lambda_3 x_1^{k_3} x_2^3 + \lambda_4 x_1^{k_4} x_2^4$ with $k_1 > k_2 > k_3 > k_4 > 1$ and $k_i|x_1| + j|x_2| = 5|x_2|$ for $1 \leq j \leq 4$. By a simple computation, we show that $k_1 > 4$, $k_2 > 3$, $k_3 > 2$ and $k > 1$, then $f_2 - x_2^5 = f_3 \in \langle f_1 \rangle$. Finally,

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1^2, x_2^5\right).$$

The case $k = 3$ or 4 is impossible, because, if we take as an example $k = 3$, then $|f_1| = 3|x_1|$ and $|f_2| = \frac{10}{3}|x_2|$, so certainly $f_2 \in (x_1)$ and $|f_1|$ is an integer multiple of $|x_2|$, which conflicts the assumption. If $k = 5$, then $|f_1| = 5|x_1|$

and $|f_2| = 2|x_2|$. As $|f_1| \leq |f_2|$, automatically leads to $5|x_1| < 2|x_2|$. Thus, $(f_1, f_2) = (x_1^5, x_2^2)$, so

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1^5, x_2^2\right).$$

iii) Suppose $|f_1|$ is neither an integer multiple of $|x_1|$ nor of $|x_2|$; then $f_1 \in (x_1) \cap (x_2)$ and f_2 contains a non zero multiple of x_1 and x_2, thus $|f_2| = k_1|x_1| = k_2|x_2|$ for $k_1 > k_2 \geq 2$. In this case, we obtain $k_2 = 3$ or $k_2 = 4$, because, if $k_2 \geq 6$, then $|f_1| = \frac{10}{6}|x_1| < 2|x_1|$, which is impossible, since $|f_1| \geq 2|x_1|$. Also, if $k_2 = 2$ or $k_2 = 5$, then $|f_1|$ is an integer multiple of $|x_1|$, which contradicts our hypothesis. But, if $k_2 = 3$, then $|f_2| = 3|x_2| = k_1|x_1|$ and we have $|f_1| \geq |x_1| + |x_2| \geq \left(1 + \frac{k_1}{3}\right)|x_1|$, as a result $4 \leq k_1 \leq 8$ (if not, we get $|f_2| \leq \frac{10}{4}|x_2|$ but $|f_2| = 3|x_2|$). We can easily show that the only possible cases are when $k_1 = 4$ and $k_1 = 7$, so we suppose $k_1 = 4$; then $|f_2| = 3|x_2| = 4|x_1|$ and $|f_1| = \frac{10}{3}|x_1| = \frac{10}{4}|x_2|$ with $f_1 = \sum \lambda_{ij} x_1^i x_2^j$, hence $i|x_1| + j|x_2| = \frac{10}{3}|x_1|$, which gives $3i + 4j = 10$. Therefore, we must have $(i, j) = (2, 1)$, so $(f_1, f_2) = (x_1^2 x_2, x_1^4 + \lambda x_2^3)$, $\lambda \in \mathbb{Q}^*$. In particular

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1^2 x_2, x_1^4 + \lambda x_2^3\right).$$

Moreover, if $k_1 = 7$ we get $|f_2| = 3|x_2| = 7|x_1|$ and $|f_1| = \frac{10}{3}|x_1| = \frac{10}{7}|x_2|$ with $f_1 = \sum \lambda_{ij} x_1^i x_2^j$, hence $i|x_1| + j|x_2| = \frac{10}{7}|x_1|$, which gives $3i + 7j = 10$. We necessarily obtain $(i, j) = (1, 1)$, so $(f_1, f_2) = \left(x_1 x_2, x_1^7 + \lambda x_2^3\right), \lambda \in \mathbb{Q}^*$. Then

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1 x_2, x_1^7 + \lambda x_2^3\right).$$

Finally, if $k_2 = 4$ then $|f_2| = 4|x_2| = k_1|x_1|$ for $k_1 > 4$, by the formula of dimension, we have $\frac{10}{4} \geq \left(1 + \frac{k_1}{3}\right)$, thus $4 < k_1 \leq 6$, if $k_1 = 5$, hence $|f_1|$ is an integer multiple of $|x_2|$, contradiction. Then $k_1 = 6$ so $|f_2| = 4|x_2| = 6|x_1|$ and $|f_1| = \frac{10}{4}|x_1| = \frac{10}{6}|x_2|$ with $f_1 = \sum \lambda_{ij} x_1^i x_2^j$, hence $i|x_1| + j|x_2| = \frac{10}{4}|x_1|$ which gives $2i + 3j = 5$, thus necessarily $(i, j) = (1, 1)$, so $(f_1, f_2) = \left(x_1 x_2, x_1^6 + \gamma x_2^4\right)$, $\gamma \in \mathbb{Q}^*$. Therefore

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2] \diagup \left(x_1 x_2, x_1^6 + \gamma x_2^4\right).$$

We consider now $|x_1| = |x_2|$. Then f_1 and f_2 are homogeneous polynomials of the second and fifth degrees respectively. Using ([14, Lemma 3.1]), we say that $f_1 = x_1^2 - a x_2^2$ for $a > 0$ and $a^{\frac{1}{2}} \in \mathbb{Q}$, as well $f_2 = x_1^5 + \sum \lambda_{ij} x_i x_j$ with $i|x_1| + j|x_2| = 5|x_1| = 5|x_2|$. So, by using f_1, we obtain $(f_1, f_2) = \left(x_1^2 - a x_2^2, x_1^5 + b x_1^4 x_2\right)$. Then we get the system

$$\begin{cases} x_1^2 - a x_2^2 = 0 \\ x_1^5 + b x_1^4 x_2 = 0 \end{cases}.$$

Using the following variables changes $x_1' = x_1 + \alpha x_2$ and $x_2' = x_1 - \alpha x_2$ for $\alpha = a^{\frac{1}{2}}$, the system can be further simplified into $\left\{ \begin{array}{c} x_1' x_2' = 0 \\ \lambda_1 x_1'^5 + \lambda_2 x_2'^5 = 0 \end{array} \right.$. Then $H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1', x_2'] \diagup (x_1'x_2', \lambda_1 x_1'^5 + \lambda_2 x_2'^5)$, consequently

$$X \sim_{\mathbb{Q}} \mathbb{S}^n_{(5)} \# \mathbb{S}^n_{(5)}.$$

• If $\boldsymbol{n = 3}$, then in this case we have $H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3] \diagup (f_1, f_2, f_3)$. Let us define two spaces V_0 and V_1 as follows:

$$V_0 = \mathbb{Q}\{1, x_1, x_2, x_3\} \diagup (f_1, f_2, f_3)$$
$$V_1 = \mathbb{Q}\left\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\right\} \diagup (f_1, f_2, f_3)$$

