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A NOTE ON DARK SOLITONS IN NONLINEAR COMPLEX GINZBURG-LANDAU EQUATIONS

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Abstract. We analyze the existence of dark solitons in nonlinear complex Ginzburg-Landau equations. We prove existence results concerned with the initial value problem for these equations in Zhidkov spaces using a new approach with splitting methods.

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1. INTRODUCTION

We consider the one-dimensional autonomous system

$$(1)\qquad \begin{cases} \partial_t u = (a+\mathrm{i}\alpha)\partial_{xx}u + \gamma u - (b+\mathrm{i}\beta)F(u), \\ u(0) = u_0 \end{cases}$$

where $u(x,t)$ is a complex valued function for $x \in \mathbb{R}$, $t > 0$, $a,\ b,\ \alpha,\ \beta,\ \gamma$ are positive real parameters and F is a continuous map. The linear term represented by $(a+i\alpha)\partial_{xx}u$ characterizes the complex Ginzburg-Landau equations (CGL). For $\alpha = 0$, (1) reduces to a non-linear reaction-diffusion equation and, for $a = 0$, to a non-linear Schrödinger equation or Gross-Pitaevskii equation. A large amount of work has been done to prove well-posedness of (1) with different non-linearities (see for instance, [1, 9, 10]).

In this paper, we analyze well-posedness for the nonlinear complex Ginzburg-Landau equation in Zhidkov spaces by applying splitting methods for abstract semlinear evolution equations [3, 5]. These techniques were used to achieve well-posedness results for the fractional reaction-diffusion equation [2]. Zhidkov Spaces, introduced by P. Zhidkov in [12], consist of functions defined on $\mathbb{R}$, bounded and uniformly continuous, with derivatives up to k order in L^2. These spaces are applied in nonlinear optics to model dark solitons - these are solutions of the form $u(x,t) = u_v(x - vt)$. For instance, in [6], dark soliton solutions are described for a complex Ginzburg-Landau equation. A typical

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example of a function in Zhidkov spaces is described in [8, 11] :

$$u_v(x) = \sqrt{1-\frac{v^2}{2}}\tanh\left(\sqrt{1-\frac{v^2}{2}}\frac{x}{\sqrt{2}}\right) + \mathrm{i}\frac{v}{\sqrt{2}}.$$

Our aim is to prove existence of solutions in Zhidkov spaces with $k = 1$, where the nonlinearity is $F(u) = |u|^2u$. Such nonlinearities appear not only in complex Ginzburg-Landau equations, but also in other equations, such as the nonlinear Schroedinger and Gross-Pitaevski equations. We use a new approach, based on a Lie-Trotter method developed recently for numerical purposes [5].

The paper is organized as follows: In Section 2, we set notations and preliminary results. In Section 3, we analyze the nonlinear problem. Finally, in Section 4, using splitting methods, we combine results from Sections 2 and 3.

2. NOTATIONS AND PRELIMINARIES

We introduce some definitions and preliminary results.

Definition 2.1. We define $C_{\mathrm{u}}(\mathbb{R})$ as the set of uniformly continuous and bounded functions on $\mathbb{R}$.

Definition 2.2. We denote the Zhidkov spaces as, for $k > d/2$,

$$X^k(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) \cap C_u(\mathbb{R}^d) : \ \partial_j \in L^2(\mathbb{R}^d), 1 \le |j| \le k\}$$

equipped with the norm:

$$\|u\|_{X^k} = \|u\|_{L^\infty} + \sum_{1\le|j|\le k} \|\partial_j u\|_{L^2}. \tag{2}$$

Remark 2.3. Zhidkov spaces are closed for the norm defined in (2), see [8].

The following definitions and proofs, given here for $x \in \mathbb{R}$, can be extended to $x \in \mathbb{R}^d$ (see [7]).

