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SOME PROPERTIES OF FUZZY NORMED LINEAR SPACES AND FUZZY RIESZ BASES

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Abstract. Some results of fuzzy frames on fuzzy Hilbert spaces from the point of view of Bag and Samanta are proved. In this paper, we define dual fuzzy frames and fuzzy Riesz bases and establish some fundamental results via dual fuzzy frames and fuzzy Ries bases. Next, we investigate the relation between fuzzy frame and fuzzy Riesz bases.

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1. INTRODUCTION

The idea of fuzzy norms on a linear space was first introduced by Katsaras [15] in 1984. Later on, many authors: Felbin [13], Cheng, Mordeson [5], Bag, Samanta [2] etc. gave different definitions of fuzzy normed linear spaces. R. Biswas [4], A. M. El-Abye, H. M. El-Hamouly [12] tried to give a meaningful definition to fuzzy inner product space and associated fuzzy norm function restricted to a real linear space. P. Mazumder and S. K. Samanta introduced the definition of fuzzy inner product space from the point of view of Bag and Samanta using fuzzy norm [2]. Recently, Daraby and et al. [8] studied some properties of fuzzy Hilbert spaces and they showed that all results in classical Hilbert spaces are immediate consequences of the corresponding results for Felbin-fuzzy Hilbert spaces. Moreover, by an example, they showed that the spectrum of the category of Felbin-fuzzy Hilbert spaces is broader than the category of classical Hilbert spaces [7].

One of the important concepts in the study of vector spaces is the concept of basis, which allows every vector to be uniquely represented as a linear combination of the basis elements. The main feature of a basis $\{x_k\}$ in a Hilbert space H is that every $x \in H$ can be represented as a linear combination of the elements x_k in the form:

$$x = \sum_{k=1}^{\infty} c_k(x) x_k. \tag{1}$$

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However, the linear independence property for a basis – which implies the uniqueness of coefficients $c_k(x)$ – is restrictive in applications. Sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external conditions demanded by applied problems. For such purposes, a more flexible type of spanning sets is needed. Frames provide this alternative. Frames are used in signal and image processing, non-harmonic Fourier series, data compression and sampling theory. Today, frame theory has applications to problems in both pure and applied mathematics, physics, engineering, computer science, etc.

Many physical systems are inherently nonlinear functions and must be described by non-linear models. But some systems have an uncertain structure and it is not possible to provide an accurate mathematical model. Therefore, to these systems, the conventional control models can not be used for solving this problems; we need to use new concepts, namely fuzzy frames theory and fuzzy wavelets. Fuzzy frame and fuzzy wavelet are inspired from frame theory, wavelet theory and fuzzy concepts. For achieving approximation functions, control and identification of nonlinear systems are presented in [3, 18] . It does not only retain the frame and wavelet properties, but also has other advantages, such as a simple structure of approximation and good interpretability of the approximation of non-linear functions.

In this paper, some results of fuzzy frames on fuzzy Hilbert spaces from the point of view of Bag and Samanta are proved. We define dual fuzzy frames and fuzzy Riesz bases and establish some fundamental results via dual fuzzy frames and fuzzy Ries bases. Next, we investigate the relation between fuzzy frame and fuzzy Riesz bases.

2. SOME PRELIMINARIES

In this section, some definitions and preliminary results, which are used in this paper, are presented.

Definition 2.1 ([2]). Let U be a linear space over the field F. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if for all $x, u \in U$ and $c \in F$ the following conditions are satisfied:

(N1) $\forall t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$;
(N2) $(\forall t \in \mathbb{R}, t > 0, N(x,t) = 1)$ iff $x = \underline{0}$;
(N3) $\forall t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s,t \in \mathbb{R}, x,u \in U, N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U,N) will be referred to as a fuzzy normed linear space.

Theorem 2.2 ([2]). *Let (U,N) be a fuzzy normed linear space. Assume further that*

(N6) $\forall t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$.

Define $\|x\|_\alpha = \bigwedge \{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$. *Then* $\{\|.\|_\alpha : \alpha \in (0,1)\}$ *is an ascending family of norms on* U *and they are called* α*-norms on* U *corresponding to the fuzzy norm* N *on* U.

DEFINITION 2.3 ([1]). Let (U, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and it is denoted by $\lim x_n$.

PROPOSITION 2.4 ([5]). *Let* (U, N) *be a fuzzy normed linear space satisfying* (N_6) *and* $\{x_n\}$ *be a sequence in* U. *Then* $\{x_n\}$ *converges to* x *iff* $x_n \to x$ *w.r.t.* $\|.\|_\alpha$, *for all* $\alpha \in (0,1)$.

DEFINITION 2.5 ([1]). Let (U, N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) > \alpha$, for all $t > 0$, and x is called the limit of $\{x_n\}$.

PROPOSITION 2.6 ([17]). *Let* (U, N) *be a fuzzy normed linear space satisfying* (N_6). *If* $\{x_n\}$ *is an* α*-convergent sequence in* (U, N), *then* $\|x_n - x\|_\alpha \to 0$ *as* $n \to \infty$.

DEFINITION 2.7 ([16]). Let U be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : U \times U \times \mathbb{C} \longrightarrow I = [0,1]$ be a mapping such that the following hold:

(FIP1) for $s, t \in \mathbb{C}, \mu(x+y, z, |t|+|s|) \geq \min\{\mu(x, z, |t|), \mu(y, z, |s|)\}$;
(FIP2) for $s, t \in \mathbb{C}, \mu(x, y, |st|) \leq \min\left\{\mu\left(x, x, |s|^2\right), \mu\left(y, y, |t|^2\right)\right\}$;
(FIP3) for $t \in \mathbb{C}, \mu(x, y, t) = \mu\left(y, x, \bar{t}\right)$;
(FIP4) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|}), \alpha (\neq 0) \in \mathbb{C}, t \in \mathbb{C}$;
(FIP5) $\mu(x, x, t) = 0, \forall t \in \mathbb{C} \backslash \mathbb{R}^+$;
(FIP6) $(\mu(x, x, t) = 1, \forall t > 0)$ iff $x = \underline{0}$;
(FIP7) $\mu(x, x, .) : \mathbb{R} \to I$ is a monotonic non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} \mu(\alpha x, x, t) = 1$.

We call μ fuzzy inner product function on U and (U, μ) fuzzy inner product space (FIP space).