So $\dim V_0 = 4$ and $\dim V_1 \geq 3$. If $f_i \notin V_1$ for $i \in \{1,2,3\}$, then $\dim V_1 = 6$, but $\dim H^*(\Lambda V, \mathrm{d}) \geq \dim(V_0 \oplus V_1 \oplus \mathbb{Q}\{\mu\}) \geq 11$, which contradicts the fact that $\dim H^*(\Lambda V, \mathrm{d}) = 10$. So we suppose for example $f_1 \in V_1$; then $f_1 = \sum_{1\leq} \lambda_{ij} x_1^i x_2^j$, if $\mathbb{Q}\mu \cap V_1^2 = \varnothing$ and it follows from duality of Poincaré that there is a space V_1' such that $\dim V_1 = \dim V_1'$ and $V_1 V_1' = \mathbb{Q}\mu$, but $\dim V_1 \geq 3$. Hence

$$\dim H^*(\Lambda V, \mathrm{d}) \geq \dim\left(V_0 \oplus V_1 \oplus V_1' \oplus \mathbb{Q}\{\mu\}\right) \geq 11$$

This leads to a contradiction; then $V_1^2 = \mathbb{Q}\mu$, thus automatically $\exists\ i \in [1,3]$ such that $x_i^2 x_j \neq 0$. So there is a space V_0' such that $\dim V_0' = \dim V$ and $V_0' V_0 = \mathbb{Q}\mu$; furthermore, $\dim H^*(\Lambda V, \mathrm{d}) \geq \dim(V_0 \oplus V_0' \oplus V_1) \geq 11$, which contradicts our assumption.

3.2. PROOF OF THEOREM 1.2

In order to study this case we first need to prove several lemmas and propositions.

PROPOSITION 3.1. *If $\dim H^*(X;\mathbb{Q}) = 10$ with $\chi_\pi(X) \neq 0$, then $\mathsf{fd}(X)$ is odd.*

Proof. Suppose that $\mathsf{fd}(X)$ is even; from the Poincaré duality we have:

$$\chi_c(X) = 1 + (-1)^{\mathrm{fd}(X)} + \sum_{i=1}^{4} 2(-1)^{|\alpha_i|} = 2 + \sum_{i=1}^{4} 2(-1)^{|\alpha_i|}$$

Thus, $\chi_c(X) \neq 0$, which contradicts the assumption, because $\chi_\pi(X) \neq 0$. □

LEMMA 3.2. *If $B = \{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \mu\}$ is a basis of $H^*(X;\mathbb{Q})$, then $|\alpha_4| < |\alpha_5|$.*

Proof. Suppose $|\alpha_4| = |\alpha_5|$; from the Poincaré duality, we have $\mu = \alpha_4\alpha_5$ then $\mathrm{fd}(X) = |\alpha_4| + |\alpha_5| = 2|\alpha_4|$. It is impossible, because $\mathrm{fd}(X)$ is odd. □

According to the duality of Poincaré, the only possible cases are:

First case: $\boldsymbol{|\alpha_1| = |\alpha_2| = |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6| = |\alpha_7| = |\alpha_8|}$

Here necessarily $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are generators of $H^*(X;\mathbb{Q})$.

If $|\alpha_1|$ is odd, then $|\alpha_5|$ is even. Let $(\Lambda V, \mathrm{d}) = (\Lambda(x_1, \ldots, x_p, y_1, \ldots, y_{n+p}), \mathrm{d})$ be the Sullivan minimal model of X. Put $\alpha_i = [y_i]$ for $i \in \{1, \ldots, 4\}$ and $W = \{[y_i y_j] \text{ for } 1 \leq i < j \leq 4\} \subset H^{\mathrm{even}}(X; \mathbb{Q})$; so $\dim W \leq 4$, and thus there exist at least two generators $z_1, z_2 \in V^{\mathrm{odd}}$ such that $\mathrm{d}z_i \in W$. Consequently $\dim W = 0$, otherwise we have $\mathrm{fd}(X) = 3|y_1|$, and

$$\begin{aligned}\sum\nolimits_{a_i \in V^{\mathrm{odd}}} |a_i| &\geq \sum_{i=1}^{4} |y_i| + |z_1| + |z_2| = 4|y_1| + 2|y_1| + 2|y_1| - 2 \\ &\geq 8|y_1| - 2 > 2\mathrm{fd}(X) - 1, \text{ impossible.}\end{aligned}$$

Then $\dim W = 0$, which implies $[y_i y_j] = 0$ for $1 \leq i < j \leq 4$. Hence there exist $z_{ij} \in V^{\mathrm{odd}}$ such that $\mathrm{d}z_{ij} = y_i y_j$ for $1 \leq i < j \leq 4$. As $\mathrm{d}(y_i z_{ij}) = \mathrm{d}(y_j z_{ij}) = 0$, we put $W_1 = \{[y_k z_{ij}], \text{ for } 1 \leq i < j \leq 4 \text{ and } k = i \text{ or } j\}$. Hence $\dim W_1 = 0$, if not, we have $\mathrm{fd}(X) = 4|y_1| - 1$, and

$$\begin{aligned}\sum\nolimits_{a_i \in V^{\mathrm{odd}}} |a_i| &\geq 4|y_1| + 4|y_1| - 1 + \sum |z_{ij}| \\ &\geq 8|y_1| - 1 + \sum |z_{ij}| > 2\mathrm{fd}(X) - 1\end{aligned}$$

This is impossible. If we continue this process, we will find an infinity of generators and cocycles, which contradicts the fact that X is elliptic.

If $|\alpha_1|$ is even, then $|\alpha_5|$ is odd, so there exist x_1, x_2, x_3, x_4 generators of even degree such that $\mathrm{d}x_i = 0$ and $\alpha_i = [x_i]$ for $1 \leq i \leq 4$. It is clear that α_k does not come from any generator of $(\Lambda V, \mathrm{d})$ (for $k \in \{5, \ldots, 8\}$), and thus, by degree reasons, we have $[x_i x_j] = 0$, so $\exists y_{ij} \in V^{\mathrm{odd}} / \mathrm{d}y_{ij} = x_i x_j$ for $1 \leq i, j \leq 4$. Therefore, we put $W_1 = \{[x_k y_{ij} - x_j y_{ik}] \ / \ 1 \leq i, j, k \leq 4\}$. Note that $W_1 \subset H^{\mathrm{odd}}(X; \mathbb{Q})$. We can easily show that $W_1 = \varnothing$, otherwise $\exists i_0, j_0, k_0 \in \{1, \ldots, 4\}$ such that $\alpha_k = [x_{k_0} y_{i_0 j_0} - x_{j_0} y_{i_0 k_0}]$ for $k \in \{5, \ldots, 8\}$ so $\mathrm{fd}(X) = 4|x_1| - 1$. Hence

$$\sum_{a_i \in V^{\mathrm{odd}}} |a_i| \geq 10|y_1| = 20|x_1| - 10 > 2\mathrm{fd}(X) - 1.$$

Since $\dim W_1 = 0$ we will get other generators of even degree of $(\Lambda V, \mathrm{d})$. Following the same approach as above, we get an infinity of generators, thus $\dim V = \infty$, which is impossible.

