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THE BOUNDEDNESS OF A CLASS OF SEMICLASSICAL FOURIER INTEGRAL OPERATORS ON BESOV SPACES

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Abstract. The aim of this paper is to discuss the Besov spaces bounds for semiclassical Fourier integral operator. We give the conditions that the symbol and the phase function must satisfy for this operator to be bounded.

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1. INTRODUCTION

For a function $u \in C_0^\infty(\mathbb{R}^n)$ a Fourier integral operator (FIO) is given by:

$$(Fu)(x) = \frac{1}{(2\pi)^n} \int \mathrm{e}^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) \mathrm{d}y \mathrm{d}\xi, \tag{1}$$

where a is the *symbol* and ϕ is the *phase function.* These operators appear naturally in the expression of the solutions of the hyperbolic partial differential, see [9, 10], and in the expression of the C^∞ solution of the associated Cauchy's problem, see [12], and appear also in the quantum mechanics, see [11, 17].

A semiclassical Fourier integral operator (with semiclassical parameter h) is defined by:

$$(Fu)(x) = \frac{1}{(2\pi h)^n} \int \mathrm{e}^{\frac{i}{h}\phi(x,y,\xi)} a(x,\xi;h) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{2}$$

If $\phi(x, y) = (x - y)\xi$, then we obtain what we call the pseudodifferenatial operator (PDO):

$$(Pu)(x) = \frac{1}{(2\pi)^n} \int \mathrm{e}^{i(x-y)\xi} a(x,\xi) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{3}$$

An interesting question is: under which conditions for a and ϕ are these operators bounded? Since 1970, many authors made efforts in the study of the boundedness of these operators on many functional spaces (such as L^p, Hölder and Besov spaces). In [1, 15, 2] the authors have studied the boundedness of FIO of the form (1) on L^p. Qingjiu in [16] has argued that a class of FIO is

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bounded on certain Besov space. However, Boudraud in [4] and Moussai in [13, 14] have discussed the boundedness on this space for the pseudodifferential operators. Harrat and Senoussaoui in [8] have proved that (2) is bounded on L^2 if the weight of the amplitude is bounded. Elong and Senoussaoui in [6] have demonstrated that (2) is bounded from L^p to L^q such that $\frac{1}{p}+\frac{1}{q}=1$. In this work we deal with the boundedness of a class of a semiclassical Fourier integral operator which is denoted by: *h-FIO* on the Besov spaces.

2. NOTATIONS AND DEFINITIONS

NOTATION 2.1.

- *$F(f)$ or $\widehat{f}$ denotes the Fourier transformation of f and $\mathcal{F}^{-1}g$ is the inverse Fourier tranformation of a function g.*
- *$\mathcal{S}$ is the Schwartz space (space of rapidly decreasing functions), and $\mathcal{S}'$ its dual space.*
- *L^∞_m denotes the function space for which the derivatives up to and including order m belong to L^∞.*
- *C^∞_0 is the space of infinitely differentiable functions having a compact support.*

Next, we give the definitions of some spaces that are useful later.

DEFINITION 2.2 (Series of Littlewood-Paley). Let C^∞-functions $\varphi_0(\xi)$ and $\varphi(\xi)$ be such that

(1) $\varphi_0(\xi)\geq 0, \quad \varphi(\xi)\geq 0;$
(2) if $\varphi_k(\xi)=\varphi(2^{-k}\xi)$, then $\sum_{k=0}^{\infty}\varphi_k(\xi)=1;$
(3) $\operatorname{supp}\varphi_0\subset\{\xi\in\mathbb{R}^n : |\xi|\leq 2\}$, and $\operatorname{supp}\varphi\subset\left\{\xi\in\mathbb{R}^n : 2^{-1}\leq|\xi|\leq 2\right\}$, so, $\operatorname{supp}\varphi_k\subset\left\{\xi\in\mathbb{R}^n : 2^{k-1}\leq|\xi|\leq 2^{k+1}\right\}$.

For any $f(x)\in\mathcal{S}'$ let $f_k(x)=f*F^{-1}[\varphi_k(\xi)]$.

So, we have the *Littlewood-Paley expansion* of $f(x)$ as follows:

$$f(x)=\sum_{k=0}^{\infty}f_k(x). \tag{4}$$

DEFINITION 2.3 (Besov spaces $B^s_{p,q}$). Let $1<p\leq\infty$, $1<q\leq\infty$ and $s\in\mathbb{R}$. Then

$$B^s_{p,q}=\left\{f\in\mathcal{S}' : \|f\|_{B^s_{p,q}}=\left[\sum_{k=0}^{\infty}\left(2^{sk}\|f_k\|_{L_p}\right)^q\right]^{\frac{1}{q}}<\infty\right\}.$$

We note that $\Lambda_s=B^s_{\infty,\infty}$ (*classical Hölder space*).