THEOREM 2.8 ([16]). *Let* U *be a linear space over* $\mathbb{C}$. *Let* μ *be a FIP on* U. *Then*

$$N(x,t) = \begin{cases} \mu(x, x, t^2) & \textit{if } t \in \mathbb{R}, t > 0, \\ 0 & \textit{if } t \leq 0 \end{cases}$$

is a fuzzy norm on U. *Now, if* μ *satisfies the following conditions:*

(FIP8) $\left(\mu\left(x, x, t^2\right) > 0, \forall t > 0\right) \Rightarrow x = \underline{0}$ *and*
(FIP9) *for all* $x, y \in U$ *and* $p, q \in \mathbb{R}$,

$$\mu\left(x+y, x+y, 2q^2\right) \bigwedge \mu\left(x-y, x-y, 2p^2\right) \geq \mu\left(x, x, p^2\right) \bigwedge \mu\left(y, y, q^2\right),$$

then $\|x\|_\alpha = \bigwedge \{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ *is an ordinary norm satisfying parallelogram law. By using the polarization identity, we can get an ordinary inner product, called the* α*-inner product, as follows:*

$$\langle x, y\rangle_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right) + \frac{1}{4} i \left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right), \forall \alpha \in (0,1).$$

Definition 2.9 ([16]). Let (U,μ) be a FIP space satisfying $(FIP8)$. The linear space U is said to be level complete if for any $\alpha \in (0,1)$ every Cauchy sequence converges w.r.t. $\|.\|_\alpha$ (the α-norm generated by the fuzzy norm N which is induced by the fuzzy inner product μ).

Definition 2.10 ([1]). Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a fuzzy linear operator, where (U, N_1) and (V, N_2) are fuzzy normed linear spaces. The mapping T is said to be strongly fuzzy bounded on U if and only if there exists a positive real number M such that

$$N_2(T(x), s) \geq N_1(x, \tfrac{s}{M}), \quad \forall x \in U, \forall s \in \mathbb{R}.$$

Definition 2.11 ([1]). Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a fuzzy linear operator where (U, N_1) and (V, N_2) are fuzzy normed linear spaces. The mapping T is said to be uniformly bounded if there exists $M > 0$ such that

$$\|Tx\|_\alpha^2 \leq M \|x\|_\alpha^1 \quad \forall \alpha \in (0,1)$$

where $\|.\|_\alpha^1$ and $\|.\|_\alpha^2$ are α-norms on N_1 and N_2, respectively.

Remark 2.12. Let us denote the set of all uniformly bounded linear operators from a fuzzy normed linear space (U, N_1) to (V, N_2) by $B(U, V)$.

Theorem 2.13 ([1]). *Let* $T : (U, N_1) \longrightarrow (V, N_2)$ *be a fuzzy linear operator where* (U, N_1) *and* (V, N_2) *are fuzzy normed linear spaces satisfying* (N_6)*. Then* T *is strongly fuzzy bounded if and only if it is uniformly bounded with respect to* α*-norms of* N_1 *to* N_2*.*

Definition 2.14 ([1]). Let (U, N_1) and (V, N_2) be two fuzzy normed linear spaces satisfying (N_6). For $T \in B(U, V)$, let

$$\|T\|_\beta' = \bigvee\nolimits_{x \in U, x \neq \underline{0}} \frac{\|Tx\|_\beta^2}{\|x\|_\beta^1}, \quad \beta \in (0,1),$$

and

$$\|T\|_\alpha = \bigvee\nolimits_{\beta \leq \alpha} \|T\|_\beta', \quad \alpha \in (0,1).$$

Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms in $B(U, V)$.

Definition 2.15 ([16]). Let (U, μ) be a FIP space. The linear space U is said to be a fuzzy Hilbert space if it is level complete.

Definition 2.16 ([16]). Let $\alpha \in (0,1)$ and (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. Now, if $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are α-fuzzy orthogonal to each other and we denote $x \perp_\alpha y$. Let M be a subset of U and $x \in U$. If $\langle x, y\rangle_\alpha = 0$, for all $y \in M$, then we say that

x is α-fuzzy orthogonal to M and we denote $x \perp_\alpha M$. The set of all α-fuzzy orthogonal elements to M is called α-fuzzy orthogonal set.

DEFINITION 2.17 ([16]). Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. If $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$, for all $\alpha \in (0,1)$, then we say that x, y are fuzzy orthogonal to each other and we denote $x \perp_\alpha y$. Thus $x \perp y$ if and only if $x \perp_\alpha y$, for all $(0,1)$. The set of all fuzzy orthogonal elements to each other is called fuzzy orthogonal set.

DEFINITION 2.18 ([17]). Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. An α-fuzzy orthogonal set M in U is said to be α-fuzzy orthonormal if the elements have α-norm 1, that is for all $x, y \in M$,

$$\langle x, y\rangle_\alpha = \begin{cases} 1, & x = y \\ 0, & x \neq y, \end{cases}$$

where $\langle ., .\rangle_\alpha$ is the inner product induced by μ.

DEFINITION 2.19 ([17]). Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. A fuzzy orthonormal set M in U is said to be fuzzy orthonormal if the elements have α-norm 1 for all $\alpha \in (0,1)$, that is for all $x, y \in M$

$$\langle x, y\rangle_\alpha = \begin{cases} 1, & x = y \\ 0, & x \neq y, \end{cases}$$

where $\langle ., .\rangle_\alpha$ is the inner product induced by μ.

PROPOSITION 2.20 ([17]). *An α-fuzzy orthonormal set and a fuzzy orthonormal set in a FIP space are linearly independent.*

3. SOME PROPERTIES OF FUZZY INNER PRODUCT SPACES

In this section, we present some properties of the space $(B(U,V), \|.\|_\alpha)$ and of fuzzy linear spaces analogous to properties of the ordinary normed spaces.

DEFINITION 3.1. Let (U, μ) and (V, μ) be two fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$. Let T be a strongly fuzzy bounded linear operator from U to V. If there exists an operator T^* from V to U such that for all $\alpha \in (0,1)$

$$\langle Tx, y\rangle_\alpha = \langle x, T^* y\rangle_\alpha, \forall x \in U, y \in V,$$

then the operator T^* is called fuzzy adjoint of operator T.

In the following example, we give that a fuzzy inner product induces a classic inner product.

EXAMPLE 3.2. Let $(U, \langle ., .\rangle)$ be a real inner product space. Define a function $\mu : U \times U \times \mathbb{C} \to [0,1]$ by

$$\mu(x, y, t) = \begin{cases} \dfrac{|t|}{|t| + \|x\|\|y\|} & \text{if } t > \|x\|\|y\|, \\ 0 & \text{if } t \leq \|x\|\|y\|, \\ 0 & \text{if } t \in \mathbb{C} \setminus \mathbb{R}^+. \end{cases}$$

Then μ is a fuzzy inner product function on U and (U,μ) is a fuzzy inner product space.

Hence, we conclude that every classic inner product induces the fuzzy inner product. In what follows, we show that $(FIP8)$ also holds. So, we have a fuzzy norm from the point of view of Bag and Samanta.