Second case: $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6| < |\alpha_7| = |\alpha_8|$

In this case $H^*(X; \mathbb{Q})$ is necessarily generated by α_1 and α_2, we therefore have several cases to discuss:

(1) If $|\alpha_1|$ is odd, we put $\alpha_1 = [y_1]$ and $\alpha_2 = [y_2]$, where y_1 and y_2 are two odd generators,, then, from the assumption, we have $|y_1| = |y_2|$. If $|\alpha_3|$ is even and $\alpha_1 \alpha_2 = 0$, then there exist two even generators x_1 and x_2 such that $\alpha_3 = [x_1]$ and $\alpha_4 = [x_2]$. Moreover, $\exists y_3 \in V^{\mathrm{odd}} / \mathrm{d}y_3 = y_1 y_2$. Therefore $Z^{\text{homogène}}(\Lambda V, \mathrm{d}) = \{x_i^n; y_i; y_1 y_2; x_i^n y_i, y_i y_3 \text{ for } 1 \leq i, j \leq 2\}$.

Lemma 3.3. *$\{\alpha_i\}_{i \in \{5,6,7,8\}}$ does not come from any generator of $(\Lambda V, d)$.*

Proof. By degree reasons, $|\alpha_7|$ and $|\alpha_8|$ are even, so we can suppose for example $\alpha_7 = [x_3]$ with x_3 is an even generator. Then $\mathrm{fd}(X) = |y_1| + |x_3|$, but $\sum_{x_i \in V^{\mathrm{even}}} |x_i| \geq |x_1| + |x_2| + |x_3| > |y_1| + |x_3| = \mathrm{fd}(X)$ (impossible). By the same argument, if $\exists y \in V^{\mathrm{odd}}/\alpha_5 = [y]$ and $\mathrm{d}y = 0$, then $\mathrm{fd}(X) = |x_1| + |y|$. But $[y_i y] = 0$ for $1 \leq i \leq 2$ (if not, we obtain $\mu = [y_1 y_2 y] = [\mathrm{d}(y y_3)] = 0$, this is absurd). Then $\exists\ z_i \in V^{\mathrm{odd}}/\mathrm{d}z_i = y_i y$ for $i = 1, 2$ and $\exists\ y_3 \in V^{\mathrm{odd}}/\ |y_3| \geq 2|x_1| - 1$. Therefore

$$\sum_{a_i \in V^{\mathrm{odd}}} |y_i| \geq |y_1| + |y_2| + |y_3| + |y| + 2|z_1|$$
$$> 4|y_1| + 3|y| + 2|x_1| - 3 > 2\mathrm{fd}\,(X) - 1,\ \text{impossible.}$$

□

Lemma 3.4. *$\alpha_k \notin \mathbb{Q}\{[x_i y_j]/1 \leq i, j \leq 2\}$ for $k = 5, 6$.*

Proof. If not, we get $\mathrm{fd}(X) = 2|x_1| + |y_1|$ and $\dim\mathbb{Q}\{[x_i y_i]/\ 1 \leq i, j \leq 2\} \leq 2$, consequently there is at least a generator x_3 of even degree such that $|x_3| = |x_1| + |y_1| - 1$. Therefore, $\sum_{x_i \in V^{\mathrm{even}}} |x_i| \geq |x_1| + |x_2| + |x_3| = 2|x_1| + |x_2| + |y_1| - 1 > \mathrm{fd}(X)$, impossible. □

So $\alpha_1 \alpha_2 \neq 0$, and thus there exists an even generator x_1 of V such that $\mathrm{d}x_1 = 0$ and $\alpha_j = [x_1]$ for $j = 3$ or $j = 4$. Thus, by the previous lemma and from the Poincaré duality, we have $[x_1 y_i] = 0$ for $1 \leq i \leq 2$, hence $\exists u_i \in V^{\mathrm{even}}/\mathrm{d}u_i = x_1 y_i$ for $i = 1, 2$. As $\mathrm{d}(y_i u_i) = 0$, we obtain $[y_i u_i] = 0$, because, if $\alpha_k \in \mathbb{Q}\{[y_i u_i]/\ 1 \leq i, j \leq 2\}$ for $k = 5$ or 6, then $\mathrm{fd}(X) = |x_1| + |u_1| + |y_1| = 2|x_1| + 2|y_1| - 1$. But $\sum_{|x_i|\ \mathrm{even}} |x_i| \geq |x_1| + 2|u_1| \geq |x_1| + 2|x_1| + 2|y_1| - 2 > \mathrm{fd}(X)$, impossible. Therefore, $[y_i u_i] = 0$ and, carrying on the same process, we get $\dim V = \infty$, which gives a contradiction, because $(\Lambda V, \mathrm{d})$ is elliptic.

On the other hand if $|\alpha_3|$ is odd, then there exist two odd generators y_3 and y_4 of ΛV such that $\alpha_3 = [y_3]$ and $\alpha_4 = [y_4]$.

Lemma 3.5. *$\{\alpha_k\}_{k \in \{5,6,7,8\}}$ does not come from any generator of $(\Lambda V, d)$.*

Proof. By contradiction, we suppose there exists $i_0 \in \{5, 6, 7, 8\}$ such that $\alpha_{i_0} = [x_1]$ where x_1 is an even generator of ΛV, and thus, from the duality of Poincaré, $\mathrm{fd}(X) \leq |x_1| + |y_3|$ and necessarily $\exists y \in V^{\mathrm{odd}}/\mathrm{d}y = x_1^2$ (because $2|x_1| > \mathrm{fd}\,(X)$) . Therefore

$$\sum_{a_i \in V^{\mathrm{odd}}} |a_i| \geq 2|y_1| + 2|y_3| + |y|$$
$$\geq 2|y_1| + 2|y_3| + 2|x_1| - 1 > 2\mathrm{fd}\,(X) - 1,\ \text{impossible.}$$

□

Lemma 3.6. *$[y_i y_j] = 0$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$.*

Proof. By absurd, from the duality of Poincaré, $\dim\mathbb{Q}\{[y_i y_j]/i \in \{1, 2\}$ and $j \in \{3, 4\}\} \leq 2$, and thus $\exists u_1, u_2 \in V^{\mathrm{odd}}/|u_i| = |y_1| + |y_3| - 1$ for $i = 1, 2$.