It is clear that the Besov spaces were introduced by interpolation spaces, see [3, 18].

DEFINITION 2.4. Let $1<p\leq\infty$, $1<q\leq\infty$ and $s\in\mathbb{R}$. Then

- $(B^s_{p,q})_{\text{comp}}$ is the collection of all $f\in B^s_{p,q}$ having a compact support.

- $(B^s_{p,q})_{\text{loc}}$ is the collection of all $f \in \mathcal{S}'$ such that $\psi f \in B^s_{p,q}$ for any $\psi \in C_0^\infty$.

For more details about these spaces (properties and equivalents norms) we refer to [19].

Definition 2.5 (Symbol class $S^{-m}_{1,\delta}$). We say that $a \in S^{-m}_{1,\delta}$ if $a(x,\xi;h) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times [1,+\infty[$ and satisfies the estimate

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi;h)| \le Ch^{-|\alpha|}(1+|\xi|)^{-m-|\alpha|+\delta|\beta|}.$$

A semiclassical Fourier integral operator with the semiclassical parameter h such that $h \ge 1$ is introduced as follows: for $f \in \mathcal{S}$

$$F_h(f)(x) = (2\pi h)^{-n} \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} a(x,\xi;h) f(y)\mathrm{d}y\mathrm{d}\xi, \tag{5}$$

where Φ is called the phase function which is a positive function and homogeneous of degree 1 in ξ.
Furthermore, we assume that $\frac{\partial^2\Phi}{\partial_{x_i}\partial_{\xi_j}} \ge 0$ or (≤ 0), the map

$$S_x = \{\xi \in \operatorname{supp} a(x,\xi;h); \Phi(x,\xi) = 1\} \ni \xi \mapsto \frac{\nabla_\xi \Phi(x,\xi)}{|\nabla_\xi \Phi(x,\xi)|} \quad \text{is} \quad 1-1 \tag{6}$$

and the symbol $a(x,\xi;h) \in S^{-m}_{1,\delta}$.
We put

$$(F_{j,h}u)(x) \equiv U_{j,h}(x) = \frac{1}{(2\pi h)^n} \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi)|\xi|^{-m} u(y)\mathrm{d}y\mathrm{d}\xi, \tag{7}$$

where $\Psi_h(2^{-j}\xi)$ is supported in $\gamma^{-1}2^{j-1} \le |\xi| \le \gamma 2^{j+1}$. To prove the main result we need the next lemmas.

Lemma 2.6. *Suppose $U_{j,h}(x)$ is as in (7) and $\Phi(x,\xi)$ satisfies the condition (6). Then, if $m > (n-1)|\frac{1}{p}-\frac{1}{2}|$, there exists a constant C such that*

$$\left[\sum_{j=0}^{\infty} 2^{sj} \|U_{j,h}\|_{L^p_{loc}}\right]^{\frac{1}{q}} \le C\|u\|_{(B^s_{p,q})_{comp}}. \tag{8}$$

Proof. Set

$$K_j(x,y) = \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi)|\xi|^{-m}\mathrm{d}\xi$$

and for $m = \frac{n-1}{2} + \epsilon$, by the method of [1, Lemma 4.2] and [6, Lemma 3.3], we have

$$\sup_{\text{near } x^0} \left[\int_{\text{near } y^0} K_{j\epsilon}(x,y)\mathrm{d}y\right] < C$$

and

$$\sup_{\text{near } y^0} \left[\int_{\text{near } x^0} K_{j\epsilon}(x,y)\mathrm{d}x\right] < C, \quad \text{for any } x^0,\ y^0 \in \mathbb{R}^n.$$

Therefore, for $u(y)$ supported near y^0, we have

$$\sup_{\text{near } x^0} \left| \int K_{j\epsilon}(x,y)u(y)\mathrm{d}y \right| \le C_\infty \|u\|_{L^\infty}$$

and

$$\int_{\text{near } x^0} \left| \int K_{j\epsilon}(x,y)u(y)\mathrm{d}y \right| \mathrm{d}x \le C_1 \|u\|_{L^1},$$

where C_∞ and C_1 are independent on j. It means that $F_{j,h} : L^\infty_{comp} \longrightarrow L^\infty_{loc}$, $u \longrightarrow U_{j,h}$ and $F_{j,h} : L^1_{comp} \longrightarrow L^1_{loc}$, $u \longrightarrow U_{j,h}$ are uniformly bounded for any $\epsilon > 0$.

