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CERTAIN SUBCLASS OF MEROMORPHICALLY UNIFORMLY CONVEX FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates, growth and distortion theorems, radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\sigma_p(\alpha, \beta, \lambda)$.

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1. INTRODUCTION

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z \in \mathcal{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions that satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $f \in A$ which are univalent in E. A function $f \in A$ is a starlike function of order $\alpha, 0 \leq \alpha < 1$, if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in E. \tag{2}$$

We denote this class by $S^*(\alpha)$.

A function $f \in A$ is a convex function of order $\alpha, 0 \leq \alpha < 1$, if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in E. \tag{3}$$

We denote this class with $K(\alpha)$.

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Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0, z \in E, \tag{4}$$

and let $T^*(\alpha) = T \cap S^*(\alpha),\ C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and some related classes possess some interesting properties and have been extensively studied by Silverman [18] and others.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $UCV(\beta, \gamma)$, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta \left|\frac{zf''(z)}{f'(z)}\right|, \tag{5}$$

where $\beta \geq 0, \gamma \in [-1, 1)$ and $\beta+\gamma \geq 0$ and it is said to be in the corresponding class denoted by $SP(\beta, \gamma)$, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{6}$$

where $\beta \geq 0, \gamma \in [-1, 1)$ and $\beta + \gamma \geq 0$. Indeed, it follows from (5) and (6) that

$$f \in UCV(\gamma, \beta) \Leftrightarrow zf' \in SP(\gamma, \beta). \tag{7}$$

For $\beta = 0$, we get the classes $K(\gamma)$ and $S^*(\gamma)$, respectively. The functions of the class $UCV(1, 0) \equiv UCV$, called uniformly convex functions, were introduced and studied by Goodman, using a geometric interpretation, in [4, 5] and, using uniformly starlike functions, in [6]. The class $SP(1, 0) \equiv SP$ is defined by Ronning [15]. The classes $UCV(1, \gamma) \equiv UCV(\gamma)$ and $SP(1, \gamma) \equiv SP(\gamma)$ were investigated by Ronning in [14]. For $\gamma = 0$, the classes $UCV(\beta, 0) \equiv \beta - UCV$ and $SP(\beta, 0) \equiv \beta - SP$ are defined by Kanas et al. in [8] and [9], respectively.

Further Ahuja et al. [1], Bharathi et al. [2], Murugusundaramurthy et al. [12] and others have studied and investigated interesting properties of the classes $UCV(\beta, \gamma)$ and $SP(\beta, \gamma)$.

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \tag{8}$$

which are regular in the domain $E = \{z \in C : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. Let $\sum_s$, $\sum^*(\alpha)$ and $\sum_k(\alpha), 0 \leq \alpha < 1$, denote the subclasses of functions in $\sum$ that are univalent, meromorphically starlike of order α and meromorphically convex of order α, respectively. $f(z)$ of the form (8) is in $\sum^*(\alpha)$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in E. \tag{9}$$

Similarly, $f \in \sum_k(\alpha)$ if and only if $f(z)$ is of the form (8) and satisfies

$$\mathrm{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha,\ z \in E. \tag{10}$$

If $\alpha = 1$, then $f(z) = \frac{1}{z}$ is the only function in $\sum^*(1)$ and $\sum_k(1)$. The classes $\sum^*(\alpha)$ and $\sum_k(\alpha)$ have been extensively studied by Pommerenke [13], Clunie [3], Royster [16] and others.

Since, to a certain extent, the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of $\sum_s$ that has properties analogous to those of $T^*(\alpha)$. Juneja et al. [7] introduced the class $\sum_p$ of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, a_m \geq 0, \tag{11}$$

$$\Sigma_p^*(\alpha) = \Sigma_p \cap \Sigma^*(\alpha). \tag{12}$$

For functions $f(z)$ in the class $\sum_p$, we define a linear operator D^n as follows:

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= (1-\lambda)f(z) + \lambda\frac{(z^2 f(z))'}{z},\ \lambda \geq 0 \\ &= (1+\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z) \\ D^2 f(z) &= D_\lambda(D^1 f(z)) \\ &\ \vdots \\ D^n f(z) &= D_\lambda(D^{n-1} f(z)) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty}[1+\lambda(m+1)]^n a_m z^m, n \in N_0 = 0, 1, 2, \dots. \end{aligned} \tag{13}$$

The classes $\sum_p^*$ and various other subclasses of $\sum$ were studied by Clunie [3] (see also [11, 13, 16, 20, 21]). Motivated by the works of Madhavi et al. [10], we define the following a new subclass $\sigma_p(\alpha, \beta, \lambda)$ of meromorphically uniformly convex functions in $\sum_p$, by making use of the generalized differential operator.

