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SENTINELS FOR AN EPIDEMIOLOGICAL SIR MODEL WITH SPATIAL DIFFUSION

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Abstract. The controllability of the classical epidemiological SIR model of Kermack and McKendrick [8] with spatial diffusion is studied as an application to the sentinel method of Lions [10], where we consider that the observation and the control have their supports in two different sets. The perturbation affects the ill population I, while the initial condition of the susceptible individuals S is incomplete. We show that we have null-controllability, which proves the existence of a sentinel for the SIR model.

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1. INTRODUCTION

Mathematical models are considered complementary tools for understanding the functioning of concrete SIR models and for predicting their evolutions. Among the first SIR models (since the beginning of the 19th century), we distinguish the model given by Kermack and Mckendrick [8]. In particular, they observe that one characteristic in the study of epidemics is the diversity of the magnitudes of the epidemics due to many factors (incomplete data). A SIR epidemic model is based on three compartments (or populations): S, compartment of susceptible individuals (healthy individuals, who can have a disease), I, compartment of infections, and R, compartment of the recovered individuals. Several studies have been carried out around the SIR model (see, for example, Brauer et al.[4], Capasso [5] or the classical book of Murray [14] and the references therein).

Here we use the model from [8] for which we add the mortality term and a spatial diffusion. Indeed, we suppose the most realistic situation of the geographical spread of epidemics. So, we add the spatial spread as a diffusive process, where the three compartments have the same diffusion coefficient D as an example. Only few papers consider this situation (see for example Abramson [1]). Moreover, we will consider the case of perturbations and incomplete data. Indeed, some of the parameters of the SIR model are not directly observable as the model is often disturbed by uncertainties. Prediction

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of the unknown data in the confrontation with real parameters is sometimes necessary. Our goal is to show that using the method of sentinel, initiated by Lions [9], [10], an interesting analysis can be done. This theoretical method works and is insensitive to the missing data.

The sentinel method of Lions is a particular least squares-like method which is adapted to the identification of parameters in ecosystems with incomplete data. The sentinel concept relies on the following three objects: a state equation, an observation function and a control function to be determined. Finding a sentinel to the SIR model is the same as to study the null-controllability. In our knowledge, there are only some papers on controllability for the SIR or SEIR problems (see for example the recent work of Löber [12] and the references therein).

Many authors use the sentinel method of Lions. See for example Ainseba et al.[2], Bodart, Demeestere [3] and the references therein. In these articles, the control function and the observation have their supports in the same set called the observatory. In this case there always exists a sentinel.

In Miloudi et al.[13] (see also [15] and [7]) the observation and the control functions can have their supports in two different sets which makes the problem of finding a sentinel non-trivial. In Miloudi et al.[13] this new method is used to find an instantaneous sentinel (see also the references therein). In the present article, we show how to apply this method to the epidemiological SIR problem of missing data.

2. POSITION OF THE PROBLEM

2.1. THE SIR DIFFUSION MODEL

The individuals are in a bounded region Ω, viewed as an open subset of $\mathbf{R^d}$, $d = 2$ or 3, with regular boundary Γ. For $T > 0$ large enough, we denote $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. Then the SIR model under consideration is the following:

$$(1) \qquad \begin{cases} \partial_t S - \delta\Delta S &= \mu N - \beta SI - \mu S & \text{in } Q, \\ \partial_t I - \delta\Delta I &= \beta SI - (\gamma+\mu)I + \lambda\widehat{I} & \text{in } Q, \\ \partial_t R - \delta\Delta R &= \gamma I - \mu R & \text{in } Q, \\ S(0) &= S^0 + \tau\widehat{S^0} & \text{in } \Omega, \\ I(0) &= I^0 & \text{in } \Omega, \\ R(0) &= R^0 & \text{in } \Omega, \\ S &= 0 & \text{on } \Sigma, \\ I &= 0 & \text{on } \Sigma, \\ R &= 0 & \text{on } \Sigma, \end{cases}$$

where $\partial_t := \frac{\partial}{\partial t}$ denotes the time first derivative and $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ the Laplacian operator, where S represents the population of individuals susceptible to infection, I the individuals infected and R, the recovered individuals. We also denote by N the total population density: $N = S + I + R$.