Then, using the Marcinkiewiez interpolation theorem, we obtain

$$\left[\int_{\text{near } x^0} \left| \int K_{j\epsilon}(x,y)u(y)\mathrm{d}y \right|^{p_0} \mathrm{d}x \right]^{\frac{1}{p_0}} \le C_{p_0} \|u\|_{L^{p_0}},$$

where $1 < p_0 < \infty$ and $C_{p_0} = C C_\infty^{1-\frac{1}{p_0}} C_1^{\frac{1}{p_0}}$, so C_{p_0} is also independent of j. It implies that the map

$$F_{j,h} : L^{p_0}_{comp} \to L^{p_0}_{loc}, \text{ for } m = \frac{n-1}{2} + \epsilon.$$

is also uniformly bounded. We consider the symbol

$$A_{j\alpha} = \Psi_h(2^{-j}\xi)|\xi|^{-m} 2^{jm} (2^j)^{(\alpha-1)(\frac{n-1}{2}+\epsilon)} \in S^{(\alpha-1)(\frac{n-1}{2}+\epsilon)},$$

where ($|\xi| \sim 2^j$) and

$$K_j(x,y) = 2^{j((1-\alpha)(\frac{n-1}{2}+\epsilon)-m)} \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} A_{j\alpha}(\xi)\mathrm{d}\xi.$$

Then $A_{j0} \in S^{-\frac{n-1}{2}-\epsilon}$, $A_{j1} \in S^0$, which means that the map $F_{j,h} : L^{p_0}_{comp} \to L^{p_0}_{loc}$, has the upper bound $C_{p_0} 2^{j(\frac{n-1}{2}+\epsilon-m)}$ and $F_{j,h} : L^2_{comp} \to L^2_{loc}$ has the upper bound $C_2 2^{-jm}$.

Assume that $1 \le p \le 2$ and

$$p_0 = \frac{\frac{1}{p} - \frac{1}{2} + \frac{\epsilon}{(n-1)}}{\frac{1}{p} - \frac{1}{2} + \frac{\epsilon}{(n-1)p}}, \qquad 1 < p_0 < p.$$

Then, by Stein's analytic interpolation theorem in [7], we obtain $F_{j,h} : L^p_{comp} \to L^p_{loc}$, where

$$\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{2}, \quad \text{i.e. } \alpha = \frac{1-\frac{1}{p}}{\frac{1}{2} + \frac{\epsilon}{n-1}}$$

and the upper bound

$$C(C_{p_0} 2^{j(\frac{n-1}{2}+\epsilon-m)})^{1-\alpha} (C_2 2^{-jm})^\alpha = C_p 2^{j((n-1)|\frac{1}{p}-\frac{1}{2}|+\epsilon-m)},$$

where C_p is independent on j.

By duality, we have $F_{j,h} : L^p_{comp} \longrightarrow L^p_{loc}$, $1 < p < \infty$, with the upper bound $C_p 2^{j((n-1)|\frac{1}{p}-\frac{1}{2}|+\epsilon-m)}$. It implies that

(9)
$$\|U_{j,h}\|_{L^p_{loc}} \leq C2^j((n-1)\left|\frac{1}{p}-\frac{1}{2}\right| + \epsilon - m)\|u\|_{L^p_{comp}}, \qquad \text{for} \epsilon > 0 \quad \text{small.}$$

Now, we take the Littlewood-Paley series of $u = \sum_{l=0}^{\infty} u_l$, so $\widehat{u}_l = \varphi_l \widehat{u}$ and

$$\text{supp } \widehat{u}_l \subset \left\{\xi : 2^{l-1} \leq |\xi| \leq 2^{l+1}\right\}, \quad \text{supp } \widehat{u}_0 \subset \{\xi : |\xi| \leq 2\}.$$

Since $\varphi_{l-1}, \varphi_l, \varphi_{l+1}$ are not equal to zero,

$$\widetilde{\varphi}_l = \varphi_{l-1} + \varphi_l + \varphi_{l+1} \qquad \varphi_{-1} = 0,$$

for $2^{l-1} \leq |\xi| \leq 2^{l+1}$ and $\widetilde{\varphi}_l \widehat{u}_l = \widehat{u}_l$ or $u_l = F^{-1}(\widetilde{\varphi}_l) * u_l$. Then

