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A NEWLY DEFINED SUBCLASS OF BI-UNIVALENT FUNCTIONS SATISFYING SUBORDINATE CONDITIONS

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Abstract. The purpose of our present paper is to introduce a newly defined subclass of bi-univalent functions satisfying subordinate conditions defined in the open unit disc. Coefficient estimates of $|a_2|$ and $|a_3|$ and the Fekete-Szegö problem for functions of this newly-defined class are established. The results of this work generalize some well known results.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. Let S be the subclass of A consisting of functions which are analytic and univalent in $\mathbb{U}$.

Here, we recall some definitions and concepts of classes of analytic functions. Let $f \in A$. Then f is said to be in the class $S(\alpha, s, t)$ if it satisfies

$$\operatorname{Re}\left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right) > \alpha,$$

for some $0 \le \alpha < 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \le 1$ and for all $z \in \mathbb{U}$. The class $S(\alpha, s, t)$ was introduced by Frasin [7]. The class $S(\alpha, 1, t)$ was introduced and studied by Owa et al.[14], and, by taking $t = -1$, the class $S(\alpha, 1, -1) \equiv S_s(\alpha)$ was introduced by Sakaguchi [15] and a corresponding element is called Sakaguchi function of order α, where as $S_s(0) = S_s$ is the class of starlike functions with respect to the symmetrical points in $\mathbb{U}$. Also, we note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ which is the familiar class of starlike functions of order $\alpha\,(0 \le \alpha < 1)$.

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We recall the principle of subordination between analytic functions. Let the functions f and g be analytic in $\mathbb{U}$. Given the functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0, \ |w(z)| < 1, \ z \in \mathbb{U}\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f \prec g \ \text{ or } \ f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Keobe One-Quarter Theorem [6] states that the range of every function in the class S contains the disk $\{w : |w| < 1/4\}$. Therefore, every $f \in S$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f); r_0(f) \geq 1/4).$$

In fact, the inverse function f^{-1} is given by

(2) $$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

An analytic function f is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.The class of analytic and bi-univalent function in $\mathbb{U}$ is denoted by Σ.

For a brief history of the functions in the class Σ, see [5, 10, 18], the pioneering work on this subject by Srivastava et al.[17], which has apparently revived the study of bi-univalent functions in recent years. Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al.[17], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [2, 9, 11, 12, 13, 18, 19, 20]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions (see, for example, [1, 8, 16, 22]). The coefficient estimate problem for each

$$|a_n| \quad (n \in \mathbb{N}\backslash\{1, 2\}, \ \mathbb{N} = \{1, 2, 3, \ldots\})$$

is still an open problem.

In [4] the class $L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β was defined as follows:

Definition 1.1. Let $f \in A$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. Then $f(z) \in L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk if and only if

$$\mathrm{Re}\frac{z[f'(z)]^\lambda}{f(z)} > \beta. \tag{3}$$

Babalola [4] proved that all pseudo-starlike functions are Bazilevic of type $\left(1 - \frac{1}{\lambda}\right)$ order $\beta^{\frac{1}{\lambda}}$ and univalent in the open unit disk $\mathbb{U}$.

Motivated by the work of Eker, Şeker [16] and Zaprawa [21], the purpose of our present paper is to introduce a subclass of bi-univalent functions satisfying subordinate conditions. Coefficient estimates for $|a_2|$ and $|a_3|$ and the Fekete-Szegö problem for functions of this newly-defined class are established.

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $\angle S_\Sigma^{\lambda,\alpha}(\varphi, s, t)$ if the following subordinations hold

$$(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz)-f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^\lambda}{(f(sz)-f(tz))'} \prec \varphi(z)$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw)-f(tw)} + \alpha\frac{(s-t)[(wg'(w))']^\lambda}{(g(sw)-f(tw))'} \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t, |\lambda| > 0, |t| \leq 1, 0 \leq \alpha \leq 1$.

Remark 1.3. (i) For $s = 1$ and $t = -1$ we get the class $\angle S_\Sigma^{\lambda,\alpha}(\varphi, 1, -1)$ of functions $f \in \Sigma$ satisfying the conditions

$$(1-\alpha)\frac{2z[f'(z)]^\lambda}{f(z)-f(-z)} + \alpha\frac{2[(zf'(z))']^\lambda}{(f(z)-f(-z))'} \prec \varphi(z)$$

and

$$(1-\alpha)\frac{2w[g'(w)]^\lambda}{g(w)-f(-w)} + \alpha\frac{2[(wg'(w))']^\lambda}{(g(w)-f(-w))'} \prec \varphi(w),$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Eker and Şeker [16].