Definition 1.1. For $-1 \leq \alpha < 1, \lambda > 0$ and $\beta \geq 1$, we let $\sigma_p(\alpha, \beta, \lambda)$ be the subclass of $\sum_p$ consisting of functions of the form (11) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right\} > \beta\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|, \tag{14}$$

where $D^n f(z)$ is given by (13).

The function class $\sigma_p(\alpha, \beta, \lambda)$ contains the class of meromorphic uniformly convex functions with positive coefficients. To illustrate this, we observe that the class $\sigma_p(\alpha, \beta, 1) = \sigma_p(\alpha, \beta)$ was studied by Madhavi et al. [10] and the class $\sigma_p(\alpha, 1, 1) = \sigma_p(\alpha)$ was studied by Thirupathi Reddy et al. [19].

The main object of the paper is to study some usual properties in geometric function theory, such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combinations, convolution properties, integral operators and $\delta-$neighbourhoods for the class $\sigma_p(\alpha, \beta, \lambda)$.

2. COEFFICIENT INEQUALITY

In this section we obtain coefficient bounds for functions f in the class $\sigma_p(\alpha, \beta, \lambda)$.

Theorem 2.1. *A function f of the form* (11) *is in $\sigma_p(\alpha, \beta, \lambda)$, if*

$$\begin{aligned}\sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n [(1 + \beta)(m + 1)\lambda + 1 - \alpha] \; |a_m| \\ \leq \; 1 - \alpha, \; -1 \leq \alpha < 1 \text{ and } \beta \geq 1.\end{aligned} \tag{15}$$

Proof. It suffices to show that

$$\beta \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| - \mathrm{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned}&\beta \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| - \mathrm{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right\} \\ &\leq (1 + \beta) \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| \\ &\leq \frac{(1 + \beta) \sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n (m + 1)\lambda |a_m||z^m|}{\frac{1}{|z|} - \sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n |a_m||z^m|}.\end{aligned}$$

Letting $z \to 1$ along the real axis, we obtain

$$\frac{(1 + \beta) \sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n (m + 1)\lambda |a_m|}{1 - \sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n |a_m|}.$$

The above expression is bounded by $(1 - \alpha)$ if

$$\sum_{m=1}^{\infty} [1 + \lambda(m + 1)]^n [(1 + \beta)(m + 1)\lambda + 1 - \alpha] \; |a_m| \; \leq \; 1 - \alpha.$$

Hence the proof is completed. □

COROLLARY 2.2. *Let the function f defined by* (11) *be in the class $\sigma_p(\alpha,\beta,\lambda)$. Then*

$$a_m \leq \frac{(1-\alpha)}{\sum\limits_{m=1}^{\infty}[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}, \ m \geq 1. \tag{16}$$

Equality holds for the function of the form

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}\ z^m. \tag{17}$$

REMARK 2.3.

(i) For $\lambda = 1$ in Theorem 2.1 and Corollary 2.2, we observe that the coefficient estimates for the functions in the class

$$|a_m| \ \leq \ \frac{(1-\alpha)}{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}$$

are same as those of Madhavi et al. [10] .

(ii) For $\beta = 1$ and $\lambda = 1$ in Theorem 2.1 and Corollary 2.2, we observe that the coefficient estimates of the functions in the class are the same as those of Thirupathi Reddy et al. [19].

3. DISTORTION THEOREMS

In this section we obtain some sharp distortion theorems for the functions of the form (11).