$$\begin{aligned}
U^l_{j,h} &= \frac{1}{(2\pi h)^n} \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi)\, |\xi|^{-m}\, u_l(y) \mathrm{d}y\mathrm{d}\xi \\
&= \frac{1}{(2\pi h)^n} \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi)\, |\xi|^{-m} \left(\int F^{-1}\left[\widetilde{\varphi}_l\right](y-z) u_l(z) dz\right) \mathrm{d}y\mathrm{d}\xi \\
&= \frac{1}{(2\pi h)^n} \int u_l(z) \left[\int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi)\, |\xi|^{-m} \right. \\
&\qquad \left. \times \left(\int F^{-1}\left[\widetilde{\varphi}_l\right](y-z)\right) \mathrm{d}y\mathrm{d}\xi\right] \mathrm{d}z,
\end{aligned}$$

$$\begin{aligned}
K^l_j(x,y) &= \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi) |\xi|^{-m} F^{-1}\left[\widetilde{\varphi}_l\right](y-z) \mathrm{d}y\mathrm{d}\xi \\
&= \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi) |\xi|^{-m} \left(\int e^{-i\langle y-z,\xi\rangle} F^{-1}\left[\widetilde{\varphi}_l\right](y-z) \mathrm{d}y\right) \mathrm{d}\xi \\
&= \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi) |\xi|^{-m} \widetilde{\varphi}_l(\xi) \mathrm{d}\xi \\
&= K^l_{j,-1} + K^l_{j,0} + K^l_{j,1},
\end{aligned}$$

where

$$K^l_{j,k} = \int \mathrm{e}^{\frac{i}{h}(\Phi(x,\xi)-\langle y,\xi\rangle)} \Psi_h(2^{-j}\xi) |\xi|^{-m} \varphi(2^{l+k}\xi) \mathrm{d}\xi, \qquad k = -1, 0, 1,$$

$$\text{supp } \Psi_h(2^{-j}\xi) \cap \text{supp } \varphi(2^{l+k}\xi) \neq \emptyset, \quad \text{when } |l-j| \leq 3,$$

and the intersection is empty when $|l-j| > 3$.

So, for fixed j, we have

$$\|U^l_{j,h}\|_{L^p_{loc}} = \begin{cases} C_p 2^{j((n-1)|\frac{1}{p}-\frac{1}{2}|}\epsilon - m)\|u_l\|_{L^p_{comp}} & , |l-j| \leq 3, \\ 0, \qquad , |l-j| > 3, \end{cases}$$

and then,

$$\left[\sum_{j=0}^{\infty}(2^{js}\|U_{j,h}\|_{L^p_{loc}})^q\right]^{\frac{1}{q}} \le \left\{\sum_{j=0}^{\infty}\left(2^{js}\sum_{l=0}^{\infty}\|U^l_{j,h}\|_{L^p_{loc}}\right)^q\right\}^{\frac{1}{q}}$$

$$\le C_p\left\{\sum_{j=0}^{\infty}2^{j((n-1)\left|\frac{1}{p}-\frac{1}{2}\right|+\epsilon-m+s)^q}\left(\sum_{k=-2}^{2}\|u_{j+k}\|_{L^p_{comp}}\right)^q\right\}^{\frac{1}{q}}$$

$$\le C_p\left\{\sum_{j=0}^{\infty}2^{j((n-1)|\frac{1}{p}-\frac{1}{2}|+\epsilon-m+s)^q}\|u_j\|^q_{L^p_{comp}}\right.$$

$$+\sum_{j=1}^{\infty}2^{(j-1)((n-1)|\frac{1}{p}-\frac{1}{2}|+\epsilon-m+s)^q}\|u_j\|^q_{L^p_{comp}}$$

$$\left.+\sum_{j=2}^{\infty}2^{(j-2)((n-1)|\frac{1}{p}-\frac{1}{2}|+\epsilon-m+s)^q}\|u_j\|^q_{L^p_{comp}}\right\}^{\frac{1}{q}}$$

$$\le C'_p\left\{\sum_{j=0}^{\infty}2^{j((n-1)|\frac{1}{p}-\frac{1}{2}|+s-m)}(2^{js}\|u_j\|_{L^p_{comp}})^q\right\}^{\frac{1}{q}}$$

with

$$m > (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|,$$

so

$$(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|+\epsilon-m<0, \quad \text{for } \ \epsilon \text{ small.}$$

Hence, we obtain (8). □

REMARK 2.7. The spectrum of $U_{j,h}$ is contained in

$$b^{-1}a^j \le |\xi| \le ba^j \text{ for some constants } a>1,\ b>0. \tag{10}$$

In fact, the nondegeneracy of Φ implies that, after a linear change of coordinates,

$$\Phi(x,\xi)=\langle x,\xi\rangle-\Phi_1(\xi)+O(|x|^2|\xi|). \tag{11}$$

Therefore, when

$$\Phi(x,\xi)=\langle x,\xi\rangle-\Phi_1(\xi),$$
$$\widehat{U}_{j,h}(\xi)=e^{\Phi_1(\xi)}\Psi_h(2^{-j}\xi)|\xi|^{-m}\widehat{u}(\xi),$$

we have

$$\text{supp }\widehat{U}_{j,h}(\xi)\subset\left\{\xi:\gamma^{-1}2^{j-1}|\xi|\le\gamma 2^{j+1}\right\}.$$

It means that $U_{j,h}$ satisfies the condition (11).