Consequently $\mathrm{fd}(X) \leq |y_1| + 2|y_3|$. Hence

$$\sum_{a_i \in V^{\mathrm{odd}}} |a_i| \geq 2|y_1| + 2|y_3| + 2|u_1| = 4|y_1| + 4|y_3| - 2 > 2\mathrm{fd}\,(X) - 1, \text{ impossible.}$$

Then there exist some generators z_{ij} of odd degree of ΛV such that $\mathrm{d}z_{ij} = y_i y_j$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, so $\mathrm{d}(y_i z_{ij})$ $=\mathrm{d}(y_j z_{ij}) = 0$. Moreover, we can easily show that $[y_i z_{ij}] = [y_j z_{ij}] = 0$, but, if we proceed in the same manner, we get $\dim V = \infty$, which gives a contradiction, because $(\Lambda V, \mathrm{d})$ is elliptic. □

(2) If $|\alpha_1|$ is even, then we can put $\alpha_i = [x_i]$, where $x_i \in V^{\mathrm{even}}$ such that $\mathrm{d}x_i = 0$ for $i = 1, 2$. Assume that $|\alpha_3|$ is even, so, by degree reasons, $\dim\mathbb{Q}\{[x_i x_j]/1 \leq i, j \leq 2\} \leq 2$, hence $\exists\ z_i \in V^{\mathrm{odd}}/|z_i| = 2|x_1| - 1$ for $i = 1, 2$. Furthermore, we have $|\alpha_5|$ and $|\alpha_6|$ are odd, then there exist two odd generators y_1 and y_2 of ΛV such that $\alpha_5 = [y_1]$ and $\alpha_6 = [y_2]$. By the formula of dimensions, we can easily show that α_7 and α_8 do not come from a generator of $(\Lambda V, \mathrm{d})$. Moreover, $\alpha_k \notin \mathbb{Q}\{\ [x_i y_j]/$ for $1 \leq i, j \leq 2\ \}$ for $k = 7$ or $k = 8$, because we obtain $\mathrm{fd}(X) = 2|x_1| + |y_1|$, and there exist at least two even generators u_1 and u_2 such that $|u_i| = |x_1| + |y_1| - 1$, for $i = 1, 2$ since $\dim\mathbb{Q}\{\ [x_i y_j]/$ for $1 \leq i, j \leq 2\ \} \leq 2$. But

$$\sum_{|x_i|\ \mathrm{even}} |x_i| \geq 2|x_1| + 2|u_1| = 4|y_1| + 2|y_1| - 2 > \mathrm{fd}\,(X)\,, \text{ impossible.}$$

Consequently, $[x_i y_j] = 0$, so $\exists\ z_{ij} \in V^{\mathrm{even}}/\mathrm{d}z_{ij} = x_i y_j$. But $\mathrm{d}(y_i z_{ij})$ $=\mathrm{d}(y_j z_{ij})$ $= 0$, then, by the formula of dimension, we show that $[y_i z_{ij}] = [y_j z_{ij}] = 0$ for $i, j = 1, 2$. Proceeding this way, we will obtain $\dim V = \infty$, (contradiction).

Remark 3.7. Using the same argument, we can prove $\dim V = \infty$, when $|\alpha_3|$ is odd, which leads to a contradiction.

Third case: $|\alpha_1| < |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6| < |\alpha_7| < |\alpha_8|$

Many cases have to be considered now:

If $|\alpha_1|$ and $|\alpha_2|$ are odd , then certainly α_1 and α_2 are generators of $H^*(X; \mathbb{Q})$. Moreover, if $|\alpha_3|$ is odd, then α_3 and α_4 are also generators of $H^*(X; \mathbb{Q})$ and we put $\alpha_i = [y_i]$ for $i \in \{1, 2, 3, 4\}$ with y_i being odd generator of V. So, if there is $i_0 \in \{5, 6, 7, 8\}$ such that $\alpha_{i_0} = [x]$ with $x \in V^{\mathrm{even}}$ and $\mathrm{d}x = 0$, then $\mathrm{fd}(X) \leq |y_3| + |x|$, but

$$\sum_{a_i \in V^{\mathrm{odd}}} |a_i| \geq |y_1| + |y_2| + 2|y_3| + 2|x| - 1 > 2\mathrm{fd}\,(X) - 1, \text{ impossible.}$$

Thus, $\alpha_k \in \mathbb{Q}\{[y_i y_j]/i = 1$ *or* 2 and $3 \leq j \leq 4\}$ for $k \in \{5, 6, 7, 8\}$; take for example $k = 5$ and $i = 1$, then $\mathrm{fd}(X) = |y_1| + 2|y_3|$, so, by degree reasons $[y_2 y_3] = [y_2 y_4] = 0$, $\exists u_1, u_2 \in V^{\mathrm{odd}}/\mathrm{d}u_1 = y_2 y_3$ and $\mathrm{d}u_2 = y_2 y_4$. But

$$\begin{aligned}\textstyle\sum_{a_i \in V^{\mathrm{odd}}} |a_i| &\geq |y_1| + |y_2| + 2|y_3| + |u_1| + |u_2| \\ &\geq |y_1| + 3|y_2| + 4|y_3| - 2 > 2\mathrm{fd}\,(X) - 1, \text{ impossible.}\end{aligned}$$

Therefore, $|\alpha_3|$ and $|\alpha_4|$ are even, and thus, if $[y_1y_2] \neq 0$, then $\exists i_0 = 3$ or 4 such that $\alpha_{i_0} = [x]$ with $x \in V^{\text{even}}$ and $\mathrm{d}x = 0$. From the Poincaré duality, we have $[y_ix] = 0$ for $i = 1$ or 2 and also there is an odd generator y of V such that $\alpha_j = [y]$ for $j = 4$ or 5, and thus $\mathrm{fd}(X) = |y| + |x| = 2|x| + |y_l|$ for $l \in \{1,2\} \setminus \{i\}$. As a result, we have $\alpha_k \in \mathbb{Q}\{[x^2],[yy_i]\}$ and $[yy_l] = 0$, so $\dim\mathbb{Q}\{[x^2],[yy_i]\} \leq 1$, hence $\exists u_1, u_2 \in V^{\text{odd}}/\mathrm{d}u_1 = yy_l$ and $\mathrm{d}u_2 = x^2 - \lambda yy_i$. But

$$\sum\nolimits_{a_i \in V^{\text{odd}}} |a_i| \geq |y_1| + |y_2| + |y| + |u_1| + |u_2|$$
$$\geq 2|y_1| + |y_2| + 2|y| + 2|x| - 2 > 2\mathrm{fd}\,(X) - 1, \text{ impossible.}$$