Definition 2.4. We denote by $U(t)$ the one parameter semigroup that solves the underlying linear equation

$$\partial_t u = (a + \mathrm{i}\alpha)\partial_{xx}u + \gamma u. \tag{3}$$

The operator can be represented by the convolution in x

$$U(t) = (4\pi t(a+\mathrm{i}\alpha))^{-1/2}e^{(-x^2/[4t(a+\mathrm{i}\alpha)])+\gamma t} * u_0 = G_t(x) * u_0$$

and the kernel G_t satisfies

$$|G_t(x)| = (4\pi t(\alpha^2+\beta^2))^{-1/2}e^{(-x^2/[4t(\alpha^2+\beta^2)])+\gamma t}.$$

Clearly, $G_t(x) \in L^1(\mathbb{R})$.

Proposition 2.5. *The one-parameter family* $\{U(t)\}_{t\ge 0}$ *of operators defined as* $U(t)u_0 = G_t * u_0$ *is a strongly continuous semigroup on* $C_u(\mathbb{R})$.

Proof. The proof is similar to [2, Proposition 2.2]. □

LEMMA 2.6. *If $u_0 \in X^1(\mathbb{R})$, then $U(t)u_0 \in X^1(\mathbb{R})$ for $t > 0$.*

Proof. As $u_0 \in L^\infty(\mathbb{R})$ and $G_t(x) \in L^1(\mathbb{R})$, using Young's inequality, we have $\|G_t * u_0\|_{L^\infty} \le \|G_t\|_{L^1} \|u_0\|_{L^\infty}$. On the other hand, we obtain

$$\|\partial_x(G_t * u_0)\| = \|G_t * \partial_x u_0\|_{L^2} \le \|G_t\|_{L^1} * \|\partial_x u_0\|_{L^2}$$

As $G_t \in L^1(\mathbb{R})$ and $\partial_x u_0 \in L^2(\mathbb{R})$ we have the result. □

REMARK 2.7. Similarly, if $x \in \mathbb{R}^d$ and we have k derivatives of $U(t)u_0$, the same procedure proves that $U(t)u_0 \in X^k(\mathbb{R}^d)$.

Next, we consider integral solutions of the problem (1). We say that $u \in C([0,T], C_{\mathrm{u}}(\mathbb{R}))$ is a mild solution of (1) if and only if u verifies

$$u(t) = U(t)u_0 + \int_0^t U(t-t')F(u(t'))\mathrm{d}t'. \tag{4}$$

If F is a locally Lipschitz map, for any $z_0 \in C_{\mathrm{u}}(\mathbb{R})$ there exists a unique solution of the equation

$$\begin{cases} \partial_t z = F(z), \\ z(0) = z_0, \end{cases} \tag{5}$$

defined in the interval $[0, T^*(z_0))$. Moreover, there exists a non-increasing function $\bar{T} : [0,\infty) \to [0,\infty)$ such that $T^*(z_0) \ge \bar{T}(|z_0|)$. The solution of (5) is a solution of the integral equation

$$z(t) = z_0 + \int_0^t F(z(t'))\mathrm{d}t'. \tag{6}$$

Also, one of the following alternatives holds:

- $T^*(z_0) = \infty$;
- $T^*(z_0) < \infty$ and $|z(t)| \to \infty$ when $t \uparrow T^*(z_0)$.

We will denote by $\mathsf{N}(t,.) : C_{\mathrm{u}}(\mathbb{R}) \to C_{\mathrm{u}}(\mathbb{R})$ the flow generated by the ordinary equation, i.e., for any $x \in \mathbb{R}$, $\mathsf{N}(t,u_0)(x)$ is the solution of the problem (5) with initial data $z_0 = u_0(x)$. Therefore, if $u(t) = \mathsf{N}(t,u_0)$, then

$$u(x,t) = u_0(x) + \int_0^t B(u(x,t'))\mathrm{d}t'.$$

We recall well-known local existence results for evolution equations.