(ii) For $s = 1$, $t = 0$, $\lambda = 1$ we get the class $\angle S_\Sigma^{1,\alpha}(\varphi, 1, 0)$ of functions $f \in \Sigma$ satisfying the conditions

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec \varphi(z)$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\frac{(wg'(w))'}{g'(w)} \prec \varphi(w),$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Ali et al.[3].

(iii) For $\alpha = 0$ we get the class $\angle S^{\lambda}_{\Sigma}((\frac{1+z}{1-z})^{\alpha}, s, t)$ of functions $f \in \Sigma$ satisfying the conditions

$$\left|\arg\left\{\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)}\right\}\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left\{\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-f(tw)}\right\}\right| < \frac{\alpha\pi}{2},$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Mazi and Opoola [11].

(iv) For $\alpha = 0$ we get the class $\angle S^{\lambda}_{\Sigma}((\frac{1+(1-2\beta)z}{1-z}), s, t)$ of functions $f \in \Sigma$ satisfying the conditions

$$\mathrm{Re}\left\{\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)}\right\} > \beta$$

and

$$\mathrm{Re}\left\{\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-f(tw)}\right\} > \beta,$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Mazi and Opoola [11].

(v) For $\alpha = 0$ we get the class $\angle S^{\lambda}_{\Sigma}((\frac{1+z}{1-z})^{\alpha}, 1, 0)$ of functions $f \in \Sigma$ satisfying the conditions

$$\left|\arg\left\{\frac{z[f'(z)]^{\lambda}}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left\{\frac{w[g'(w)]^{\lambda}}{g(w}\right\}\right| < \frac{\alpha\pi}{2},$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Joshi et al.[9].

(vi) For $\alpha = 0$ we get the class $\angle S^{\lambda}_{\Sigma}((\frac{1+(1-2\beta)z}{1-z}), 1, 0)$ of functions $f \in \Sigma$ satisfying the conditions

$$\mathrm{Re}\left\{\frac{z[f'(z)]^{\lambda}}{f(z)}\right\} > \beta$$

and

$$\mathrm{Re}\left\{\frac{w[g'(w)]^{\lambda}}{g(w)}\right\} > \beta,$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Joshi et al. [9].

(vii) For $s = 1$, $t = 0$ $\lambda = 1$ we get the class $\angle S^{1,0}_{\Sigma}(\varphi, 1, 0)$ of functions $f \in \Sigma$ satisfying the conditions

$$\frac{zf'(z)}{f(z)} \prec \varphi(z)$$

and

$$\frac{wg'(w)}{g(w)} \prec \varphi(w),$$

where the function $g = f^{-1}$ is defined by (2) and was studied by Ali et al.[3].

2. COEFFICIENT ESTIMATES

Let φ be an analytic function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$. Also, let $\varphi(\mathbb{U})$ be starlike with respect to 1 and symmetric with respect to the axis. Thus, φ has the Taylor series expansion

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0). \tag{4}$$

For functions in the class $\angle S_{\Sigma}^{\lambda,\alpha}(\varphi, s, t)$ the following estimates are obtained:

THEOREM 2.1. *Let the function f given by* (1) *be in the class $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\begin{vmatrix}[(\lambda - 2\lambda(s+t-\lambda) - st) + \alpha((s^2+4st+t^2) \\ \quad - 6\lambda(s+t-\lambda))]B_1^2 - (1+\alpha)^2(2\lambda - s - t)^2(B_2 - B_1)\end{vmatrix}}} \tag{5}$$

and

$$|a_3| \leq \frac{B_1^2}{(1+\alpha)^2(2\lambda - s - t)^2} + \frac{B_1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)} \tag{6}$$

Proof. Let $f \in \angle S_{\Sigma}^{\lambda,\alpha}(\varphi, s, t)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, satisfying

$$(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz) - f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^{\lambda}}{(f(sz) - f(tz))'} = \varphi(u(z)) \tag{7}$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw) - f(tw)} + \alpha\frac{(s-t)[(wg'(w))']^{\lambda}}{(g(sw) - f(tw))'} = \varphi(v(w)). \tag{8}$$

Define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

or, equivalently,

$$u(z) = \frac{p_1 - 1}{p_1 + 1} = \frac{1}{2}\left((c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right) \tag{9}$$

and

$$u(z) = \frac{p_2 - 1}{p_2 + 1} = \frac{1}{2}\left((b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \cdots\right) \tag{10}$$