Consequently, $[y_1y_2] = 0$, thus there is an odd generator z of V such that $\mathrm{d}z = y_1y_2$, hence we can put $\alpha_3 = [x_1]$ and $\alpha_4 = [x_2]$ with x_1 and x_2 are two even generators of V. Therefore, $\alpha_k \in \mathbb{Q}\{[y_ix_1],[y_ix_2]$ for $i = 1$ or $2\}$ with $k \in \{5,6\}$, hence $\mathrm{fd}(X) \leq |y_2| + 2|x_1|$ and, by degree reasons, $[y_jx_1] = [y_jx_2] = 0$ for $j \in \{1,2\} \setminus \{i\}$. Then $\exists\ z_1, z_2 \in V^{\text{even}}/\mathrm{d}z_1 = y_jx_1$ and $\mathrm{d}z_2 = y_jx_2$, but

$$\sum_{|a_i| \text{ even}} |a_i| \geq 2|x_1| + 2|z_1| = 4|x_1| + 2|y_j| - 2 > \mathrm{fd}\,(X)\,.$$

If $|\alpha_2|$ is even, then there exists an even generator x_1 of V such that $\alpha_2 = [x_1]$. Moreover, suppose that $|\alpha_3|$ is also even so $\alpha_k \in \mathbb{Q}\{[x_1^2]\}$ for $k = 3$ or 4 and $\alpha_j = [x_2]$ for $j \in \{3,4\}\backslash k$ where x_2 is an even generator of $(\Lambda V, \mathrm{d})$. We can easily show that α_5 and α_6 do not come from generators of V, so, by degree arguments, $\alpha_k \in \mathbb{Q}\{[y_1x_1^2],[y_1x_2]\}$for $k \in \{5,6\}$. Therefore, $\mu = [y_1x_1^4]$ and $\mathrm{fd}(X) = 4|x_1| + |y_1|$, then $\dim\mathbb{Q}\{[x_1^4],[x_2^2],[x_1^2x_2]\} \leq 2$ and $[x_1^3] \neq 0$, which contradicts the fact that $\dim H^{\text{even}}(X;\mathbb{Q}) = 5$. But, if $|\alpha_3|$ is odd, then we can take $\alpha_3 = [y_1x_1]$ (the justification will be the same for α_4) and there is an odd generator y_2 of V such that $\alpha_4 = [y_2]$. By the Poincaré duality, we have $[y_1y_2] = 0$, so $\exists\ u_1 \in V^{\text{odd}}/\mathrm{d}u_1 = y_1y_2$, but $\mathrm{d}y_1u_1 = \mathrm{d}y_2u_1 = 0$ and, applying the Poincaré duality again, we get $[y_1u_1] = 0$, then $\exists\ u_2 \in V^{\text{odd}}/\mathrm{d}u_2 = y_1u_1$. Carrying on the same process we will obtain $\dim V = \infty$, which is impossible.

Following the same approach, the case where $|\alpha_2|$ is even will be proven impossible.

Remark 3.8. There are other cases, but as they can all be disproved in the same way, we found it redundant to tackle them all and we restricted our study to the previous cases.

Fourth case: $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6| < |\alpha_7| < |\alpha_8|$

To discuss the remaining case, we will use the degree of nilpotency of α_1.

Lemma 3.9. *The only possible cases are when $\alpha_1^i \neq 0$ for $i \leq 4$.*

Proof. If $\alpha_1^i \neq 0$ for $i \geq 5$, then we put $\alpha_1 = [x_1]$ with $x_1 \in V^{\text{even}}$ such that $\mathrm{d}x_1 = 0$. Therefore, $\{1, [x_1], [x_1^2], [x_1^3], [x_1^4], [x_1^5], \ldots\} \subset H^{\text{even}}(X;\mathbb{Q})$ and then $\dim H^{\text{even}}(X;\mathbb{Q}) \geq 6$, but this contradicts the fact that $\dim H^{\text{even}}(X;\mathbb{Q}) = 5$. ☐

- If $\boldsymbol{\alpha_1^4 \neq 0}$, but $\boldsymbol{\alpha_1^5 = 0}$:

PROPOSITION 3.10. *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6| < |\alpha_7| < |\alpha_8|$ *and if* $\alpha_1^4 \neq 0$ *but* $\alpha_1^5 = 0$, *then* X *has the rational homotopy type of* $X \sim_{\mathbb{Q}} \mathbb{S}^{2n+1} \times \mathbb{S}^m_{(4)}$.

Proof. By the hypothesis, we have $H^{\text{even}}(X;\mathbb{Q}) = \{1, [x_1], [x_1^2], [x_1^3], [x_1^4]\}$ and then, necessarily, there is a generator y of odd degree such that $\mathrm{d}y = 0$. Therefore, $H^{\text{odd}}(X;\mathbb{Q}) = \{[y], [yx_1], [yx_1^2], [yx_1^3], [yx_1^4]\}$, thus $\mathrm{fd}(X) = 4|x_1| + |y|$. More precisely, $(\Lambda V, \mathrm{d}) = (\Lambda(x, y_1, y), \mathrm{d}) = (\Lambda(x_1, y) \otimes (\Lambda y, 0))$ with $\mathrm{d}x_1 = \mathrm{d}y = 0$ and $\mathrm{d}y_1 = x_1^5$. Consequently, $X \sim_{\mathbb{Q}} \mathbb{S}^{2n+1} \times \mathbb{S}^m_{(4)}$. ☐

- If $\boldsymbol{\alpha_1^3 \neq 0}$, but $\boldsymbol{\alpha_1^4 = 0}$:

In general, the minimal model of X is given as $(\Lambda V, \mathrm{d}) = (\Lambda(x_1, x_2, \ldots), \mathrm{d})$ with $|x_1| < |x_2| < \ldots$, that means $|x_1| = \min\{|x_i| / \mathrm{d}x_i = 0\}$; then we get two cases:

1) If $|x_2|$ is even and $\mathrm{d}x_2 = 0$, then $\{1, [x_1], [x_2], [x_1^2], [x_1^3], [x_1x_2], [x_2^2], \ldots\} \subset H^{\text{even}}(X;\mathbb{Q})$. Since $\dim H^{\text{even}}(X;\mathbb{Q}) = 5$, there exist two odd generators y_1 and y_2 of V such that

$$\begin{cases} \mathrm{d}y_1 = x_1x_2 + \lambda_1 x_1^n \\ \mathrm{d}y_2 = x_2^2 + \lambda_2 x_1^n \end{cases}$$

But we have $|x_1| < |x_2|$ and then $n = 3$, hence

$$\begin{cases} \mathrm{d}y_1 = x_1x_2 + \lambda_1 x_1^3 \\ \mathrm{d}y_2 = x_2^2 + \lambda_2 x_1^3 \end{cases} \Longrightarrow \begin{cases} \mathrm{d}y_1 = x_1\left(x_2 + \lambda_1 x_1^2\right) \\ \mathrm{d}y_2 = x_2^2 + \lambda_2 x_1^n \end{cases}$$

Put $x_1' = x_2 + \lambda_1 x_1^2$; then $\mathrm{d}y_1 = x_1' x_1$ and, by a simple computation, we get $\mathrm{d}y_2 = x_1'^2 + \lambda_2 x_1^3$.