The term $\lambda\widehat{I}$ is the perturbation around the infected population term, $\tau\widehat{S^0}$ is the missing term of susceptible infected population. Here, λ and τ are real parameters, they are not known. The rest are known parameters and have the following meaning: β is the rate of infection, γ the healing rate, μ is the mortality rate and δ is the diffusion parameter, representing the same spread of epidemics in the three classes S, I and R, for simplicity.

If we sum the first three equations in the above SIR system, we obtain the simplified system:

$$
(2) \qquad \begin{cases} \partial_t N - \delta \Delta N &= \lambda \widehat{I} & \text{in } Q, \\ N(0) &= N^0 + \tau \widehat{S^0} & \text{in } \Omega, \\ N &= 0 & \text{on } \Sigma, \end{cases}
$$

where $N^0 = S^0 + I^0 + R^0$. Hence, system (2) is considered.

We suppose that $\left\|\widehat{I}\right\|_{L^2(Q)} \leq 1$ and $\left\|\widehat{S^0}\right\|_{L^2(\Omega)} \leq 1$. The initial data are S^0, I^0 and R^0 which are known functions and belong to $L^2(\Omega)$. Then we know that there exists a unique solution to the heat equation (2) such that:

$$
(3) \qquad N \in \mathcal{C}^0\left([0,T[; L^2(\Omega)\right) \cap \mathcal{C}^0\left(]0,T[; H^2(\Omega) \cap H_0^1(\Omega)\right)
$$

(see Lions-Magenes [11] for details).

2.2. THE SENTINEL

We now consider the following question: *Is it possible to obtain information about the term of infected population $\lambda\widehat{I}$, insensitive to the missing term of susceptible population $\tau\widehat{S^0}$?*

A partial answer can be obtained from the least squares method. The method consists in taking the unknowns $\{\lambda\widehat{I}, \tau\widehat{S^0}\} = \{v, w\}$ as control variables, but there is no real possibility to find v or w independently. The notion of sentinel provides the right response for this type of problem as we will explain later in Section 3.

Naturally, in order to be able to obtain some information, one must observe the state of the infected population. We then consider an observation system: we associate a non-empty open subset $O \subset \Omega$ to it, called observatory, and an observation of N on O, during a time T, and choose h_0 such that

$$
h_0 \in L^2(O \times (0,T)).
$$

Now, let ω be an open and non-empty subset of Ω ($\omega \subset \Omega$, $\omega \neq O$) and denote $N(t,x;\lambda,\tau) := N(\lambda,\tau)$. Given a control function $v \in L^2(\omega \times (0,T))$, we finally define

$$
(4) \qquad \mathcal{S}(\lambda,\tau) = \int_0^T \int_O h_0 \, N(\lambda,\tau) \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\omega v N(\lambda,\tau) \, \mathrm{d}x\mathrm{d}t.
$$

The problem consists in finding v such that the sentinel-control pair $(\mathcal{S}, v)$ satisfies the following conditions:

- the sentinel $\mathcal{S}$ is insensitive of first order with respect the to missing terms $\tau\widehat{S^0}$, which means

$$\frac{\partial \mathcal{S}}{\partial \tau}(0,0) = 0 \qquad \text{for all } \widehat{S^0}; \tag{5}$$

- the control function v has the property of minimal norm in $L^2(\omega \times (0,T))$ in the following sense:

$$\|v\|_{L^2(\omega\times(0,T))} = \min_{u \in L^2(\omega\times(0,T))} \|u\|. \tag{6}$$

Remark 2.1. Lions's sentinel $\mathcal{S}$ corresponds to the case $\omega = O$, so that:

$$\mathcal{S}(\lambda, \tau) = \int_0^T \int_O (h_0 - v) N(\lambda, \tau)\, \mathrm{d}x\mathrm{d}t. \tag{7}$$

In this case there exists always a sentinel satisfying (5) defined by $v = -h_0$.