THEOREM 3.1. *Let the function f defined by* (11) *be in the class $\sigma_p(\alpha,\beta,\lambda)$. Then, for $0 < |z| = r < 1$,*

$$\begin{aligned} &\frac{1}{r} - \frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}\ r \leq |f(z)| \\ &\leq \frac{1}{r} + \frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]} r \end{aligned} \tag{18}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}\ z, \quad for \ \ z = r, ir. \tag{19}$$

Proof. Suppose that f is in $\sigma_p(\alpha,\beta,\lambda)$. In view of Theorem 2.1, we have

$$\begin{aligned} &(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]\ \sum_{m=1}^{\infty} a_m \\ &\leq \sum_{m=1}^{\infty}[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha] \leq 1-\alpha, \end{aligned}$$

which yields $\sum\limits_{m=1}^{\infty} a_m \leq \frac{1-\alpha}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}$.
Consequently, we obtain

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \leq \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m \\ &\leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m \\ &\leq \frac{1}{r} + \frac{1-\alpha}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}\ r.\end{aligned}$$

Also,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m \\ &\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \\ &\geq \frac{1}{r} - \frac{1-\alpha}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}\ r.\end{aligned}$$

Hence, (18) follows. □

Theorem 3.2. *Let the function f defined by* (11) *be in the class* $\sigma_p(\alpha, \beta, \lambda)$. *Then, for* $0 < |z| = r < 1$,

$$\begin{aligned}\frac{1}{r^2} - \frac{1-\alpha}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]} &\leq |f'(z)| \\ &\leq \frac{1}{r^2} + \frac{1-\alpha}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}.\end{aligned}$$

The result is sharp, the extremal function being of the form (17)

Proof. From Theorem 2.1, we have

$$\begin{aligned}&(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]\sum_{m=1}^{\infty} m a_m \\ &\leq \sum_{m=1}^{\infty}[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha] \leq 1-\alpha,\end{aligned}$$

which yields

$$\sum_{m=1}^{\infty} m a_m \leq \frac{1-\alpha}{[1+2\lambda]^n[2\lambda(1+\beta)+1-\alpha]}.$$

Consequently, we obtain

$$\begin{aligned}|f'(z)| &\leq \left| \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1} \right| \\ &\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \\ &\leq \frac{1}{r^2} + \frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}.\end{aligned}$$

Also,

$$\begin{aligned}|f'(z)| &\geq \left| \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1} \right| \\ &\geq \frac{1}{r^2} \ - \sum_{m=1}^{\infty} m a_m \\ &\geq \frac{1}{r^2} + \frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+1-\alpha]}.\end{aligned}$$

This completes the proof. □

Remark 3.3.

(i) For the choice of $\lambda = 1$ in Theorems 3.1 and 3.2, we observed that the sharp distortion theorems for the functions in the class,

$$\begin{aligned}\frac{1}{r} - \frac{(1-\alpha)}{3^n[3+2\beta-\alpha]} r \ &\leq |f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)}{3^n[3+2\beta-\alpha]} r \\ \frac{1}{r^2} - \frac{(1-\alpha)}{3^n[3+2\beta-\alpha]} &\leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n[3+2\beta-\alpha]}.\end{aligned}$$

coincide with those of Madhavi et al. [10] .

(ii) For the choice of $\beta = 1$ and $\lambda = 1$ in Theorems 3.1 and 3.2, we observe that the sharp distortion theorems for the functions in the class coincide with those of Thirupathi Reddy et al. [19].

4. CLASS PRESERVING INTEGRAL OPERATORS

In this section we consider the class preserving integral operators of the form (11) .

Theorem 4.1. *Let the function f defined by* (11) *be in the class $\sigma_p(\alpha, \beta, \lambda)$. Then*

$$f(z) = cz^{-c-1} \int_0^z t^c f(t) \mathrm{d}t = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} \ a_m z^m, \ c > 0, \tag{20}$$

is in $\sigma_p(\delta,\beta,\lambda)$, *where*

$$\delta(\alpha,\beta,c,\lambda)=\frac{[2\lambda(1+\beta)+(1-\alpha)](c+2)-c(1-\alpha)[2\lambda(1+\beta)+1]}{[2\lambda(1+\beta)(1-\alpha)](c+2)-(1-\alpha)c}. \tag{21}$$

The result is sharp for $f(z)=\frac{1}{z}+\frac{(1-\alpha)}{(1+2\lambda)^n[2\lambda(1+\beta)+(1-\alpha)]}z$.