(FIP8) $\mu(x,x,t^2) > 0, \forall t > 0 \Rightarrow t > \|x\|^2 \quad \forall t > 0 \Rightarrow x = \underline{0}$.

$$\begin{aligned}
\|x\|_\alpha &= \bigwedge \{t : \mu(x,x,t^2) \geq \alpha\} \\
&= \bigwedge \left\{t : \frac{|t|^2}{|t|^2 + \|x\|^2} \geq \alpha\right\} \\
&= \sqrt{\frac{\alpha}{1-\alpha}}\|x\|.
\end{aligned}$$

It is clear that $(FIP9)$ holds. By using the polarization identity, the α-inner product follows from the classic inner product.

$$\begin{aligned}
\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 &= \frac{\alpha}{1-\alpha}\|x-y\|^2 + \frac{\alpha}{1-\alpha}\|x+y\|^2 \\
&= \frac{\alpha}{1-\alpha}(\|x-y\|^2 + \|x+y\|^2) \\
&= \frac{\alpha}{1-\alpha}(2\|x\|^2 + 2\|y\|^2) \\
&= 2(\|x\|_\alpha^2 + \|y\|_\alpha^2).
\end{aligned}$$

so, we have

$$\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 = \frac{\alpha}{1-\alpha}(2\|x\|^2 + 2\|y\|^2) = 2(\|x\|_\alpha^2 + \|y\|_\alpha^2).$$

It follows that

$$\begin{aligned}
&\langle x,y\rangle_\alpha = \frac{1}{4}(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2) + \frac{i}{4}(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2) \\
&= \frac{\alpha}{4(1-\alpha)}(\|x+y\|^2 - \|x-y\|^2) + \frac{\alpha i}{4(1-\alpha)}(\|x+iy\|^2 - \|x-iy\|^2) \\
&= \frac{\alpha}{1-\alpha}\langle x,y\rangle.
\end{aligned}$$

Example (3.2) shows that the fuzzy inner product implies the classic inner product.

LEMMA 3.3. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. For any $\alpha \in (0,1)$ and $y,z \in U$, if $\langle x,y\rangle_\alpha = \langle x,z\rangle_\alpha$ for all $x \in U$, then $y = z$.*

Proof. Using Example 3.2 we have $\langle x, y\rangle_\alpha = \dfrac{\alpha}{1-\alpha}\langle x, y\rangle$. Following functional analysis, if $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$. So, if $\langle x, y\rangle_\alpha = \langle x, z\rangle_\alpha$, then $y = z$. $\square$

THEOREM 3.4. *Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. Let T be a fuzzy linear operator on (U, μ). Then T^* is also a linear operator on (U, μ) and the following properties hold:*

(i) $(T^*)^* = T$;
(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$;
(iii) $(\lambda T)^* = \overline{\lambda} T^*, \quad \forall \lambda \in \mathbb{C}$;
(iv) $(ST)^* = T^* S^*$.

Proof. Suppose that $y_1, y_2 \in U$ and $\lambda, \beta \in \mathbb{C}$. For each $x \in U$ and by using $\langle x, y\rangle_\alpha = \frac{\alpha}{1-\alpha}\langle x, y\rangle$ we have:

$$\begin{aligned}
\langle x, T^*(\lambda y_1 + \beta y_2)\rangle_\alpha &= \frac{\alpha}{1-\alpha}\langle x, T^*(\lambda y_1 + \beta y_2)\rangle \\
&= \frac{\alpha}{1-\alpha}\langle Tx, \lambda y_1 + \beta y_2\rangle \\
&= \frac{\alpha}{1-\alpha}\overline{\lambda}\langle Tx, y_1\rangle + \frac{\alpha}{1-\alpha}\overline{\beta}\langle Tx, y_2\rangle \\
&= \frac{\alpha}{1-\alpha}\langle x, \lambda T^* y_1\rangle + \frac{\alpha}{1-\alpha}\langle x, \beta T^* y_2\rangle \\
&= \frac{\alpha}{1-\alpha}\langle x, \lambda T^* y_1 + \beta T^* y_2\rangle \\
&= \langle x, \lambda T^* y_1 + \beta T^* y_2\rangle_\alpha.
\end{aligned}$$

It follows from Lemma 3.3 that $T^*(\lambda y_1 + \beta y_2) = \lambda T^* y_1 + \beta T^* y_2$, that is, T^* is linear.

For each $x, y \in U$

$$\langle y, (T^*)^* x\rangle_\alpha = \langle T^* y, x\rangle_\alpha = \overline{\langle x, T^* y\rangle_\alpha} = \overline{\langle Tx, y\rangle_\alpha} = \langle y, Tx\rangle_\alpha.$$

Hence, $(T^*)^* = T$, so we have (i).

For proving (ii), obviously we have

$$\begin{aligned}
\langle x, (T_1 + T_2)^* y\rangle_\alpha &= \langle (T_1 + T_2)x, y\rangle_\alpha \\
&= \langle T_1 x, y\rangle_\alpha + \langle T_2 x, y\rangle_\alpha \\
&= \langle x, T_1^* y\rangle_\alpha + \langle x, T_2^* y\rangle_\alpha \\
&= \langle x, (T_1^* + T_2^*) y\rangle_\alpha.
\end{aligned}$$

For each $\alpha \in (0, 1)$ and $\lambda \in \mathbb{C}$, we have

$$\langle \lambda Tx, y\rangle_\alpha = \lambda\langle Tx, y\rangle_\alpha = \lambda\langle x, T^* y\rangle_\alpha = \langle x, \overline{\lambda} T^* y\rangle_\alpha, \text{ so we get (iii).}$$

For each $x, y \in U$,

$$\langle STx, y\rangle_\alpha = \langle Tx, S^* y\rangle_\alpha = \langle x, T^* S^* y\rangle_\alpha.$$

Therefore, $(ST)^* = T^*S^*$. $\square$

Definition 3.5. Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. A bijective strongly fuzzy bounded linear operator $T:(U,\mu)\longrightarrow(U,\mu)$ is unitary if $TT^*=T^*T=I$.