It is clear that p_1 and p_2 are analytic in $\mathbb{U}$ and $p_1(0) = p_2(0) = 1$; since $u, v : \mathbb{U} \to \mathbb{U}$, the functions p_1 and p_2 have positive real part in $\mathbb{U}$ and hence $|b_i| \leq 2$ and $|c_i| \leq 2$. In view (5), (6), (9) and (10), we have

$$(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^\lambda}{(f(sz) - f(tz))'} = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{11}$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw) - f(tw)} + \alpha\frac{(s-t)[(wg'(w))']^\lambda}{(g(sw) - f(tw))'} = \varphi\left(\frac{p_1(w) - 1}{p_2(w) + 1}\right). \tag{12}$$

Using (9) and (10) together with (4), we obtain

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots \tag{13}$$

and

$$\varphi\left(\frac{p_1(w) - 1}{p_1(w) + 1}\right) = 1 + \frac{1}{2}B_1 b_1 w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) w^2 + \cdots. \tag{14}$$

Since

$$\begin{aligned} &(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^\lambda}{(f(sz) - f(tz))'} \\ &= 1 + [(1+\alpha)(2\lambda - s - t)]\, a_2 z + (1+3\alpha)((s^2 + 2st + t^2) \\ &- 2\lambda(s + t - \lambda + 1))a_2^2 z^2 + (1+2\alpha)(3\lambda - s^2 - st - t^2)a_3 z^2 + \cdots \end{aligned} \tag{15}$$

and

$$\begin{aligned} &(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw) - f(tw)} + \alpha\frac{(s-t)[(wg'(w))']^\lambda}{(g(sw) - f(tw))'} \\ &= 1 - [(1+\alpha)(2\lambda - s - t)]a_2 w \\ &+ ((6\lambda - s^2 - t^2) - 2\lambda(s + t - \lambda + 1) \\ &- \alpha(6\lambda(s + t - \lambda - 1) + (s-t)^2))a_2^2 w^2 \\ &- (1+2\alpha)(3\lambda - s^2 - t^2 - st)a_3 w^2 + \cdots, \end{aligned} \tag{16}$$

it follows from (11)-(16) that

$$(1+\alpha)(2\lambda - s - t)a_2 = \frac{1}{2}B_1 c_1, \tag{17}$$

$$\begin{aligned} &(1+3\alpha)((s^2 + 2st + t^2) - 2\lambda(s + t - \lambda + 1))a_2^2 \\ &+ (1+2\alpha)(3\lambda - s^2 - st - t^2)a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \end{aligned} \tag{18}$$

$$(19)\qquad -(1+\alpha)(2\lambda-s-t)a_2=\frac{1}{2}B_1b_1,$$

and

$$(20)\qquad \begin{aligned}&(6\lambda-s^2-t^2)-2\lambda(s+t-\lambda+1)-\alpha(6\lambda(s+t-\lambda-1)+(s-t)^2)a_2^2\\&-(1+2\alpha)(3\lambda-s^2-t^2-st)a_3=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2.\end{aligned}$$

See that (17) and (19) together yield:

$$(21)\qquad c_1=-b_1 \quad \text{and} \quad 2(1+\alpha)^2(2\lambda-s-t)^2=\frac{1}{4}(b_1^2+c_1^2).$$

By adding (20) to (18), we obtain

$$(22)\qquad \begin{aligned}&[(2\lambda-4\lambda(s+t-\lambda)-2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]a_2^2\\&=\frac{1}{2}B_1(b_2+c_2)+\frac{1}{4}(b_1^2+c_1^2)(B_2-B_1).\end{aligned}$$

By using (21) and (22), we find that

$$(23)\qquad a_2^2=\frac{B_1^3(b_2+c_2)}{\begin{aligned}&2[2\lambda-4\lambda(s+t-\lambda)-2st+2\alpha((s^2+4st+t^2)\\&-6\lambda(s+t-\lambda))]B_1^2-4(1+\alpha)^2(2\lambda-s-t)^2(B_2-B_1)\end{aligned}}$$

which, in view of the well-known inequalities $|b_2|\le 2$ and $|c_2|\le 2$ for functions with positive real part, gives us the desired estimate for $|a_2|$, as asserted in (5).

In order to find the bound for $|a_3|$, by subtracting (20) from (18) and using (21), we obtain

$$(24)\qquad a_3=\frac{B_1^2b_1^2}{4(1+\alpha)^2(2\lambda-s-t)^2}+\frac{B_1(c_2-b_2)}{4(1+2\lambda)(3\lambda-s^2-st-t^2)}$$

and, applying $|b_i|\le 2$ and $|c_1|\le 2$ $(i=1,2)$ again, we get

$$|a_3|\le\frac{B_1^2}{(1+\alpha)^2(2\lambda-s-t)^2}+\frac{B_1}{(1+2\alpha)(3\lambda-s^2-st-t^2)}.$$

This completes the proof. □

For $s=1$ and $t=-1$, the class $\angle S_\Sigma^{\lambda,\alpha}(\phi,s,t)$ reduces to the class studied by Eker and Seker [16]. For functions in this class we have the following corollary.