Proof. Suppose that $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_p(\alpha,\beta,\lambda)$. We have

$$f(z)=cz^{-c-1}\int\limits_0^z t^c f(t)\mathrm{d}t=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{c}{c+m+1}\ a_m z^m,\ c>0.$$

It is sufficient to show that

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\delta]}{1-\delta}\frac{c}{c+m+1}a_m\le 1. \tag{22}$$

Since f is in $\sigma_p(\alpha,\beta,\lambda)$, we have

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{1-\alpha}|a_m|\le 1. \tag{23}$$

Thus (22) is satisfied, if

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\delta]}{1-\delta}\frac{c}{c+m+1}$$
$$\le\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{1-\alpha}.$$

We obtain

$$\delta\le\frac{[(1+\beta)(m+1)\lambda+1-\alpha](c+m+1)-c[(1+\beta)(m+1)\lambda+1](1-\alpha)}{[(1+\beta)(m+1)\lambda+1-\alpha](c+m+1)-c(1-\alpha)}$$
$$=G(m)$$

A simple computation shows that G is increasing and $G(m)\ge G(1)$. Using this, the result follows. $\square$

5. CONVEX LINEAR COMBINATIONS AND CONVOLUTION PROPERTIES

In this section, we obtain sharp results for f meromorphically convex of order δ and necessary and sufficient conditions for f to be in the class $\sigma_p(\alpha,\beta,\lambda)$. Also we prove that the convolution is in the class $\sigma_p(\alpha,\beta,\lambda)$.

THEOREM 5.1. *If the function* $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty} a_m z^m$ *is in* $\sigma_p(\alpha,\beta,\lambda)$, *then* f *is meromorphically convex of order* $\delta(0\le\delta<1)$ *in* $|z|<r=r(\alpha,\beta,\delta)$, *where* $r(\alpha,\beta,\delta)=\inf\limits_{n\ge1}\left\{\frac{(1-\delta)(m+2)^n[(1+\beta)(1+m)+1-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}$. *The result is sharp.*

Proof. Let $f(z)$ be in $\sigma_p(\alpha,\beta,\lambda)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty}[1+\lambda(m+1)]^n[(1+\beta)(1+m)\lambda+1-\alpha]|a_m| \le (1-\alpha). \tag{24}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \le (1-\delta)$, for all $|z|<r=r(\alpha,\beta,\delta,\lambda)$, where $r(\alpha,\beta,\delta,\lambda)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty} m a_m z^{m-1}}\right| \le \frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty} m a_m|z|^{m+1}}.$$

This is bounded by $(1-\delta)$, if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \le 1. \tag{25}$$

By (24), it follows that (25) is true, if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \le \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{1-\alpha}|a_m|,\ m\ge 1,$$

or

$$|z| \le \left\{\frac{(1-\delta)[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{26}$$

Setting $|z|=r(\alpha,\beta,\delta,\lambda)$ in (26), the result follows. The result is sharp for the function

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}z^m,\ m\ge 1.$$

□

THEOREM 5.2. *Let* $f_0(z)=\frac{1}{z}$ *and* $f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}\cdot z^m,\ m\ge 1$. *Then* $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m$ *is in the class* $\sigma_p(\alpha,\beta,\lambda)$ *if and only if it can be expressed in the form* $f(z)=\omega_0 f_0(z)+\sum\limits_{m=1}^{\infty}\omega_m f_m(z)$, *where* $\omega_0\ge 0$, $\omega_m\ge 0$, $m\ge 1$, *and* $\omega_0+\sum\limits_{m=1}^{\infty}\omega_m=1$.

Proof. Let $f(z)=\omega_0 f_0(z)+\sum\limits_{m=1}^{\infty}\omega_m f_m(z)$ with $\omega_0\geq 0$, $\omega_m\geq 0$, $m\geq 1$, and $\omega_0+\sum\limits_{m=1}^{\infty}\omega_m=1$. Then

$$\begin{aligned} f(z)&=\omega_0 f_0(z)+\sum_{m=1}^{\infty}\omega_m f_m(z)\\ &=\frac{1}{z}+\sum_{m=1}^{\infty}\omega_m\frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}z^m \end{aligned}$$

Since

$$\begin{aligned} &\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}\\ &\cdot\omega_m\frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}\\ &=\sum_{m=1}^{\infty}\omega_m=1-\omega_0\leq 1, \end{aligned}$$

by Theorem 2.1, f is in the class $\sigma_p(\alpha,\beta,\lambda)$.