Remark 2.2. Definition (4) gives a generalization of Lions's sentinel to the case when the observation and the control are having their supports in two different sets (see [13] and [15] for more details). It is also a more realistic hypothesis to have a smaller control set. In the following, we consider:

$$v \subset\subset O. \tag{8}$$

The system under consideration being linear, the state N has a differentiable dependence on τ and λ. We denote:

$$N_\tau = \frac{\partial N}{\partial \tau}(0,0) = \lim_{\tau \to 0} \left(\frac{N(0,\tau) - N(0,0)}{\tau} \right).$$

Then $N_\tau = S_\tau + I_\tau + R_\tau$ is a solution of the system:

$$\begin{cases} \partial_t N_\tau - \delta\, \Delta N_\tau &= 0 \quad \text{in} \quad Q, \\ N_\tau(0) &= \widehat{S}^0 \quad \text{in} \quad \Omega, \\ N_\tau &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{9}$$

Recall that the solution N_τ has the same regularity (3).

3. INFORMATION GIVEN BY THE SENTINEL

We suppose that the density of the population N is observed on O with the observation:

$$N_{\text{obs}} = m_0.$$

Moreover, we suppose that N has a differentiable dependence on τ and λ of the first order and that we can formally write:

$$S(\lambda, \tau) \simeq S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0),$$

since, by definition, $\frac{\partial S}{\partial \tau}(0,0) = 0$. Therefore, we can write:

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) = \int_Q (h_0 \chi_O + v\chi_\omega)\, m_0 \,\mathrm{d}x\mathrm{d}t - S(0,0).$$

If we denote by N_0 the density of the population calculated at $\lambda = \tau = 0$, then

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) = \int_Q (h_0 \chi_O + v\chi_\omega)\,(m_0 - N_0) \,\mathrm{d}x\mathrm{d}t,$$

which contains the information on $\lambda \widehat{I}$.

We point out that N_0 is a solution to the heat problem:

$$\left\{ \begin{array}{llll} \partial_t N_0 - \delta\, \Delta N_0 & = & 0 & \text{in} \quad Q, \\ N_0\,(0) & = & I^0 + R^0 & \text{in} \quad \Omega, \\ N_0 & = & 0 & \text{on} \quad \Sigma, \end{array} \right. \tag{10}$$

where we suppose that $I^0, R^0 \in L^2(\Omega)$ and that it is the unique solution of the regularity in (3).

4. FROM SENTINELS TO NULL-CONTROLLABILITY

Now, we show that v is such that (5)–(6) are equivalent to a null-controllability problem. We transform the insensitivity condition (5) as follows:

We introduce the adjoint state by defining the function $q = q(t,x)$ as a solution of the following adjoint problem:

$$\left\{ \begin{array}{llll} -\partial_t q - \delta\, \Delta q & = & h_0 \chi_O + v\chi_\omega & \text{in} \quad Q, \\ q\,(T) & = & 0 & \text{in} \quad \Omega, \\ q & = & 0 & \text{on} \quad \Sigma. \end{array} \right. \tag{11}$$

Since $h_0 \in L^2((0,T) \times O)$ and $v \in L^2((0,T) \times \omega)$, using the method of transposition [11], we can show that the adjoint problem (11) admits a unique solution

$$q \in L^2\,(Q) \cap C\left([0,T]\,; H^{-1}\,(\Omega)\right).$$

In the following, and, for simplicity, we denote

$$L = \frac{\partial}{\partial t} - \delta \Delta \quad \text{and its adjoint} \quad L^* = -\frac{\partial}{\partial t} - \delta \Delta.$$

The proposition below shows that the existence of a sentinel is equivalent to a controllability problem.

PROPOSITION 4.1. *Let q be the solution of the adjoint problem* (11). *Then the problem of existence of a sentinel insensitive to the missing term is equivalent to a null-controllability problem, that is to* (11) *together with:*

$$q(0) = 0 \quad in\, \Omega. \tag{12}$$

Proof. We multiply the first equation of the system (11) by N_τ and integrate by parts over Q to have:

$$\begin{aligned}-\int_Q N_\tau\, L^*q\,\mathrm{d}x\mathrm{d}t \;&=\; -\int_\Omega q(T)N_\tau(T)\,\mathrm{d}x + \int_\Omega q(0)N_\tau(0)\,\mathrm{d}x\\ &\quad -\int_Q q\, LN_\tau\,\mathrm{d}x\mathrm{d}t + \int_\Sigma \frac{\partial q}{\partial \nu}N_\tau \mathrm{d}\sigma - \int_\Sigma q\frac{\partial N_\tau}{\partial t}\mathrm{d}\sigma.\end{aligned}$$