LEMMA 3.11. *We can assume without loosing generality that there are two odd generators* y_1 *and* y_2 *such that* $\begin{cases} dy_1 = x_1x_2 \\ dy_2 = x_2^2 + x_1^3 \end{cases}$.

Proof. According to the above, we found: $\begin{cases} \mathrm{d}y_1 = x_1x_2 \\ \mathrm{d}y_2 = x_2^2 + \lambda_2 x_1^3 \end{cases}$, we multiply $\mathrm{d}y_2$ by λ_2^2 and we replace x_1 and x_2 by $\lambda_2 x_1$ and $\lambda_2 x_2$, respectively. Then we get

$$\begin{cases} \mathrm{d}y_1 = x_1x_2 \\ \mathrm{d}y_2 = x_2^2 + x_1^3 \end{cases}.$$

☐

Since $\mathrm{fd}(X)$ is odd, there is a generator of odd degree such that $\mathrm{d}y = 0$, therefore, $\mu = [x_1^3 y] = [x_2^2 y]$, and thus $\mathrm{fd}(X) = |y| + 3|x_1| = 2|x_2| + |y|$. Hence, the minimal model of X is given by: $(\Lambda V, \mathrm{d}) = (\Lambda(x_1, x_2, y_1, y_2), \mathrm{d}) \otimes (\Lambda y, 0)$ with $\mathrm{d}x_1 = \mathrm{d}x_2 = 0$, $\mathrm{d}y_1 = x_1 x_2$, $\mathrm{d}y_2 = x_2^2 + x_1^3$ and $\mathrm{d}y = 0$. So $X \sim_{\mathbb{Q}} \mathbb{S}^{2p+1} \times Y$ with $\dim H^{\mathrm{even}}(Y; \mathbb{Q}) = 5$, in particular, from [10], $Y \sim_{\mathbb{Q}} \mathbb{S}^n_{(3)} \# \mathbb{S}^m_{(2)}$. Finally,

$$X \sim_{\mathbb{Q}} \mathbb{S}^{2p+1} \times \mathbb{S}^n_{(3)} \# \mathbb{S}^m_{(2)}$$

2) If the second generator that comes right after x_1 is odd, then we can put $\mathrm{d}y_1 = x_1^4$ and $\mathrm{d}y_2 = x_1^i$ for $i \geq 4$ with $y_1, y_2 \in V^{\mathrm{odd}}$.

Lemma 3.12. *There exists an odd generator y of V such that $dy = 0$.*

Proof. We have $\mathrm{d}y_2 = \lambda_1 x_1^i$ but $\mathrm{d}x_1^i = \mathrm{d}\omega$ for $i \geq 4$, then $\mathrm{d}y_2 = \mathrm{d}(\lambda_1 \omega)$, so $\mathrm{d}(y_2 - \lambda_1 \omega) = 0$. We put $y = y_2 - \lambda_1 \omega$ and we obtain the result. □

Automatically $[x_1^3 y] \in H^{\mathrm{odd}}(X; \mathbb{Q})$, so we have two possibilities: the first one is when $\mathrm{fd}(X) = 3|x_1| + |y|$. If $V^{\mathrm{even}} \neq \mathbb{Q}\{x_1\}$, so $\exists\ x_2 \in V^{\mathrm{even}} / |x_1| < |y| < |x_2|$ with $\mathrm{d}x_2 = \lambda x_1^i y_1$, and $\exists\ y_2 \in V^{\mathrm{odd}} / |y_2| > 2|x_2| - 1$. But

$$\sum\nolimits_{a_i \in V^{\mathrm{odd}}} |a_i| \geq |y_1| + |y_2| + |y| > 2\mathrm{fd}(X) - 1$$

If $V^{\mathrm{even}} = \mathbb{Q}\{x_1\}$, then $\exists\ y, y_1, y_2, \ldots \in V^{\mathrm{odd}} / \mathrm{d}y = 0$, $\mathrm{d}y_1 = x_1^4$, $\mathrm{d}y_2 = x_1^i + \lambda y y_1$ for $i \geq 4$. Since $\mathrm{d}y_2^2 = 0$, we get $\lambda = 0$, so $\mathrm{d}y_2 = x_1^i$, but $\mathrm{d}\left(x_1^{i-4} y_1 - y_2\right) = 0$, thus we put $y_2' = x_1^{i-4} y_1 - y_2$ so $\mathrm{d}y_2' = 0$. Therefore, from the Poincaré duality, we obtain
$\mathbb{Q}\{[y], [x_1 y], [x_1^2 y], [x_1^3 y], [y_2'], [x_1 y_2'], [x_1^2 y_2'], [x_1^3 y_2']\} \in H^{\mathrm{odd}}(X; \mathbb{Q})$. Then, by the formula of dimension, we must get a generator x_2 of even degree such that $\mathrm{d}x_2 = x_1 y_2' + \lambda x_1^i y$, but this contradicts the assumption $(V^{\mathrm{even}} = \mathbb{Q}\{x_1\})$. Since the two possibilities are impossible, $\mathrm{fd}(X) > 3|x_1| + |y|$, hence, $\exists\ x_2 \in V^{\mathrm{even}} / |x_1| < |y| < |x_2|$ and $\mathrm{d}x_2 = \lambda x_1^i y$ (for $i > 3$ or $\lambda = 0$), but $\mathrm{d}\left(x_2 x_1^j y\right) = 0$ for $j \leq 3$. Then, from the Poincaré duality again, $\mu = [x_1^3 x_2 y]$, consequently $[x_1^i x_2] \neq 0$ for $1 \leq i \leq 3$, which gives $\dim H^{\mathrm{even}}(X; \mathbb{Q}) > 5$.

• If $\boldsymbol{\alpha_1^2 \neq 0}$, but $\boldsymbol{\alpha_1^3 = 0}$:

We put $\alpha_1 = [x_1]$ with $x_1 \in V^{\mathrm{even}} / \mathrm{d}x_1 = 0$ and $|x_1| = \min\{|x_i| / \mathrm{d}x_i = 0\}$.