Conversely, suppose that the function f is in the class $\sigma_p(\alpha,\beta,\lambda)$. Since

$$a_m\leq\frac{(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}z^m, m\geq 1,$$

we have

$$\omega_m=\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}a_m$$

and $\omega_0=1-\sum\limits_{m=1}^{\infty}\omega_m$. It follows that $f(z)=\omega_0 f_0(z)+\sum\limits_{m=1}^{\infty}\omega_m f_m(z)$. This completes the proof of the theorem. □

For the functions $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m$ and $g(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}b_m z^m$ that belong to $\sum_p$, we denote by $(f*g)(z)$ the convolution of $f(z)$ and $g(z)$ defined by

$$(f*g)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m b_m z^m$$

THEOREM 5.3. *If the functions $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m$ and $g(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}b_m z^m$ are in the class $\sigma_p(\alpha,\beta,\lambda)$, then $(f*g)(z)$ is in the class $\sigma_p(\alpha,\beta,\lambda)$.*

Proof. Suppose f and g are in $\sigma_p(\alpha,\beta,\lambda)$. By Theorem 2.1, we have

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}a_m\le 1$$

$$\text{and}\quad \sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}b_m\le 1\ .$$

Since f and g are regular are in E, so is $(f*g)$. Furthermore,

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}a_mb_m\\
&\le\sum_{m=1}^{\infty}\left\{\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}\right\}^2a_mb_m\\
&\le\left(\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}a_m\right)\cdot\\
&\left(\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)\lambda+1-\alpha]}{(1-\alpha)}b_m\right)\\
&\le 1.
\end{aligned}$$

Hence, by Theorem 2.1, $(f*g)$ is in the class $\sigma_p(\alpha,\beta,\lambda)$. □

Remark 5.4.

(i) For $\lambda=1$ in Theorems 5.1, 5.2 and 5.3, we observe that the the results coincide with those of Madhavi et al. [10].

(ii) For $\beta=1$ and $\lambda=1$ in Theorems 5.1, 5.2 and 5.3, we observe that the the results coincide with those of Thirupathi Reddy et al. [19].

6. NEIGHBORHOODS FOR THE CLASS $\sigma_P(\alpha,\beta,\gamma,\lambda)$

In this section, we define the $\delta-$neighborhood of a function f and establish a relation between the $\delta-$neighborhood and the $\sigma_p(\alpha,\beta,\gamma,\lambda)$ class of a function.

Definition 6.1. A function $f\in\sum_p$ is said to be in the class $\sigma_p(\alpha,\beta,\gamma,\lambda)$, if there exists a function $g\in\sigma_p(\alpha,\beta,\lambda))$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<(1-\gamma),\ z\in E,\ 0\le\gamma<1. \tag{27}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4] and Ruschweyh [17], we defined the $\delta-$neighborhood of a function $f\in\sum_p$ by

$$N_\delta(f)=\left\{g\in\Sigma_p\ :\ g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_mz^m\text{ and }\sum_{m=1}^{\infty}m|a_m-b_m|\le\delta\right\}. \tag{28}$$

Theorem 6.2. *If* $g \in \sigma_p(\alpha, \beta, \lambda)$ *and*

$$\gamma = 1 - \frac{\delta[2\lambda(1+\beta)+1-\alpha]}{2\lambda(1+\beta)} \tag{29}$$

then $N_\delta(g) \subset \sigma_p(\alpha, \beta, \gamma, \lambda)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (28) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta, \tag{30}$$

which implies the coefficient inequality: $\sum\limits_{m=1}^{\infty} |a_m - b_m| \leq \delta,\ m \in \mathbb{N}$. Since $g \in \sigma_p(\alpha, \beta, \lambda)$, we have $\sum\limits_{m=1}^{\infty} b_m = \frac{1-\alpha}{2\lambda(1+\beta)+1-\alpha}$. So $\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta[2\lambda(1+\beta)+1-\alpha]}{2\lambda(1+\beta)} = 1 - \gamma$, provided γ is given by (29). Hence, by Definition 6.1, $f \in \sigma_p(\alpha, \beta, \gamma)$ for γ given by (29), which completes the proof of theorem. □

Remark 6.3.
- (i) For $\lambda = 1$ in Theorem 6.2, we observe that the results coincide with those of Madhavi et al. [10] .
- (ii) For $\beta = 1$ and $\lambda = 1$ in Theorem 6.2, we observe that the results coincide with those of Thirupathi Reddy et al. [19].

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