LEMMA 2.8. *For series h_j, $j = 0, 1, \ldots$, suppose that the spectrum of h_j is contained in $b^{-1}a^{j-1} \leq |\xi| \leq ba^{j+1}$ for some $a > 1$, $b > 0$, we have*

$$\left\| \sum_{j=0}^{\infty} h_j \right\|_{B^s_{p,q}} \leq C \left[\sum_{j=0}^{\infty} (2^{sj} \left|\left|h_j\right|\right|_{L_p})^q \right]^{\frac{1}{q}}. \tag{12}$$

Moreover, if $s > 0$, (12) holds when the spectrum of h_j is contained in $|\xi| \leq ba^j$.

Proof. Put

$$H_k(x) = H * F^{-1} [\varphi_k(\xi)] = \sum_{j=0}^{\infty} h_j * F^{-1} [\varphi_k(\xi)]$$

and take the Littlewood-Paley series of $H = \sum_{j=0}^{\infty} h_j$ as follows: $\sum_{j=0}^{\infty} H_k(x)$ such that

$$(h_j * \widehat{F^{-1} [\varphi_k(\xi)]}) = \widehat{h}_j \varphi_k(\xi),$$

and suppose $b = a$. Then we have

$$\widehat{h}_j \varphi_k(\xi) = \widehat{h}_k \varphi_k(\xi), \quad (j = k), \quad \text{or} = 0, \quad (j \neq k),$$

so,

$$H_k(x) = h_k * F^{-1} [\varphi_k(\xi)] .$$

Then

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} h_j \right\|_{B^s_{p,q}} &= \left[\sum_{k=0}^{\infty} \left(2^{sk} \|H_k\|_{L_p} \right)^q \right]^{\frac{1}{q}} \\ &= \left[\sum_{k=0}^{\infty} \left(2^{sk} \|h_k * F^{-1} [\phi_k(\xi)]\|_{L_p} \right)^q \right]^{\frac{1}{q}} \\ &\leq C \left[\sum_{k=0}^{\infty} \left(2^{sk} \|h_k\|_{L_p} \right)^q \right]^{\frac{1}{q}}, \end{aligned}$$

i.e. (12) holds. By [7, Lemma 5], we obtain the second part of this lemma. □

LEMMA 2.9. *For any $a(x, \xi; h) \in S^{-m}_{1,\delta}$, any $r > 0$ and large $N > 0$ there exists a positive convergent series $\sum_{k \in \mathbb{Z}^n} u_k$ and a sequence $\{a_k(x, \xi; h)\}$ such that*

$$a(x, \xi; h) = \sum_{k \in \mathbb{Z}^n} u_k a_k(x, \xi; h), \tag{13}$$

where

$$a_k(x, \xi; h) = \sum_{j=0}^{\infty} M_{k,j}(2^j \delta x) \Psi_{k,h}(2^{-j}\xi)|\xi|^{-m},$$

$$\Psi_{k,h} \in C^\infty,\ \text{supp}\ \Psi_{k,h} \subset \left\{\xi : \frac{1}{3} \le |\xi| \le 3\right\}$$

and

$$\|M_{k,j}\|_{\Lambda_r} \le C, \qquad \|\Psi_{k,h}\|_{L^\infty_{N-n-1}} \le C'^{-|\alpha|},$$

where C and C' are independent on j and k.

Proof. Set $|\xi|^{+m} a(x,\xi;h) = A(x,\xi;h)$, so, $A(x,\xi;h) \in S^0_{1,\delta}$. For $A(x,\xi;h)$, let

$$\lambda(\xi) \in C_0^\infty, \quad \text{supp}\,\lambda(\xi) \subset \left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}$$

and

$$\sum_{j=0}^{\infty} \lambda(2^{-j}\xi) = 1 \text{ at } \xi \neq 0;\ \theta(\xi) \in C_0^\infty,$$

$$\text{supp}\ \theta(\xi) \subset \left\{\xi : \frac{1}{3} \le |\xi| \le 3\right\} \quad \text{and} = 1 \quad \text{in} \quad \frac{1}{2} \le |\xi| \le 2.$$