Lemma 3.13. *$\exists y_1 \in V^{odd} / dy_1 = x_1^3$.*

Proof. We have $x_1^2 \notin \mathrm{d}(V)$ and $x_1^3 \in \mathrm{d}(\Lambda V)$, if there is $\omega \in \Lambda^{\leq 2} V / \mathrm{d}\omega = x_1^3$. Then $\omega = z + x_1 z_1 + \ldots$ with $z, z_1 \in V^{\mathrm{odd}}$, thus $\mathrm{d}\omega = \mathrm{d}z + x_1 \mathrm{d}z_1 = x_1^3$; then, by degree reasons, we get $z_1 = 0$ and $\mathrm{d}\omega = P(x_i)$, so ω is a generator. □

Therefore, we have three possibilities:

1) $V^{\mathrm{even}} = \mathbb{Q}\{x_1\}$, so the minimal model of X is given by $(\Lambda V, \mathrm{d}) = (\Lambda(x_1, y_1, y_2 \ldots), \mathrm{d})$ with $|x_1| < |y_1| < |y_2| \ldots$ and $\mathrm{d}y_1 = x_1^3$, $\mathrm{d}y_2 = \beta x_1^j$ for $j \geq 3$. Thus, $\exists\ y_3 \in V^{\mathrm{odd}} \mathrm{d}y_3 = 0$ (because $\mathrm{d}(\beta x_1^{j-3} y_1 - y_2) = 0$, so we put

$y_3 = \beta x_1^{j-3} y_1 - y_2$). In addition, if there is a generator y_4 of odd degree such that $\mathrm{d}y_4 = \gamma x_1^i + \lambda x_1^j y_1 y_3$ for $i > j$, then $\lambda = 0$ (because $\mathrm{d}^2 y_4 = 0$). By the same justification and by a variable change, we obtain a generator y_5 of odd degree such that $\mathrm{d}y_5 = 0$, and thus $\mathbb{Q}\{[y_3], [x_1 y_3], [x_1^2 y_3], [y_5], [x_1 y_5], [x_1^2 y_5]\} \in H^{\text{odd}}(X;\mathbb{Q})$, but this contradicts the fact that $\dim H^{\text{odd}}(X;\mathbb{Q}) = 5$.

2) If $\{x_1, x_2\} \subset V^{\text{even}}$ with $\mathrm{d}x_1 = \mathrm{d}x_2 = 0$ and $|x_1| < |x_2|$, then $\mathbb{Q}\{1, [x_1], [x_1^2], [x_2], [x_1 x_2], [x_1^2 x_2], [x_2^2], \dots\} \in H^{\text{even}}(X;\mathbb{Q})$, thus many possibilities are left to be studied. If $[x_1^2 x_2] = 0$ and $[x_2^2] = 0$, then $\exists\, y_2 \in V^{\text{odd}} / \mathrm{d}y_2 = x_2^2$.

Lemma 3.14. *$\exists y_3 \in V^{odd} / dy_3 = x_1^2 x_2$.*

Proof. Let $\omega \in \Lambda^{\leq 2} V / \mathrm{d}\omega = x_1^2 x_2$ and $\omega = z + x_1 z_1 + x_2 z_2 + \dots$ with $z, z_1, z_2 \in V^{\text{odd}}$, thus $\mathrm{d}\omega = \mathrm{d}z + x_1 \mathrm{d}z_1 + x_2 \mathrm{d}z_2 = x_1^2 x_2$, then, by degree reasons, we get $z_1 = z_2 = 0$ and $\mathrm{d}\omega = P(x_i)$, so ω is a generator. □

Consequently, there exist three generators of odd degree such that $\mathrm{d}y_1 = x_1^3$, $\mathrm{d}y_2 = x_2^2$ and $\mathrm{d}y_3 = x_1^2 x_2$. Thus, we have

$$(\Lambda V, \mathrm{d}) \to \left(\Lambda(x_2, y_2), \overline{\mathrm{d}}\right) \otimes \left(\Lambda(x_1, y_1, \dots), \overline{\mathrm{d}}\right)$$

with $\dim H^*\left(\Lambda(x_2, y_2), \overline{\mathrm{d}}\right) = 2$ and $\dim H^*\left(\Lambda(x_1, y_1, \dots), \overline{\mathrm{d}}\right) = 5$, but, according to the classification in [5], there is no space Y with a model of this form $\left(\Lambda(x_1, y_1, \dots), \overline{\mathrm{d}}\right)$, given $\mathrm{d}x_1 = 0$ and $\mathrm{d}y_1 = x_1^3$ such that $\dim H^*(Y;\mathbb{Q}) = 5$. But, if $[x_1 x_2] = 0$ and $[x_2^3] = 0$, then we can show similarly that there exist two generators y_2 and y_3 of odd degree such that $\mathrm{d}y_2 = x_1 x_2$ and $\mathrm{d}y_3 = x_2^3$. According to the Poincaré duality, we have $B = \{1, [x_1], [x_1^2], [x_2], [x_2^2], [\omega_1], [x_2 \omega_1], [\omega_2], [x_1 \omega_2], [x_2^2 \omega_1]\}$ is a basis of $H^*(X;\mathbb{Q})$. It is easy to show that ω_1 and ω_2 are not generators of V; also, we have $\mathrm{d}(x_2 y_1 - x_1^2 y_2) = 0$ and $\mathrm{d}(x_1 y_3 - x_2^2 y_2) = 0$, thus we can consider $\omega_1 = x_2 y_1 - x_1^2 y_2$ and $\omega_2 = x_1 y_3 - x_2^2 y_2$. Let $\omega \in Z^{\text{even}}$ then $\omega = P_1 + P_2 y_1 y_2 + P_3 y_1 y_3 + P_4 y_2 y_3$, where $P_i \in \mathbb{Q}[x_1, x_2]$ for $1 \leq i \leq 4$. An easy computation shows that $\mathrm{d}\omega = (-P_2 \mathrm{d}y_2 - P_3 \mathrm{d}y_3) y_1 + (P_2 \mathrm{d}y_1 - P_4 \mathrm{d}y_3) y_2 + (P_3 \mathrm{d}y_1 + P_4 \mathrm{d}y_2) = 0$, leading us to the system

$$\begin{cases} P_2 x_1 x_2 + P_3 x_2^3 = 0 \\ P_2 x_1^3 - P_4 x_2^3 = 0 \\ P_3 x_1^3 + P_4 x_1 x_2 = 0 \end{cases}$$