Theorem 3.6. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and let $\{e_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal sequence in U. Then the following statements are equivalent:*

(i) *$\{e_k\}_{k=1}^{\infty}$ is complete fuzzy orthonormal;*
(ii) *if $\langle x,e_k\rangle_\alpha=0$ for $k\in\mathbb{N}$, then $x=\underline{0}$;*
(iii) *For every $x\in U, x=\sum_{k=1}^{\infty}\langle x,e_k\rangle_\alpha e_k$ for all $\alpha\in(0,1)$ and hence*

$$\langle x,e_k\rangle_\alpha=\langle x,e_k\rangle_\beta,\quad \forall\alpha,\beta\in(0,1)\,;$$

i.e. x is independent of α.
(iv) *For every $x,y\in U$, $\langle x,y\rangle_\alpha=\sum_{k=1}^{\infty}\langle x,e_k\rangle_\alpha\langle e_k,y\rangle_\alpha$ for all $\alpha\in(0,1)$.*
(v) *For every $x\in U$, $\|x\|_\alpha^2=\sum_{k=1}^{\infty}|\langle x,e_k\rangle_\alpha|^2$ for all $\alpha\in(0,1)$ and hence*

$$\|x\|_\alpha^2=\|x\|_\beta^2,\quad \forall\alpha,\beta\in(0,1)\,.$$

Proof. (i)$\to$(ii). Let $\{e_k\}_{k=1}^{\infty}$ be a complete fuzzy orthonormal sequence and $\langle x,e_k\rangle_\alpha=0$ for $k\in\mathbb{N}$ and $\alpha\in(0,1)$. Set for a fixed α_0, $e^{\alpha_0}=\dfrac{x}{\|x\|_{\alpha_0}}$. Then $\|e^{\alpha_0}\|_{\alpha_0}^2=\langle e^{\alpha_0},e^{\alpha_0}\rangle_{\alpha_0}=1$ and $\langle e^{\alpha_0},e_k\rangle_{\alpha_0}=0$ for $k\in\mathbb{N}$. Therefore, we get an α_0-fuzzy orthonormal sequence $\{e^{\alpha_0},e_1,e_2,\cdots\}$ of which $\{e_1,e_2,\cdots\}$ is proper subset, a contraction to completeness. Therefore, $e^{\alpha_0}=\underline{0}$, thus $x=\underline{0}$.

(ii)$\to$(iii). Suppose $\langle x,e_k\rangle_\alpha=0$ for $k\in\mathbb{N}$, implies $x=\underline{0}$. Therefore,

$$\left\langle x-\textstyle\sum_{k=1}^{\infty}\langle x,e_k\rangle_\alpha e_k,e_j\right\rangle_\alpha=0\quad\forall j\in\mathbb{N},\forall\alpha\in(0,1),$$

i.e.

$$x-\textstyle\sum_{k=1}^{\infty}\langle x,e_k\rangle_\alpha e_k=0,\quad\forall\alpha\in(0,1).$$

Thus,

$$x=\textstyle\sum_{k=1}^{\infty}\langle x,e_k\rangle_\alpha e_k=\sum_{k=1}^{\infty}\langle x,e_k\rangle_\beta e_k,\quad\forall\alpha,\beta\in(0,1).$$

Therefore, we have

$$\textstyle\sum_{k=1}^{\infty}(\langle x,e_k\rangle_\alpha-\langle x,e_k\rangle_\beta)e_k=0,\quad\forall\alpha,\beta\in(0,1).$$

Since $\{e_k\}_{k=1}^{\infty}$ is linearly independent,

$$\langle x,e_k\rangle_\alpha-\langle x,e_k\rangle_\beta=0,\quad k\in\mathbb{N},\forall\alpha,\beta\in(0,1),$$

i.e.

$$\langle x,e_k\rangle_\alpha=\langle x,e_k\rangle_\beta,\quad k\in\mathbb{N},\forall\alpha,\beta\in(0,1).$$

(iii)$\to$(iv). Suppose $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha e_k$. For all $x, y \in U$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}\langle x, y\rangle_\alpha &= \left\langle \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha e_k, y\right\rangle_\alpha \\ &= \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha\langle e_k, y\rangle_\alpha.\end{aligned}$$

(iv)$\to$(v). Since $\langle x, y\rangle_\alpha = \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha\langle e_k, y\rangle_\alpha$, we write for $\alpha \in (0,1)$

$$\begin{aligned}\|x\|_\alpha^2 &= \langle x, x\rangle_\alpha = \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha\langle e_k, x\rangle_\alpha \\ &= \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha\overline{\langle x, e_k\rangle_\alpha} \\ &= \sum_{k=1}^{\infty}|\langle x, e_k\rangle_\alpha|^2.\end{aligned}$$

Now, from (iii), we have $\langle x, e_k\rangle_\alpha = \langle x, e_k\rangle_\beta$ for all $k \in \mathbb{N}$ and $\alpha, \beta \in (0,1)$. Using this we get

$$\|x\|_\alpha^2 = \sum_{k=1}^{\infty}|\langle x, e_k\rangle_\alpha|^2 = \sum_{k=1}^{\infty}|\langle x, e_k\rangle_\beta|^2 = \|x\|_\beta^2.$$

(v)$\to$(i). Suppose (v) holds and $\{e_k\}_{k=1}^{\infty}$ is not complete. Then we get for an $\alpha \in (0,1)$, a proper subset $\{e_l, e_1, e_2, \cdots\}$ of the set $\{e_1, e_2, \cdots\}$ that $\|e_l\|_\alpha^2 = \langle e_l, e_l\rangle_\alpha = 1$ and $\langle e_l, e_k\rangle_\alpha = 0$ for $k \in \mathbb{N}$. Now,

$$\|e_l\|_\alpha^2 = \sum_{k=1}^{\infty}|\langle e_l, e_k\rangle_\alpha|^2 = 0.$$

Therefore, $e_l = \underline{0}$. □

Definition 3.7. Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. A countable family of elements $\{x_k\}_{k=1}^{\infty}$ in U is a fuzzy frame for U if there exist constants $A, B > 0$ such that for all $x \in U$ and $\alpha \in (0,1)$:

$$A\|x\|_\alpha^2 \leq \sum_{k=1}^{\infty}|\langle x, x_k\rangle_\alpha|^2 \leq B\|x\|_\alpha^2. \tag{2}$$

The numbers A and B are called fuzzy frame bounds. Fuzzy frame bounds are not unique. The optimal lower fuzzy frame bound is the supremum over all lower fuzzy frame bounds and the optimal upper fuzzy frame bound is the infimum over all upper fuzzy frame bounds. Note that the optimal fuzzy frame bounds are actually fuzzy frame bounds. If $\|x_k\|_\alpha = 1$, the fuzzy frame is normalized. A fuzzy frame $\{x_k\}_{k=1}^{\infty}$ is tight if $A = B$ and in the case $A = B = 1$ we call it *Parseval fuzzy frame.* In the case the upper inequality in

(2) holds, $\{x_k\}_{k=1}^{\infty}$ is called *fuzzy Bessel sequence.* It follows from the definition that if $\{x_k\}_{k=1}^{\infty}$ is a fuzzy frame for (U,μ), then $\overline{span}\,\{x_k\}_{k=1}^{\infty} = U$.