Set

$$A_j(x,\xi;h) = \lambda(\xi) A(2^{-j\delta}x, 2^j\xi; h),$$

so

$$A(x,\xi;h) = \sum_{j=0}^{\infty} A_j(2^{j\delta}x, 2^{-j}\xi; h).$$

Then take the Fourier series of $A_j(x,\xi;h)$:

$$A_j(x,\xi;h) = \sum_{k\in\mathbb{Z}^n} C_{kj}(x) e^{\frac{i}{h}k.\xi}\theta(\xi), \quad \xi \in \mathbb{R}^n,$$

let $u_k = (1+|k|^2)^{-\frac{n+1}{2}}$, $\Psi_{k,h}(\xi) = (1+|k|^2)^{\frac{n+1-N}{2}} e^{\frac{i}{h}k.\xi}\theta(\xi)$, and $M_{kj}(x) = (1+|k|^2)^{\frac{N}{2}}.C_{kj}(x)$. Then we have

$$A(x,\xi;h) = \sum_{k\in\mathbb{Z}^n} u_k \left(\sum_{j=0}^{\infty} M_{kj}(2^{j\delta}x)\Psi_{k,h}(2^{-j}\xi)\right)$$

and $a(x,\xi;h)$ has the decomposition (13) .

For any $r>0$, let $|\beta| = [r]$, so $r = |\beta| + r_1$ and $0 \le r_1 < 1$.

$$\begin{aligned}
\|M_{kj}\|_{\Lambda_r} &= \left\|(2\pi)^{-n}\int_{|\xi|<2} (I-\Delta_\xi)^{\frac{N}{2}} A_j(x,\xi;h) \mathrm{e}^{\frac{i}{h}k.\xi}\mathrm{d}\xi\right\|_{\Lambda_r} \\
&\le C \sup_{|\alpha|\le N, \xi\in\mathbb{R}^n} \left\|\partial_\xi^\alpha A_j(x,\xi;h)\right\|_{\Lambda_r} \\
&\le C \sup_{|\alpha|\le N, \xi\in\mathbb{R}^n} \left\|\partial_x^\beta\partial_\xi^\alpha A_j(x,\xi;h)\right\|_{\Lambda_{r_1}} \\
&\le C \sup_{|\alpha|\le N, \xi\in\mathbb{R}^n} \left|\partial_x^{\beta+1}\partial_\xi^\alpha A_j(x,\xi;h)\right|
\end{aligned}$$

$$\leq \quad C \sup_{|\alpha|\leq N, \xi\in\mathbb{R}^n} \left(\left| \partial_x^{\beta+1} \partial_\xi^\alpha A(x,\xi;h) \right| 2^{-j\delta(|\beta|+1)} 2^{j|\alpha|} \right) \leq C,$$

C is independent on j and k.

We take $N > n+1$, so, when $|\alpha| \leq N-n-1$, we have

$$|\partial_\xi^\alpha \Psi_{k,h}(\xi)| \leq C h^{-|\alpha|} \|\theta\|_{L^\infty_{N-n-1}} \leq C',$$

where C' is also independent on k. □

THEOREM 2.10. *Suppose that:*

- *the phase function $\Phi(x,\xi)$ of semi classical Fourier integral operator* (5) *satisfies the conditions* (6) *and* (10)*,*
- *$a \in S^{-m}_{1,\delta}$,*
- *and $m \geq (n-1)|\frac{1}{p} - \frac{1}{q}|$.*

Then h-FIO preserves locally the Besov spaces $B^s_{p,q}$ boundedness, i.e.

$$F_h : (B^s_{p,q})_{comp} \to (B^s_{p,q})_{loc}, \tag{14}$$

where $s > 0$ at $\delta = 1$, $s \in \mathbb{R}$ when $\delta < 1$.

Proof. By Lemma 2.9, it is sufficient to prove the boundedness for

$$a(x,\xi;h) = \sum_{j=0}^{\infty} M_j(2^{j\delta}x) \Psi_h(2^{-j}\xi) |\xi|^{-m}.$$

Set $A_r = \sup_{j\geq 0} \|M_j\|_{\Lambda_r}$. First, we consider the case of $\delta = 1$. We take $r > s > 0$ and let $M_j = \sum_{j=0}^{\infty} M_{jl}$ be the Littlewood-Paley series of M_j, where

$$\operatorname{supp} \widehat{M}_{jl} \subset \left\{ \xi : 2^{j-1} \leq |\xi| \leq 2^{j+1} \right\} \ \ j \neq 0,$$

and

$$\operatorname{supp} \ \widehat{M_{j0}} \subset \{\xi : |\xi| \leq 2\}.$$

We note that $M_j \in \Lambda_r$ implies $\|M_{jl}\|_\infty \leq C A_r 2^{-lr}$. Put $N_{jk}(x) = M_{j(k-j)}(2^j x)$, so,

$$\operatorname{supp} \widehat{N}_{jk} \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$$

and

$$\|N_{jk}\|_\infty \leq C A_r 2^{r(j-k)}. \tag{15}$$

Therefore,

$$(F_h u)(x) = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} N_{j,k}(x) U_{j,h}(x).$$