THEOREM 2.8. *There exists a function $T^* : C_{\mathrm{u}}(\mathbb{R}) \to \mathbb{R}_+$ such that, for $u_0 \in C_{\mathrm{u}}(\mathbb{R})$, there is a unique $u \in C([0,T^*(u_0)), C_{\mathrm{u}}(\mathbb{R}))$ mild solution of (1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$;
- $T^*(u_0) < \infty$ *and* $\lim_{t\uparrow T^*(u_0)} |u(t)| = \infty$.

Proof. See [4, Theorem 4.3.4]. □

PROPOSITION 2.9. *Under the conditions of the above theorem, we have the following statements:*

(1) $T^* : C_{\mathrm{u}}(\mathbb{R}) \to \mathbb{R}_+$ *is lower semi-continuous;*
(2) *If* $u_{0,n} \to u_0$ *in* $C_{\mathrm{u}}(\mathbb{R})$ *and* $0 < T < T^*(u_0)$, *then* $u_n \to u$ *in the Banach space* $C([0,T], C_{\mathrm{u}}(\mathbb{R}))$.

Proof. See [4, Proposition 4.3.7]. □

3. THE NON-LINEAR EQUATION

In this section, we study the solution for the non-linear problem (5), that is the equation

$$\begin{cases} \partial_t z = -(b + \mathrm{i}\beta)|z|^2 z, \\ z(0) = z_0. \end{cases} \tag{7}$$

LEMMA 3.1. *If* $u_0(x) = z_0 \in X^1(\mathbb{R})$, *then the solution of the problem* (7) *satisfies* $z(t) \in X^1(\mathbb{R})$ *for* $t \in (0, T^*(z_0))$.

Proof. We first remark that, if $|u|^2 \in L^\infty(\mathbb{R})$, then $u \in L^\infty$. Indeed, using (7), multiplying by $\bar{z}$ and applying real part on both sides, we have

$$\mathrm{Re}\,(\partial_t z \bar{z}) = \frac{\mathrm{Re}\,(\partial_t z \bar{z} + z \partial_t \bar{z})}{2} = \frac{\partial_t (z\bar{z})}{2} = \frac{\partial_t |z|^2}{2} = -\mathrm{Re}\,(b + \mathrm{i}\beta)\,|z|^4. \tag{8}$$

This is an ODE for $\rho(t) = |z(t)|^2$, we have $\begin{cases} \partial_t \rho = -2b\rho^2 \\ \rho(0) = \rho_0 \end{cases}$ with solution $\rho(t) = \rho_0/(2b\rho_0 t + 1)$.

Then $|u|^2 \in L^\infty(\mathbb{R})$ and therefore $u \in L^\infty(\mathbb{R})$. On the other hand, suppose that $\partial_x u_0 \in L^2(\mathbb{R})$. Then, taking the spatial derivative, $\partial_{tx} z = -(b + i\beta)(|z|^2 \partial_x z + 2\mathrm{Re}\,(\bar{z}\partial_x z) z)$, multiplying by $\partial_x \bar{z}$ and applying real part on both sides, we have

$$\mathrm{Re}\,((\partial_{tx} z)\partial_x \bar{z}) = -\mathrm{Re}\,(b + \mathrm{i}\beta)\left(|z|^2 |\partial_x z|^2 + 2\mathrm{Re}\,(\bar{z}\partial_x z)\, z \partial_x \bar{z}\right).$$

In the same way as in (8), we have $\partial_{tx}|z|^2 \le C_b |z|^2 \partial_x |z|^2$. Since $|u|^2 \in L^\infty$, $\partial_{tx}|z|^2 \le C\partial_x |z|^2$. For the integral equation, using Grönwall's inequality, we obtain

$$\left\|\partial_x |z|^2\right\|_{L^2} \le \left\|\partial_x |z_0|^2\right\|_{L^2} + C\int_0^t \partial_x |z|^2 \mathrm{d}t' \le e^{Ct}\left\|\partial_x |z_0|^2\right\|_{L^2}.$$

Then $\partial_x |z|^2 \in L^2(\mathbb{R})$ and $z(t) \in X^1(\mathbb{R})$. □

4. CGL SOLUTION

In this section, we apply Lemma 2.6 from Section 2 related to the linear problem (3) and Lemma 3.1 from Section 3 related to the nonlinear problem (7). In order to obtain well-posedness for the solution $u(t)$ of equation (1), we recall convergence results from [5], concerning the splitting method.