As the polynomials x_1^3 and x_2^3 are relatively prime, there exists $Z \in \mathbb{Q}[x_1, x_2]$ such that $P_2 = -x_2^3 Z$, $P_3 = x_1 x_2 Z$ and $P_4 = -x_1^3 Z$ so $\omega = P_1 - \mathrm{d}(Z y_1 y_2 y_3)$. Moreover, let $\omega \in Z^{\text{odd}}$; then $\omega = P_1 y_1 + P_2 y_2 + P_3 y_3 + P y_1 y_2 y_3$, where $P_i \in \mathbb{Q}[x_1, x_2]$, thus $\mathrm{d}\omega = P_1 x_1^3 + P_2 x_1 x_2 + P_3 x_2^3 + P\mathrm{d}(y_1 y_2 y_3) = 0$, which implies $P = 0$ and $P_1 x_1^3 + P_2 x_1 x_2 + P_3 x_2^3 = 0$. So there exist $Z_1, Z_2 \in \mathbb{Q}[x_1, x_2]$ such that $P_1 = x_2 Z_1$, $P_2 = -x_1^2 Z_1 - x_2^2 Z_2$ and $P_3 = x_1 Z_2$. Then $\omega =$

$Z_1\left(x_2y_1-x_1^2y_2\right)+Z_2\left(x_1y_3-x_2^2y_2\right)$. Thus, $Z^{\text{odd}}=<\omega_1,\omega_2>$, consequently,

$$\begin{aligned}H^{\text{odd}}\left(X;\mathbb{Q}\right)&=\mathbb{Q}\left[x_1,x_2\right]<\omega_1,\omega_2>\diagup_{B^{\text{even}}}\\&=\mathbb{Q}\left\{\left[\omega_1\right],\left[x_2\omega_1\right],\left[\omega_2\right],\left[x_1\omega_2\right],\left[x_2^2\omega_1\right],\left[x_1^2\omega_2\right]\right\}\end{aligned}$$

Notice that $[x_2^2\omega_1]=[x_1^2\omega_2]$. Therefore, we get this quasi-isomorphism: $(\Lambda V,\mathrm{d})\rightarrow(\Lambda W,\mathrm{d}')$ with $(\Lambda W,\mathrm{d}')=\left(\Lambda\left(x_1,x_2,y_1,y_2,y_3\right),\mathrm{d}'\right)$ and $\mathrm{d}'x_1=\mathrm{d}'x_2=0$, $\mathrm{d}'y_1=x_1^3$, $\mathrm{d}'y_2=x_1x_2$, $\mathrm{d}'y_3=x_2^3$. Finally, $X\sim_{\mathbb{Q}}E$, where E is the total space of this fibration

$$\mathbb{S}^p\rightarrow E\rightarrow\mathbb{S}^n\times\mathbb{S}^m.$$

3) If $\{x_1,x_2\}\subset V^{\text{even}}/\mathrm{d}x_1=0$ and $\mathrm{d}x_2\neq0$, then $\exists\ y\in V^{\text{odd}}/\mathrm{d}y=0$ with $|x_1|<|y|<|x_2|$ and $\mathrm{d}x_2=x_1^iy$ for $i\geq3$. Thus,

$$\mathbb{Q}\left\{[y],[x_1y],\left[x_1^2y\right],[x_2y],[x_1x_2y],\left[x_1^2x_2y\right]\right\}\in H^{\text{odd}}\left(X;\mathbb{Q}\right),$$

which gives $\dim H^{\text{odd}}\left(X;\mathbb{Q}\right)>5$, so it is impossible.

- Finally, if $\boldsymbol{\alpha_1^2=0}$:

PROPOSITION 3.15. *If* $|\alpha_1|<|\alpha_2|<|\alpha_3|<|\alpha_4|<|\alpha_5|<|\alpha_6|<|\alpha_7|<|\alpha_8|$ *and if* $\alpha_1^2=0$, *then* X *has the rational homotopy type of* $\mathbb{S}^{2n+1}\times\mathbb{S}^{2k}_{(4)}$ *or that of* $\mathbb{S}^{2p+1}\times\mathbb{S}^n_{(3)}\#\mathbb{S}^m_{(2)}$.

Proof. Since $\alpha_1^2=0$, $H^*\left(X;\mathbb{Q}\right)$ is generated by α_1 and α_2. We establish this case according to the degree of nilpotency of α_2. If $\alpha_2^i\neq0$ for $i\geq5$, then we get a contradiction, because we obtain $\dim H^{\text{even}}\left(X;\mathbb{Q}\right)>5$. If $\alpha_2^4\neq0$ and $\alpha_2^5=0$, then in this case necessarily $|\alpha_1|$ is odd, so

$$X\sim_{\mathbb{Q}}\mathbb{S}^{2n+1}\times\mathbb{S}^m_{(4)}$$

Moreover, if $\alpha_2^3\neq0$ and $\alpha_2^4=0$, we suppose that $|\alpha_1|$ is odd and, if $\alpha_1\alpha_2=0$, then we put $\alpha_1=[y_1]$ and $\alpha_2=[x_1]$, where $x_1\in V^{\text{even}}$ and $y_1\in V^{\text{odd}}$. Then there exists a generator x_3 of even degree such that $\mathrm{d}x_3=y_1x_1$, hence $\mathrm{d}(x_3y_1)=0$, then, accordingis even to the duality of Poincaré, we have $[x_3y_1]=0$. Thus, this assumption is false, because it leads us to $\dim V=\infty$. Thus, automatically $[y_1x_1]\neq0$, so $X\sim_{\mathbb{Q}}\mathbb{S}^{2p+1}\times Y$ with $\dim H^*\left(Y;\mathbb{Q}\right)=5$ and $M\left(Y\right)=\left(\Lambda\left(x_1,x_2,y_2,y_3,\ldots\right),\mathrm{d}'\right)$. Then, from [5], $Y\sim_{\mathbb{Q}}\mathbb{S}^n_{(3)}\#\mathbb{S}^m_{(2)}$. Finally,

$$X\sim_{\mathbb{Q}}\mathbb{S}^{2p+1}\times\mathbb{S}^n_{(3)}\#\mathbb{S}^m_{(2)}$$

Also, if $|\alpha_1|$ is even, we find easily $\sum_{|x_i|\text{ odd}}|x_i|>\mathrm{fd}(X)$. If $\alpha_2^2\neq0$ and $\alpha_2^3=0$, by the argument of last case, we obtain the same result. Finally, the case where $\alpha_2^2=0$ is impossible. □

Now, the proofs of Corollary 1.3 and 1.4 are immediate consequences of Theorems 1.1 and 1.2.

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Université Hassan II
Département de Mathématiques et d'Informatique
Casablanca, Maroc
E-mail: saloua.chouingou@gmail.com
E-mail: rhilali@hotmail.fr