Consider now a vector space U equipped with a fuzzy frame $\{x_k\}_{k=1}^{\infty}$ and define a linear mapping

$$T : (l^2(\mathbb{N}),\mu) \longrightarrow (U,\mu), \quad T\{\beta_k\}_{k=1}^{\infty} = \textstyle\sum_{k=1}^{\infty} \beta_k x_k$$

T is usually called the pre-fuzzy frame operator or the *fuzzy synthesis operator.* The adjoint operator is given by

$$T^* : (U,\mu) \longrightarrow (l^2(\mathbb{N}),\mu), \quad T^*x = \{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty},$$

and it called the *fuzzy analysis operator.* Composing T with its adjoint T^*, we obtain the fuzzy frame operator,

$$S : (U,\mu) \longrightarrow (U,\mu), \quad Sx = TT^*x = \sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha x_k.$$

Note that in terms of the fuzzy frame operator we have

$$\langle Sx, x\rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2, \quad \forall x \in U, \forall \alpha \in (0,1).$$

THEOREM 3.8. *Let $\{x_k\}_{k=1}^{\infty}$ be a fuzzy frame in fuzzy Hilbert space (U,μ) satisfying $(FIP8)$ and $(FIP9)$ with fuzzy frame operator S. Then the following holds:*

i) *S is invertible and self-adjoint.*

ii) *Every $x \in U$ can be represented as*

$$x = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k\rangle_\alpha x_k, \quad x \in U, \tag{3}$$

and

$$x = \sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha S^{-1}x_k, \quad x \in U. \tag{4}$$

Both series converges (w.r.t. $\|.\|_\alpha$; $\alpha \in (0,1)$, where $\|.\|_\alpha$ are the α-norms of N induced by U) iff $\sum_{k=1}^{\infty} |\langle x, S^{-1}x_k\rangle_\alpha|^2$ converges.

Proof. i) Since $S^* = (TT^*)^* = TT^* = S$, the operator S is self-adjoint. For invertibility of S, first, we show that S is one to one. By definition, one has to show if $Sx = 0$, then $x = \underline{0}$. If for all $x \in U$, $Sx = 0$, then $0 = \langle Sx, x\rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2$.

$$A\|x\|_\alpha^2 \leq \textstyle\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 = 0$$

$$A\|x\|_\alpha^2 = 0 \Rightarrow \|x\|_\alpha^2 = 0 \Rightarrow x = \underline{0}.$$

So, S is injective and actually implies that S^* is surjective and $S = S^*$, thus S is surjective , but let us give a direct proof. The fuzzy frame condition implies that $\overline{span}\,\{x_k\}_{k=1}^{\infty} = U$. So, the fuzzy synthesis operator T is surjective.

Given $x \in U$ we can therefore find $y \in l^2(\mathbb{N})$ such that $Ty = x$, we can choose $y \in N_T^{\perp} = R_{T^*}$, so it follows that $R_S = R_{TT^*} = U$. This shows that S is invertible. ii) Every $x \in U$ has the representation

$$x = SS^{-1}x = TT^*S^{-1}x = \textstyle\sum_{k=1}^{\infty} \langle S^{-1}x, x_k \rangle_\alpha x_k$$

using that S is self-adjoint, we arrive at

$$x = \textstyle\sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle_\alpha x_k.$$

Suppose $\Phi_n = \sum_{k=1}^{n} \langle x, S^{-1}x_k \rangle_\alpha x_k$ and $\varphi_n = \sum_{k=1}^{n} |\langle x, S^{-1}x_k \rangle_\alpha|^2$. Then for all $\alpha \in (0,1)$ and $n > m$

$$\|\Phi_n - \Phi_m\|_\alpha^2 = \left\langle \textstyle\sum_{k=1}^{n} \langle x, S^{-1}x_k \rangle_\alpha x_k, \sum_{k=1}^{m} \langle x, S^{-1}x_k \rangle_\alpha x_k \right\rangle_\alpha,$$

i.e.

$$\|\Phi_n - \Phi_m\|_\alpha^2 = |\langle x, S^{-1}x_{m+1} \rangle_\alpha|^2 + |\langle x, S^{-1}x_{m+2} \rangle_\alpha|^2 + \cdots + |\langle x, S^{-1}x_n \rangle_\alpha|^2,$$

i.e.

$$\|\Phi_n - \Phi_m\|_\alpha^2 = \varphi_n - \varphi_m, \quad \forall \alpha \in (0,1).$$

Hence, Φ_n is Cauchy w.r.t. $\|.\|_\alpha$, for all $\alpha \in (0,1)$ iff φ_n is Cauchy in $\mathbb{R}$. Hence, Φ_n is Cauchy iff φ_n is Cauchy in $\mathbb{R}$. The expansion (4) is proved similarly, using $x = S^{-1}Sx$. □

Theorem 3.8 shows that all information about a given vector $x \in U$ is contained in the sequence $\{\langle x, S^{-1}x_k \rangle_\alpha\}_{k=1}^{\infty}$. The numbers $\langle x, S^{-1}x_k \rangle_\alpha$ are called fuzzy frame coefficients.

Note that, since $S : (U, \mu) \longrightarrow (U, \mu)$ is bijective, the sequence $\left\{S^{-1}x_k\right\}_{k=1}^{\infty}$ is also a fuzzy frame; it is called the canonical dual fuzzy frame of $\{x_k\}_{k=1}^{\infty}$.

Lemma 3.9. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$. If $\{x_k\}_{k=1}^{\infty}$ is a tight fuzzy frame with fuzzy bound A, then the canonical dual fuzzy frame is $\left\{A^{-1}x_k\right\}_{k=1}^{\infty}$ and*

$$x = \frac{1}{A} \textstyle\sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha x_k, \quad \forall x \in U, \alpha \in (0,1).$$

Proof. If $\{x_k\}_{k=1}^{\infty}$ is a tight fuzzy frame with fuzzy frame bound A and fuzzy frame operator S, by Definition 3.7, for all $x \in U$ and $\alpha \in (0,1)$ we have

$$\langle Sx, x \rangle_\alpha = \textstyle\sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 = A\|x\|_\alpha^2 = \langle Ax, x \rangle_\alpha.$$

This implies that $S = AI$. Thus, S^{-1} is equal to A^{-1} and the result follows from Theorem 3.8. □

Example 3.10. Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$ and let $\{e_k\}_{k=1}^{2}$ be a fuzzy orthonormal sequence in U. Suppose $x_1 = e_1$, $x_2 = e_1 - e_2$, $x_3 = e_1 + e_2$. Then $\{x_k\}_{k=1}^{3}$ is a fuzzy frame for U. Using the definition of the fuzzy frame operator

$$Sx = \textstyle\sum_{k=1}^{3} \langle x, x_k \rangle_\alpha x_k,$$

and noting that $\langle x,y\rangle_\alpha = \dfrac{\alpha}{1-\alpha}\langle x,y\rangle$, we obtain that

$$\begin{aligned} Se_1 &= \langle e_1,x_1\rangle_\alpha x_1 + \langle e_1,x_2\rangle_\alpha x_2 + \langle e_1,x_3\rangle_\alpha x_3 \\ &= \frac{\alpha}{1-\alpha}\left(\langle e_1,x_1\rangle x_1 + \langle e_1,x_2\rangle x_2 + \langle e_1,x_3\rangle x_3\right) \\ &= \frac{\alpha}{1-\alpha}\left(e_1 + e_1 - e_2 + e_1 + e_2\right) \\ &= \frac{3\alpha}{1-\alpha}e_1, \end{aligned}$$

and

$$\begin{aligned} Se_2 &= \langle e_2,x_1\rangle_\alpha x_1 + \langle e_2,x_2\rangle_\alpha x_2 + \langle e_2,x_3\rangle_\alpha x_3 \\ &= \frac{\alpha}{1-\alpha}\left(\langle e_2,x_1\rangle x_1 + \langle e_2,x_2\rangle x_2 + \langle e_2,x_3\rangle x_3\right) \\ &= \frac{2\alpha}{1-\alpha}e_2. \end{aligned}$$