In fact,

$$(F_h u)(x) = (2\pi h)^{-n} \int \mathrm{e}^{\frac{i}{h}\Phi(x,\xi) - \langle y,\xi\rangle} a(x,\xi;h) u(y) \mathrm{d}y \mathrm{d}\xi$$

and

$$a(x,\xi;h) = \sum_{j=0}^{\infty} M_j(2^{j\delta}x) \Psi_h(2^{-j}\xi) |\xi|^{-m}.$$

Then

$$\begin{aligned}
(F_h u)(x) &= (2\pi h)^{-n}\int e^{\frac{i}{h}\Phi(x,\xi)-\langle y,\xi\rangle}\sum_{j=0}^{\infty}M_j(2^{j\delta}x)\Psi_h(2^{-j}\xi)|\xi|^{-m}u(y)\mathrm{d}y\mathrm{d}\xi\\
&= \sum_{j=0}^{\infty}M_j(x)U_{j,h}(x)=\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}M_{jl}(x)U_{j,h}(x)\\
&= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}M_{j(k-j)}(x)U_{j,h}(x)=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}N_{kj}(x)U_{j,h}(x)\\
&= \sum_{v=0}^{4}\left(\sum_{j=0}^{\infty}N_{j,j+v}U_{j,h}\right)+\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-5}N_{j,k}U_{j,h}\right)\\
&= G_1+G_2.
\end{aligned}$$

We note that $F_{jk}\equiv N_{jk}U_{j,h}$ has the support required in Lemma 2.8.

So, by Lemma 2.6, Lemma 2.8 and (15), we have

$$\begin{aligned}
\|G_1\|^q_{(B^s_{pq})_{loc}} &= \|\sum_{v=0}^{4}\sum_{j=0}^{\infty}N_{j,j+v}U_{j,h}\|^q_{(B^s_{pq})_{loc}}\\
&\le \sum_{v=0}^{4}\|\sum_{j=0}^{\infty}N_{j,j+v}U_{j,h}\|^q_{(B^s_{pq})_{loc}}\\
&\le C\sum_{v=0}^{4}\left(\sum_{j=0}^{\infty}(2^{js}\|N_{j,j+v}U_{j,h}\|_{L^p_{loc}})^q\right)\\
&\le C'\left(\sum_{j=0}^{\infty}\left(2^{js}\|U_{j,h}\|_{L^p_{loc}}\right)^q\right)\\
&\le C''\|u\|^q_{(B^s_{pq})_{comp}},
\end{aligned}$$

and

$$\begin{aligned}
\|G_2\|^q_{(B^s_{pq})_{loc}} &= \|\sum_{k=5}^{\infty}\sum_{j=0}^{k-5}N_{j,k}U_{j,h}\|^q_{(B^s_{pq})_{loc}}\\
&\le C\sum_{k=5}^{\infty}\|\sum_{j=0}^{h-5}N_{j,k}U_{j,h}\|^q_{(B^s_{pq})_{loc}}\\
&\le C'\sum_{k=5}^{\infty}\left(\left(2^{h's}\|\sum_{j=0}^{h-5}N_{j,k}U_{j,h}\|_{L^p_{loc}}\right)^q\right)
\end{aligned}$$

$$\leq C'' \left(\sum_{h'=5}^{\infty} \left(\sum_{j=0}^{h'-5} 2^{h's} 2^{(j-h')r} \|U_{j,h}\|_{L^p_{loc}} \right)^q \right)$$

$$\leq C'' \sum_{h'=5}^{\infty} \left(2^{h'(s-r)} \sum_{j=0}^{h'-5} 2^{j(r-s)} 2^{js} 2^{(j-h')r} \|U_{j,h}\|_{L^p_{loc}} \right)^q .$$

By [4, Lemma 3], since $s < r$, the norm l^q of the sequence

$$\left\{ 2^{h'(s-r)} \sum_{j=0}^{h'-5} 2^{j(r-s)} 2^{js} 2^{(j-h')r} \|U_{j,h}\|_{L^p_{loc}} \right\}$$

is controlled by

$$C \left(\sum_{j=0}^{\infty} \left(2^{js} \|U_{j,h}\|_{L^p_{loc}} \right) \right)^{\frac{1}{q}} .$$

So,

$$\|G_2\|^q_{(B^s_{pq})_{loc}} \leq C'' \left(\sum_{j=0}^{\infty} \left(2^{js} \|U_{j,h}\|_{L^p_{loc}} \right) \right) \leq C''' \|u\|^q_{(B^s_{pq})_{comp}} .$$

Secondly, we consider the case $0 \leq \delta < 1$.