Since N_τ and q are solutions of the systems (9) and (11), we obtain:

$$\int_0^T\int_\Omega N_\tau(h_0\chi_O + v\chi_\omega)\mathrm{d}x\mathrm{d}t = \int_\Omega q(0)\widehat{S^0}\mathrm{d}x, \qquad \forall\ \widehat{S^0}\in L^2(\Omega)$$

If the sentinel exists, then we have:

$$\int_\Omega q(0)\widehat{S^0}\mathrm{d}x = 0, \quad \forall\ \widehat{S^0}\in L^2(\Omega).$$

That is: $q(0) = 0$ in Ω. The reciprocal is immediate. □

5. EXISTENCE OF A SENTINEL FOR THE SIR MODEL

We define

$$\mathcal{V} = \left\{\rho \in \mathcal{C}^\infty(\overline{Q}),\ \rho = 0 \text{ on}\,\Sigma\right\}.$$

We use the classical Carleman inequality to solve the null-controllability problem (11)–(12). Then we have the following proposition:

PROPOSITION 5.1. *Denote by $Q_\omega = (0,T)\times\omega$. Then there is a constant $C = C(\Omega,\omega) > 0$ such that for any $\rho\in\mathcal{V}$, we have:*

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,\mathrm{d}x\mathrm{d}t \le C\left[\int_Q |L\rho|^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_\omega}|\rho|^2\,\mathrm{d}x\mathrm{d}t\right], \tag{13}$$

where $\theta\in C^2(Q)$ is positive with $\frac{1}{\theta}$ bounded.

Proof. For a proof of this classical result, known as the observability inequality for the heat equation, we refer to the work of Fursikov and Imanuvilov [6], where they use the Carleman estimates. □

In the following, we show that there is a control v such that the conditions (11)–(12) are satisfied. We define a bilinear form from $\mathcal{V}\times\mathcal{V}$ to $\mathbb{R}$ by

$$a(\rho,\rho) = \int_Q L\rho\, L\rho'\,\mathrm{d}x\mathrm{d}t + \int_{Q_\omega}\rho\rho'\,\mathrm{d}x\mathrm{d}t, \quad \rho,\rho'\ \in\ \mathcal{V}.$$

It is easy to verify that $a(.,.)$ is bilinear, symmetric and positive. We verify that it is a scalar product too. Indeed, if $a(\rho,\rho) = 0$, then

$$\int_Q |L\rho|^2\,\mathrm{d}x\mathrm{d}t = 0 \qquad \text{and} \qquad \int_{Q_\omega}|\rho|^2\,\mathrm{d}x\mathrm{d}t = 0.$$

Then, using the observability inequality of Fursikov and Imanuvilov (13), we deduce:

$$\int_Q \frac{1}{\theta^2} |\rho|^2 \,\mathrm{d}x\mathrm{d}t = 0 \quad \text{so that} \quad \rho = 0 \text{ in } Q.$$

We denote by V the Hilbert space which is the completion of $\mathcal{V}$ with respect to the norm:

$$\|\rho\|_V^2 = a(\rho,\rho) = \|L\rho\|^2_{L^2(Q)} + \|\rho\|^2_{L^2(Q_\omega)}.$$

Lemma 5.2. *We suppose that $h_0 \in L^2(Q)$ and that $\theta h_0 \in L^2(Q)$. Then the linear mapping $\mathcal{L}$ defined by*

$$\begin{array}{rccl} \mathcal{L}: & V & \to & \mathbb{R} \\ & \rho & \longmapsto & \mathcal{L}(\rho) = \displaystyle\int_Q h_0 \chi_{\mathcal{O}}\, \rho \, dxdt \end{array}$$

is continuous on V.