Thus, $S^{-1}e_1 = \dfrac{1-\alpha}{3\alpha}e_1, \quad S^{-1}e_2 = \dfrac{1-\alpha}{2\alpha}e_2$. By linearity, the canonical dual fuzzy frame is

$$\begin{aligned} \left\{S^{-1}x_k\right\}_{k=1}^3 &= \left\{S^{-1}x_1, S^{-1}x_2, S^{-1}x_3\right\} \\ &= \left\{S^{-1}e_1, S^{-1}e_1 - S^{-1}e_2, S^{-1}e_1 + S^{-1}e_2\right\} \\ &= \left\{\frac{1-\alpha}{3\alpha}e_1, \frac{1-\alpha}{3\alpha}e_1 - \frac{1-\alpha}{2\alpha}e_2, \frac{1-\alpha}{3\alpha}e_1 + \frac{1-\alpha}{2\alpha}e_2\right\}. \end{aligned}$$

Via Theorem 3.8, the representation of $x \in U$ in terms of fuzzy frame is given by

$$\begin{aligned} x &= \sum_{k=1}^{3}\langle x, S^{-1}x_k\rangle_\alpha x_k \\ &= \frac{1-\alpha}{3\alpha}\langle x,e_1\rangle_\alpha e_1 + \langle x, \frac{1-\alpha}{3\alpha}e_1 - \frac{1-\alpha}{2\alpha}e_2\rangle_\alpha (e_1-e_2) \\ &\quad + \langle x, \frac{1-\alpha}{3\alpha}e_1 + \frac{1-\alpha}{2\alpha}e_2\rangle_\alpha (e_1+e_2) \\ &= 3(\frac{1-\alpha}{3\alpha})\langle x,e_1\rangle_\alpha e_1 + 2(\frac{1-\alpha}{2\alpha})\langle x,e_2\rangle_\alpha e_2 \\ &= \frac{1-\alpha}{\alpha}\langle x,e_1\rangle_\alpha e_1 + \frac{1-\alpha}{\alpha}\langle x,e_2\rangle_\alpha e_2 \\ &= \langle x,e_1\rangle e_1 + \langle x,e_2\rangle e_2. \end{aligned}$$

We identify now a characterization of all dual fuzzy frames $\{y_k\}_{k=1}^\infty$ associated with a given fuzzy frame $\{x_k\}_{k=1}^\infty$. Since $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are assumed to be fuzzy Bessel sequence, we can consider the associated pre-fuzzy frame operators; we will denote the pre-fuzzy frame operator for $\{x_k\}_{k=1}^\infty$ by T

and the pre-fuzzy frame operator for $\{y_k\}_{k=1}^{\infty}$ by L. Two fuzzy Bessel sequence $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are dual fuzzy frame if

$$x = \sum_{k=1}^{\infty} \langle x, y_k \rangle_\alpha x_k, \quad \forall x \in U, \alpha \in (0,1). \tag{5}$$

In terms of the fuzzy operator S and L, (5) means that $TL^* = I$. We begin with a lemma, which shows that the roles of $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ can be interchanged and that lower fuzzy frame condition automatically is satisfied for fuzzy Bessel sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$, if (5) holds.

LEMMA 3.11. *Let (U,μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0,1)$, $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ be fuzzy Bessel sequences in U. Then the following are equivalent:*

(i) *for all $x \in U$, $x = \sum_{k=1}^{\infty} \langle x, y_k \rangle_\alpha x_k$.*
(ii) *for all $x \in U$, $x = \sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha y_k$.*
(iii) *for all $x, y \in U$, $\langle x, y \rangle_\alpha = \sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha \langle y_k, y \rangle_\alpha$.*

Proof. Using Theorem 3.6, the proof is straightforward. □

In case the equivalent conditions are satisfied, then $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are dual fuzzy frames for U.

LEMMA 3.12. *Let (U,μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$, let $\{x_k\}_{k=1}^{\infty}$ be a fuzzy frame for U and $\{e_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal sequence in U. Then dual fuzzy frames for $\{x_k\}_{k=1}^{\infty}$ form a family $\{y_k\}_{k=1}^{\infty} = \{Le_k\}_{k=1}^{\infty}$, where $L : (U,\mu) \longrightarrow (U,\mu)$ is an invertible strongly fuzzy bounded operator and $LT^* = I$.*

Proof. If L is a strongly fuzzy bounded and $LT^* = I$, then L is surjective. We have the fuzzy bounded operator L on (U,μ), hence $\{y_k\}_{k=1}^{\infty}$ is a fuzzy Bessel sequence. Let $L^{-1} : (U,\mu) \longrightarrow (U,\mu)$ denote the inverse of L. For $y \in U$ we have that

$$y = LL^{-1}y = \textstyle\sum_{k=1}^{\infty} (L^{-1}y)_k y_k,$$

where $(L^{-1}y)_k$ denotes the k-th coordinate of $L^{-1}y$. Thus, for $\alpha \in (0,1)$ we have

$$\begin{aligned}
\|y\|_\alpha^4 &= |\langle y, y \rangle_\alpha|^2 \\
&= |\langle \sum_{k=1}^{\infty} (L^{-1}y)_k y_k, y \rangle_\alpha|^2 \\
&\leq \sum_{k=1}^{\infty} |(L^{-1}y)_k|^2 \sum_{k=1}^{\infty} |\langle y, y_k \rangle_\alpha|^2 \\
&\leq \|L^{-1}\|_\alpha^2 \|y\|_\alpha^2 \sum_{k=1}^{\infty} |\langle y, y_k \rangle_\alpha|^2.
\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty} |\langle y, y_k\rangle_\alpha|^2 \geq \frac{1}{\|L^{-1}\|_\alpha^2}\|y\|_\alpha^2, \quad \forall y \in U, \alpha \in (0,1),$$

i.e. $\{y_k\}_{k=1}^{\infty}$ is a fuzzy frame. Note that in terms of $\{e_k\}_{k=1}^{\infty}$,

$$T^*x = \{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty} = \sum_{k=1}^{\infty}\langle x, x_k\rangle_\alpha e_k;$$

thus, for all $x \in U$ and $\alpha \in (0,1)$,

$$x = LT^*x = \sum_{k=1}^{\infty}\langle x, x_k\rangle_\alpha y_k,$$

i.e. $\{y_k\}_{k=1}^{\infty}$ is a dual fuzzy frame of $\{x_k\}_{k=1}^{\infty}$. For the other implication, assume that $\{y_k\}_{k=1}^{\infty}$ is a dual fuzzy frame of $\{x_k\}_{k=1}^{\infty}$. Then the pre-fuzzy frame operator L for $\{y_k\}_{k=1}^{\infty}$ satisfies the condition. In fact, $\{y_k\}_{k=1}^{\infty} = \{Le_k\}_{k=1}^{\infty}$ and, by Lemma 3.11, $LT^* = I$. □

We finish this section by an example for the canonical dual fuzzy frame and non-canonical dual fuzzy frame.