Since $S^{-m}_{1,\delta} \subset S_{1,\delta}$, it is sufficient to prove this theorem for $s < 0$.

For fixed $s < 0$, let $r > 0$ such that $(\delta - 1)r < s < 0$, and

$$M_j = \sum_{l=0}^{\infty} M_{jl}$$

be the Littlewood-Paley series of M_j, where

$$\text{supp } \widehat{M_{jl}} \subset \left\{ \xi ; 2^{(1-\delta)(l-1)} \leq |\xi| \leq 2^{(1-\delta)(l+1)} \right\} \quad (l \neq 0)$$

and

$$\text{supp } \widehat{M_{j0}} \subset \left\{ \xi : |\xi| \leq 2^{1-\delta} \right\}, \text{and} \quad \|M_{jl}\|_\infty \leq C A_r 2^{(\delta-1)rl}.$$

In virtue of condition (11), it is easy to see that there exists $v = v(\delta) > 0$ such that

$$\text{supp } (M_{jl} \widehat{(2^{j\delta} x)} U_{j,h}) \subset \left\{ \xi : b^{-1} a^{j-1} \leq |\xi| \leq b a^{j+1} \right\} \quad \text{when} \quad j \geq 1 + v,$$

and

$$\text{(16)} \qquad \begin{array}{l} \text{supp } (M_{jl} \widehat{(2^{j\delta} x)} U_{j,h}) \subset \{ \xi : |\xi| \leq ba \}, \\ \text{when} \quad j < 1 + v \quad \text{for some} \quad a > 1,\ b > 0. \end{array}$$

Now, we decompose $F_h u$ as follows:

$$(F_h u)(x) = \sum_{j=v}^{\infty} \left(\sum_{l=0}^{j-v} M_{jl}(2^{j\delta} x) U_{j,h} \right) + \sum_{l=0}^{\infty} \left(\sum_{j=0}^{l+v-1} M_{jl}(2^{j\delta} x) U_{j,h} \right)$$

$$= \quad G_1 + G_2.$$

Now, we estimate G_1 and G_2. We have

$$\begin{aligned}
\|G_1\|^q_{(B^s_{pq})_{loc}} &= \|\sum_{j=v}^{\infty}\left(\sum_{l=0}^{j-v} M_{jl}(2^{j\delta}x)U_{j,h}\right)\|^q_{(B^s_{pq})_{loc}} \\
&\le C\left(\sum_{j=v}^{\infty}\left(2^{js}\|\sum_{l=0}^{j-v} M_{jl}(2^{j\delta}x)U_{j,h}\|_{L^p_{loc}}\right)^q\right) \\
&\le C'\left(\sum_{j=v}^{\infty}\left(2^{js}\left(\sum_{l=0}^{j-v} 2^{(\delta-1)rl}\right)\|U_{j,h}\|_{L^p_{loc}}\right)^q\right) \\
&\le C''\left(\sum_{j=v}^{\infty}\left(2^{js}\|U_{j,h}\|_{L^p_{loc}}\right)^q\right) \\
&\le C'''\|u\|^q_{(B^s_{pq})_{comp}}.
\end{aligned}$$

Since $B^{s_1}_{pq} \subset B^{s_2}_{pq}$, $s_1 < s_2$ and $s < 0$, $t = s + (1-\delta)r > 0$, by using Lemma 2.9, (16) and [4, Lemma 3], we have

$$\begin{aligned}
\|G_2\|^q_{(B^s_{pq})_{loc}} &\le \|G_2\|^q_{(B^t_{pq})_{loc}} \\
&= C'\left(\sum_{l=0}^{\infty}\left(2^{lt}\|\sum_{j=0}^{l+v-1} M_{jl}(2^{j\delta}x)U_{j,h}\|_{L^p_{loc}}\right)^q\right) \\
&\le C''\left(\sum_{l=0}^{\infty}\left(2^{ls}\|\sum_{j=0}^{l+v-1} 2^{-sj}\left(2^{sj}\|U_{j,h}\|_{L^p_{loc}}\right)^q\right)\right) \\
&\le C'''\sum_{j=0}^{\infty}\left(2^{sj}\|U_{j,h}\|_{L^p_{loc}}\right)^q \\
&\le C\|u\|^q_{(B^s_{pq})_{comp}}.
\end{aligned}$$

□

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