Proof. Using the hypothesis and the Cauchy-Schwarz inequality, we obtain

$$|\mathcal{L}(\rho)| \leq \left(\int_Q |\theta h_0 \chi_{\mathcal{O}}|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_Q \frac{1}{\theta^2} |\rho|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \leq C\sqrt{a(\rho,\rho)} = C\|\rho\|_V.$$

Then $\mathcal{L}$ is continuous on V. □

Remark 5.3. The space V is a weighted Hilbert space. Indeed, if we define $H_\theta(Q)$ as follows:

$$H_\theta(Q) = \left\{\rho \in L^2(Q) \text{ such that } \int_Q \frac{1}{\theta^2}|\rho|^2 \,\mathrm{d}x\mathrm{d}t < \infty\right\},$$

equipped with the norm $\|\rho\|_\theta = \left(\int_Q \frac{1}{\theta^2} |\rho|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}$, then, using the observability inequality (13), we obtain:

$$\|\rho\|_\theta \leq C\|\rho\|_V.$$

This shows that V is continuously imbedded into $H_\theta(Q)$.

As a consequence, we have the following:

Corollary 5.4. *We assume that the hypothesis of Lemma 5.2 is satisfied. Then there exists a unique function $\overline{\rho} \in V$ solution to the problem:*

$$a(\overline{\rho},\rho) = \int_0^T \int_\Omega h_0 \chi_{\mathcal{O}} \rho \,\mathrm{d}x\mathrm{d}t, \quad \forall \rho \in V. \tag{14}$$

Proof. Since the application $\mathcal{L}$ is linear and continuous on V and since the bilinear, symmetric form $a(.,.)$ is continuous and coercive on $V \times V$, by the application of Lions's theorem [11], there exists a unique $\overline{\rho} \in V$ solution to the variational problem (14). □

Proposition 5.5. *Let* $\overline{\rho} \in V$ *be the unique solution of* (14). *Then the couple* $(\overline{v}, \overline{q})$ *given by*

$$\overline{v} = -\overline{\rho}\chi_{\omega} \tag{15}$$

and by

$$\overline{q} = L\overline{\rho} \tag{16}$$

is the unique solution to the null-controllability problem (11)–(12).

Proof. Indeed, if $\overline{\rho} \in V$ is the solution of (14), then

$$\int_Q L\overline{\rho}\, L\rho \,\mathrm{d}x\mathrm{d}t + \int_{Q_\omega} \overline{\rho}\rho \,\mathrm{d}x\mathrm{d}t = \int_Q h_0\chi_{\mathcal{O}}\rho \,\mathrm{d}x\mathrm{d}t, \qquad \forall \rho \in V.$$

We put $\overline{v} = -\overline{\rho}\chi_\omega$ and $\overline{q} = L\overline{\rho}$. Then we have

$$\int_Q \overline{q}\, L\rho \,\mathrm{d}x\mathrm{d}t = \int_Q (h_0\chi_{\mathcal{O}} + \overline{v}\chi_\omega)\, \rho \,\mathrm{d}x\mathrm{d}t, \qquad \forall \rho \in V. \tag{17}$$

Now, we integrate formally by parts (for $\rho \in \mathcal{V} \subset V$) in the left hand side of (17) to obtain:

$$\begin{aligned}\int_Q \overline{q}\, L\rho \,\mathrm{d}x\mathrm{d}t \quad &= \quad \int_Q \rho\, L^*\overline{q} \,\mathrm{d}x\mathrm{d}t + \int_\Omega \rho(T)\overline{q}(T)\,\mathrm{d}x - \int_\Omega \rho(0)\overline{q}(0)\,\mathrm{d}x \\ &\quad + \int_\Sigma \frac{\partial\rho}{\partial\nu}\,\overline{q}\,\mathrm{d}\sigma - \int_\Sigma \frac{\partial\overline{q}}{\partial\nu}\,\rho\,\mathrm{d}\sigma\end{aligned}$$

This, successively implies that $q(0) = q(T) = 0$ and $q_{|\Sigma} = 0$. We then obtain:

$$L^*\overline{q} = h_0\chi_{\mathcal{O}} + \overline{v}\chi_\omega \qquad \text{on } Q.$$

That is (11)–(12). □

6. CONCLUDING REMARKS

The infections diseases represents one of the richest areas in mathematical biology. In this paper the controllability problem of the classical epidemiological SIR model with spacial diffision and the existence of a sentinel SIR model have been considered. Conditions for the existence of a problem solution are then derived.

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