Example 3.13. Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$ and let $\{e_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal sequence in U. Consider the fuzzy frame

$$\{x_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, e_4, \cdots\},$$

with fuzzy bounds $A = \dfrac{\alpha}{1-\alpha}$, $B = \dfrac{2\alpha}{1-\alpha}$. The canonical dual fuzzy frame is given by

$$\left\{S^{-1}x_k\right\}_{k=1}^{\infty} = \left\{\frac{1+\alpha}{2\alpha}e_1, \frac{1+\alpha}{2\alpha}e_1, e_2, e_3, e_4, \cdots\right\}.$$

As examples of non-canonical dual fuzzy frames, we have

$$\{y_k\}_{k=1}^{\infty} = \left\{0, \frac{1-\alpha}{\alpha}e_1, e_2, e_3, e_4, \cdots\right\},$$

and

$$\{y_k\}_{k=1}^{\infty} = \left\{\frac{1-\alpha}{3\alpha}e_1, \frac{2-2\alpha}{3\alpha}e_1, e_2, e_3, e_4, \cdots\right\}.$$

4. FUZZY FRAMES AND FUZZY RIESZ BASES

In this section, we note that all fuzzy Riesz bases are fuzzy frames.

Proposition 4.1. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$ and let $\{e_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal sequence in U. If the series $\sum_{k=1}^{\infty}\beta_k e_k$ is α-convergent w.r.t. N induced by μ, then $x \in U$ has expansion*

$$x = \sum_{k=1}^{\infty}\langle x, e_k\rangle_\alpha e_k, \quad \forall \alpha \in (0,1), \tag{6}$$

i.e.

$$\beta_k = \langle x, e_k\rangle_\alpha = \langle x, e_k\rangle_\beta, \quad \forall \alpha, \beta \in (0,1)\,,$$

where $\langle .,. \rangle$ *denotes the* α*-inner product induced by* μ*,* x *denotes the sum of* $\sum_{k=1}^{\infty} \beta_k e_k$*. Hence,*

$$x = \textstyle\sum_{k=1}^{\infty} \langle x, e_k\rangle_\alpha e_k = \sum_{k=1}^{\infty} \langle x, e_k\rangle_\beta e_k, \quad \forall \alpha, \beta \in (0,1)\,.$$

Proof. Since $\sum_{k=1}^{\infty} \beta_k e_k$ is α-convergent. So it is convergent w.r.t. $\|.\|_\alpha$. Let $S_n = \sum_{k=1}^{n} \beta_k e_k$. Taking fuzzy inner product with S_n and e_j and using the definition of fuzzy orthonormality we have

$$\langle S_n, e_j\rangle_\alpha = \beta_j, \quad \forall \alpha \in (0,1) \forall j = 1.2, \cdots, k.$$

Now $S_n \to x$ w.r.t. $\|.\|_\alpha$, hence

$$\langle S_n, e_j\rangle_\alpha \longrightarrow \langle x, e_j\rangle_\alpha = \beta_j.$$

Therefore $x = \sum_{k=1}^{\infty} \beta_k e_k = \sum_{k=1}^{\infty} \langle x, e_k\rangle_\alpha e_k$, for all $\alpha \in (0,1)$. By Theorem 3.6 we have

$$\beta_k = \langle x, e_k\rangle_\alpha = \langle x, e_k\rangle_\beta, \quad \forall \alpha, \beta \in (0,1)\,.$$

□

THEOREM 4.2. *Let* (U, μ) *be a fuzzy Hilbert spaces satisfying* $(FIP8)$ *and* $(FIP9)$ *and* $\alpha \in (0,1)$ *and let* $\{e_k\}_{k=1}^{\infty}$ *be a fuzzy orthonormal sequence in* U*. Then the fuzzy orthonormal bases for* U *are precisely the sets* $\{Te_k\}_{k=1}^{\infty}$*, where* $T : (U, \mu) \longrightarrow (U, \mu)$ *is a unitary operator.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal basis for U. Define the linear operator

$$T : (U, \mu) \longrightarrow (U, \mu), \quad T\left(\textstyle\sum_{k=1}^{\infty} \beta_k e_k\right) = \sum_{k=1}^{\infty} \beta_k x_k.$$

Then T is a fuzzy bounded and bijective mapping and $x_k = Te_k$. If for $x, y \in U$, we write $x = \sum_{k=1}^{\infty} \langle x, e_k\rangle_\alpha e_k$ and $y = \sum_{k=1}^{\infty} \langle y, e_k\rangle_\alpha e_k$, then, via the definition of T and Theorem 3.6 for $\alpha \in (0,1)$, we have

$$\begin{aligned}
\langle T^*Tx, y\rangle_\alpha &= \langle Tx, Ty\rangle_\alpha \\
&= \langle \sum_{k=1}^{\infty} \langle x, e_k\rangle_\alpha x_k, \sum_{k=1}^{\infty} \langle y, e_k\rangle_\alpha x_k\rangle_\alpha \\
&= \sum_{k=1}^{\infty} \langle x, e_k\rangle_\alpha \overline{\langle y, e_k\rangle_\alpha} \\
&= \langle x, y\rangle_\alpha.
\end{aligned}$$

This implies that $T^*T = I$. Since T is surjective, it is unitary. On the other hand, if T is a given unitary operator, then

$$\langle Te_i, Te_j\rangle_\alpha = \langle T^*Te_i, e_j\rangle_\alpha = \langle e_i, e_j\rangle_\alpha,$$

i.e. $\{Te_k\}_{k=1}^{\infty}$ is fuzzy orthonormal. □

In Theorem 4.2, we characterized all orthonormal bases in terms of unitary operators that follow from fuzzy orthonormal bases. Formally, the definition of a fuzzy Riesz basis appears by a weak condition on the linear operator:

DEFINITION 4.3. A *fuzzy Riesz basis* for fuzzy Hilbert space (U,μ) which satisfying $(FIP8)$ and $(FIP9)$ is a family of the form $\{Te_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is a fuzzy orthonormal basis for U and $T:(U,\mu)\longrightarrow(U,\mu)$ is a uniformly bounded bijective operator.