Corollary 2.2. *If $f(z)$ from (1) is in the class $\angle S_\Sigma^{\lambda,\alpha}(\phi,1,-1)$, then*

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{|[(2\lambda^2+\lambda-1)+2\alpha(3\lambda^2-1)]B_1^2-4\lambda^2(1+\alpha)^2(B_2-B_1)|}}$$

and

$$|a_3|\le\frac{B_1^2}{4\lambda^2(1+\alpha)^2}+\frac{B_1}{(1+2\alpha)(3\lambda-1)}.$$

For $s = 1$, $t = 0$ and $\lambda = 1$, the class $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$ reduces to the class studied by Ali et al.[3]. For functions in this class we have the following corollary.

Corollary 2.3. *If $f(z)$ from (1) is in the class $\angle S_{\Sigma}^{1,\alpha}(\phi, 1, 0)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)|B_1^2 - (1+\alpha)(B_2 - B_1)|}}$$

and

$$|a_3| \leq \frac{B_1^2}{(1+\alpha)^2} + \frac{B_1}{2(1+2\alpha)}$$

For $s = 1$, $t = 0$ and $\alpha = 0$, the class $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$ reduces to the class of λ-pseudo bi-convex functions with respect to symmetrical points. For functions in this class we have the following corollary.

Corollary 2.4 ([16]). *If $f(z)$ from (1) is in the class $\angle S_{\Sigma}^{\lambda,0}(\phi, 1, -1)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(2\lambda^2 + \lambda - 1)B_1^2 - 4\lambda^2(B_2 - B_1)|}}$$

and

$$|a_3| \leq \frac{B_1^2}{4\lambda^2} + \frac{B_1}{(3\lambda - 1)}.$$

For $s = 1$, $t = 0$ and $\alpha = 1$, the class $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$ reduces to the class of λ-pseudo bi-starlike functions with respect to symmetrical points. For functions in this class we have the following corollary.

Corollary 2.5. *If $f(z)$ from (1) is in the class $\angle S_{\Sigma}^{\lambda,1}(\phi, 1, -1)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(8\lambda^2 + \lambda - 3)B_1^2 - 16\lambda^2(B_2 - B_1)|}}$$

and

$$|a_3| \leq \frac{B_1^2}{16\lambda^2} + \frac{B_1}{3(3\lambda - 1)}.$$

3. FEKETE-SZEGÖ PROBLEM

In this section, we provide the Fekete-Szegö inequalities for functions of the class $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let the function $f(z)$ given in (1) be in $\angle S_{\Sigma}^{\lambda,\alpha}(\phi, s, t)$. Then*

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} B_1|h(\mu)|, |h(\mu)| \geq \dfrac{1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)} \\ \dfrac{B_1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)}, |h(\mu)| \leq \dfrac{1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)}, \end{cases} \quad (25)$$

where

$$h(\mu) = \frac{B_1^2(1-\mu)}{[(\lambda - 2\lambda(s+t-\lambda) - st) + \alpha((s^2+4st+t^2) - 6\lambda(s+t-\lambda))]B_1^2 - (1+\alpha)^2(2\lambda - s - t)^2(B_2 - B_1)}.$$

Proof. From (21) we have $c_1 = -b_1$. Subtracting (18) and (20) and applying (21), we have

$$a_3 = a_2^2 + \frac{B_1(c_2 - b_2)}{4(1+2\alpha)(3\lambda - s^2 - st - t^2)}. \tag{26}$$

From (23) we have

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{2[2\lambda - 4\lambda(s+t-\lambda) - 2st + 2\alpha((s^2+4st+t^2) - 6\lambda(s+t-\lambda))]B_1^2 - 4(1+\alpha)^2(2\lambda - s - t)^2(B_2 - B_1)} \tag{27}$$

From (26) and (27) it follows that

$$a_3 - \mu a_2^2 = \frac{B_1}{4}\left[\left(h(\mu) + \frac{1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)}\right)c_2 + \left(h(\mu) - \frac{1}{(1+2\alpha)(3\lambda - s^2 - st - t^2)}\right)b_2\right],$$

where

$$h(\mu) = \frac{B_1^2(1-\mu)}{[(\lambda - 2\lambda(s+t-\lambda) - st) + \alpha((s^2+4st+t^2) - 6\lambda(s+t-\lambda))]B_1^2 - (1+\alpha)^2(2\lambda - s - t)^2(B_2 - B_1)}.$$

Since all B_i are real and $B_1 > 0$, assertion (25) follows. This completes the proof of the theorem.

□

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