THEOREM 4.1. *If* $u_0 \in X^1(\mathbb{R})$ *and* u *is the solution of* (1), *then* $u(t) \in X^1(\mathbb{R})$ *for all* $t \in (0, T^*(u_0))$.

Proof. For $t \in [0, \min\{T^*(u_0)\})$, let $n \in \mathbb{N}$, $h = t/n$ and the sequences $\{W_{h,k}\}_{0 \le k \le n}, \{V_{h,k}\}_{1 \le k \le n}$ given by $W_{h,0} = u_0$,

$$V_{h,k+1} = U(h) W_{h,k}, \tag{9a}$$

$$W_{h,k+1} = \mathsf{N}(h, V_{h,k+1}), \quad k = 0, \dots, n-1. \tag{9b}$$

We claim that $W_{h,k} \in X^1(\mathbb{R})$ for $k = 0, \dots, n$. Clearly, the assertion is true for $k = 0$. If $W_{h,k-1} \in X^1(\mathbb{R})$, from Lemma 3.1, we have $\mathsf{N}(h, V_{h,k-1}) \in X^1(\mathbb{R})$. Using Lemma 2.6, we can see that $V_{h,k} = W(h)(\mathsf{N}(h, V_{h,k-1})) \in X^1(\mathbb{R})$. We now recall [5, Theorem 3.1] that assures us that $W_{h,n} \to u(t)$ when $n \to \infty$. As $X^1(\mathbb{R})$ is closed, we obtain the result. □

REFERENCES

[1] I.S. Aranson and L. Kramer, *The world of the complex Ginzburg-Landau equation*, Rev. Modern Phys., **74** (2002), 99–143.

[2] A.T. Besteiro and D.F. Rial, *Global existence for vector valued fractional reaction-diffusion equations*, preprint (2018), arXiv:1805.09985.

[3] J.P. Borgna, M. De Leo, D. Rial and C. Sanchez de la Vega, *General splitting methods for abstract semilinear evolution equations*, Commun. Math. Sci., **13** (2015), 83–101.

[4] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, in *Oxford Lecture Series Mathematics and Applications*, Vol. 13, Clarendon Press, 1999.

[5] M. De Leo, D. Rial and C. Sanchez de la Vega, *High-order time-splitting methods for irreversible equations*, IMA J. Numer. Anal., **36** (2015), 1842–1866.

[6] N. Efremidis, K. Hizanidis, H.E. Nistazakis, D.J. Frantzeskakis and B.A. Malomed, *Stabilization of dark solitons in the cubic Ginzburg-Landau equation*, Phys. Rev. E, **63** (2000), 7410–7414.

[7] K.J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York, 1999.

[8] C. Gallo, *Schrödinger group on Zhidkov spaces*, Adv. Differential Equations, **9** (2004), 509–538.

[9] J. Ginibre and G. Velo, *The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. I. Compactness methods*, Phys. D, **95** (1996), 191–228.

[10] J. Ginibre and G. Velo, *The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. II. Contraction methods*, Comm. Math. Phys., **187** (1997), 45–79.

[11] Y.S. Kivshar and B. Luther-Davies, *Dark optical solitons: physics and applications*, Phys. Rep., **298** (1998), 81–197.

[12] P. Zhidkov, *The Cauchy problem for a nonlinear Schrödinger equation* (in Russian), Joint Inst. for Nuclear Research, Dubna (USSR), Lab. of Computing Techniques and Automation, **P5-87-373** (1987), 1–18.

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