A fuzzy Riesz basis $\{f_k\}_{k=1}^{\infty}$ is actually a basis; this follows form the proof of Theorem 4.4, which we state in the following. Note that the expansion (6) of the elements $f\in U$ in terms of a fuzzy Riesz basis is more involved than the expression (6), which we obtained via fuzzy orthonormal bases.

THEOREM 4.4. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\{f_k\}_{k=1}^{\infty}$ be a fuzzy Riesz basis for (U,μ). Then $\{f_k\}_{k=1}^{\infty}$ is a fuzzy Bessel sequence. Furthermore, there exists a sequence $\{g_k\}_{k=1}^{\infty}$ in (U,μ) such that*

$$f=\sum_{k=1}^{\infty}\langle f,g_k\rangle_{\alpha}f_k,\quad \forall f\in U,\forall\alpha\in(0,1). \tag{7}$$

The sequence $\{f_k\}_{k=1}^{\infty}$ is also a fuzzy Bessel basis and the series (7) converges (w.r.t. $\|.\|_{\alpha}$, where $\|.\|_{\alpha}$ are α-norms of N induced by μ).

Proof. According to the definition, we can write $\{f_k\}_{k=1}^{\infty}=\{Te_k\}_{k=1}^{\infty}$,where T is a bounded bijective operator and $\{e_k\}_{k=1}^{\infty}$ is a fuzzy orthonormal basis. Let now $f\in U$ and $\alpha\in(0,1)$. By expanding $T^{-1}f$ in the fuzzy orthonormal basis $\{e_k\}_{k=1}^{\infty}$ we have

$$T^{-1}f=\textstyle\sum_{k=1}^{\infty}\langle T^{-1}f,e_k\rangle_{\alpha}e_k=\sum_{k=1}^{\infty}\langle f,(T^{-1})^{*}e_k\rangle_{\alpha}e_k,\quad\forall\alpha\in(0,1).$$

Therefore, with $g_k=(T^{-1})^{*}e_k$,

$$f=TT^{-1}f=\sum_{k=1}^{\infty}\langle f,(T^{-1})^{*}e_k\rangle_{\alpha}Te_k=\sum_{k=1}^{\infty}\langle f,g_k\rangle_{\alpha}f_k.$$

Since the operator $(T^{-1})^{*}$ is uniformly bounded and bijective, $\{g_k\}_{k=1}^{\infty}$ is a fuzzy Riesz basis by definition.

For $f\in U$ and for all $\alpha\in(0,1)$,

$$\sum_{k=1}^{\infty}|\langle f,f_k\rangle_{\alpha}|^2=\sum_{k=1}^{\infty}|\langle f,Te_k\rangle_{\alpha}|^2=\|T^{*}f\|_{\alpha}^2\le\|T^{*}\|_{\alpha}^2\|f\|_{\alpha}^2=\|T\|_{\alpha}^2\|f\|_{\alpha}^2.$$

This proves that a fuzzy Riesz basis is a fuzzy Bessel sequence. Thus, the series (7) converges (w.r.t. $\|.\|_{\alpha}$; $\alpha\in(0,1)$, where $\|.\|_{\alpha}$ are α-norms of N which are induced by μ). We complete the proof by showing that the sequence $\{g_k\}_{k=1}^{\infty}$ constructed in the proof is the only one that satisfies 7. For that purpose, we first note that if

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle_\alpha f_k = \sum_{k=1}^{\infty} \langle f, g_k \rangle_\beta f_k, \quad \forall \alpha, \beta \in (0,1)$$

for some coefficients $\langle f, g_k \rangle_\alpha$ and $\langle f, g_k \rangle_\beta$, then necessarily $\langle f, g_k \rangle_\alpha = \langle f, g_k \rangle_\beta$. By Theorem 3.6, we have $\langle f, g_k \rangle_\alpha = \langle f, g_k \rangle_\beta, \quad \forall \alpha, \beta \in (0,1)$. □

The sequence $\{g_k\}_{k=1}^{\infty}$ satisfying (7) is called the fuzzy dual Riesz basis of $\{f_k\}_{k=1}^{\infty}$. Let us find the fuzzy dual of $\{g_k\}_{k=1}^{\infty}$. With the notation used in the proof of Theorem 4.4, we have that the fuzzy dual of $\{f_k\}_{k=1}^{\infty} = \{Te_k\}_{k=1}^{\infty}$ is given by $\{g_k\}_{k=1}^{\infty} = \{(T^{-1})^* e_k\}_{k=1}^{\infty}$; thus, the fuzzy dual of $\{g_k\}_{k=1}^{\infty}$ is

$$\left\{ \left(\left((T^{-1})^* \right)^{-1} \right)^* e_k \right\}_{k=1}^{\infty} = \{Te_k\}_{k=1}^{\infty} = \{f_k\}_{k=1}^{\infty}.$$

That is $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are fuzzy duals of each other.

Proposition 4.5. *Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and let $\{f_k\}_{k=1}^{\infty} = \{Te_k\}_{k=1}^{\infty}$ be a fuzzy Riesz basis for (U, μ). Then there exist constants $A, B > 0$ such that*

$$A\|f\|_\alpha^2 \le \sum_{k=1}^{\infty} |\langle f, f_k \rangle_\alpha|^2 \le B\|f\|_\alpha^2, \quad \forall f \in U, \alpha \in (0,1).$$

The largest possible value for the constant A is $\dfrac{1}{\|T^{-1}\|_\alpha^2}$ and the smallest possible value for B is $\|T\|_\alpha^2$.

Proof. The fuzzy Riesz basis $\{Te_k\}_{k=1}^{\infty}$ is a fuzzy Bessel sequence with optimal upper bound $\|T\|_\alpha^2$, by the estimate in Theorem 4.4. The result about the lower bound is a consequence of

$$\|f\|_\alpha = \|(T^*)^{-1} T^* f\|_\alpha \le \|(T^*)^{-1}\|_\alpha \|T^* f\|_\alpha = \|T^{-1}\|_\alpha \|T^* f\|_\alpha.$$

□

5. CONCLUSION

In this paper, we consider fuzzy inner product space introduced by Bag and Samanta. Some important concepts viz. α- fuzzy orthonormal set, complete fuzzy orthonormal set etc. have been introduced. We define dual fuzzy frames and fuzzy Riesz bases and establish some fundamental results via dual fuzzy frames and fuzzy Ries bases. Next, we investigate the relation between fuzzy frames and fuzzy Riesz bases. We establish Bessels inequality and the Riesz representation theorem in fuzzy setting. We think that these results will be helpful for the researchers to develop fuzzy functional analysis, especially for frame theory.

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