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SOME PROPERTIES OF THE RESOLVENT OF STURM-LIOUVILLE OPERATORS ON UNBOUNDED TIME SCALES

BILENDER P. ALLAHVERDIEV and HÜSEYIN TUNA

Abstract. In this article, we investigate the resolvent operator of Sturm-Liouville problem on unbounded time scales. We obtain integral representations for the resolvent of this operator. Later, we discuss some properties of the resolvent operator, such as Hilbert-Schmidt's kernel property and compactness. Finally, we give a formula for the Titchmarsh-Weyl function of the Sturm-Liouville problem on unbounded time scales.

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1. INTRODUCTION

Recently the study of dynamic equations on time scales has attracted much interest, since it is an unification of the theory of differential equations and the theory of difference equations. The study of dynamic equations on time scales has led to several important applications, e.g. in the study of heat transfer, insect population models, epidemic models, stock market and neural networks (see [2, 13, 18, 25, 26]). However, there are only a few studies concerning spectral problems for operators on time scales.

Motivated by the works mentioned above, we intend in this paper to study the resolvent operator of the Sturm-Liouville problem on unbounded time scales. We obtain integral representations in terms of the spectral function for the resolvent of this operator. Later, we investigate some properties of the resolvent operator, such as the Hilbert-Schmidt kernel property, compactness. Finally, we give a formula for the Titchmarsh-Weyl function of the Sturm-Liouville problem on unbounded time scales. In the classical Sturm-Liouville equation, the integral representation of the resolvent was first proved by H. Weyl in 1910. Similar theorems were proved in [4, 22, 27].

In [17], Huseynov investigated the classical concepts of Weyl limit point and limit circle cases for the second order linear dynamic equations on time

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scales. In [15] and [16], the author proves the existence of a spectral measure for the second-order delta dynamic equation and the one-dimensional Schrödinger equation on a semi-infinite time scale interval. A Parseval equality and an expansion with the eigenfunctions formula are established in terms of the spectral measure. In [12], Guseinov established some expansion results for a Sturm-Liouville problem on semi-bounded time scale intervals. In [28], the author proves the completeness of the system of eigenfunctions for dissipative Sturm-Liouville operators. In [1], Agarwal et al. give an oscillation theorem and establish Rayleigh's principle for Sturm-Liouville eigenvalue problems on time scales with separated boundary conditions. In [6], it is studied the existence of the positive solutions for the second-order superlinear semipositone Sturm-Liouville boundary value problems on general time scales. In [10], the authors obtain a min–max characterization of the eigenvalues of the Sturm-Liouville problems on time scales and various eigenfunction expansions for the functions in a suitable function space. In [14], the author examines Green's function for an nth-order focal boundary value problem on time scales. In [29], it is studied the periodic and antiperiodic boundary value problem for the second-order symmetric linear equation on time scales. By properties of the eigenvalues of the Dirichlet boundary value problem and some oscillation results, existence of eigenvalues of these two different boundary value problems is proved and the number of the eigenvalues is calculated. In [3], we study properties of the spectrum of a Sturm-Liouville operator on time scales.

First, we recall some fundamental concepts on time scales and we refer to [7, 8, 9, 11, 13, 17, 21] for more details.

Definition 1.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \ t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \ t \in \mathbb{T}.$$

It is convenient to have the graininess operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left scattered, if $\mu_\rho(t) \neq 0$, and left dense, if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered, if $\mu_\sigma(t) \neq 0$, and right dense, if $\mu_\sigma(t) = 0$. We introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$, which are derived from the time scale $\mathbb{T}$, as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 1.2. A function f on $\mathbb{T}$ is said to be Δ-*differentiable at some point* $t \in \mathbb{T}^k$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \ s \in U.$$

Analogously, a function f on $\mathbb{T}$ is said to be ∇-differentiable at some point $t \in \mathbb{T}_k$, if there is a number $f^{\nabla}(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|, \ s \in U.$$

One can show (see [8]) that

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)), \ f^{\nabla}(t) = f^{\Delta}(\rho(t))$$

for continuously differentiable functions.

EXAMPLE 1.3. If $\mathbb{T} = \mathbb{R}$, then we have

$$\sigma(t) = t, \ f^{\Delta}(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then we have

$$\sigma(t) = t + 1, \ f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^k : q > 1, \ k \in \mathbb{N}_0\right\}$, then we have

$$\sigma(t) = qt, \ f^{\Delta}(t) = \frac{f(qt) - f(t)}{qt - t}.$$

DEFINITION 1.4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^k$, then F is a Δ-*antiderivative* of f. In this case, the integral is given by the formula

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \ \text{ for } a, b \in \mathbb{T}.$$

Analogously, let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\nabla}(t) = f(t)$, for all $t \in \mathbb{T}_k$, then F is a Δ-*antiderivative* of f. Then we define the ∇-integral, by the formula

$$\int_a^b f(t)\,\nabla t = F(b) - F(a) \ \text{ for } a, b \in \mathbb{T}.$$

Let $L^2_{\nabla}(\mathbb{T})$ be the space of all functions defined on $\mathbb{T}$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2\,\nabla t\right)^{1/2} < \infty.$$

Let $\mathbb{T}$ be a time scale which is bounded from below and unbounded from above such that $\inf \mathbb{T} = a > -\infty$ and $\sup \mathbb{T} = \infty$. We will denote $\mathbb{T}$ also as $[a, \infty)_{\mathbb{T}}$.

The space $L^2_{\nabla}[a, \infty)_{\mathbb{T}}$ is a Hilbert space with the inner product (see [24])

$$(f, g) := \int_a^{\infty} f(t)\,\overline{g(t)}\nabla t, \ f, g \in L^2_{\nabla}[a, \infty)_{\mathbb{T}}.$$

The Wronskian of $y(.), z(.)$ is defined by (see [8])

$$W_t(y, z) := p(t)\left[y(t)\,z^{\Delta}(t) - y^{\Delta}(t)\,z(t)\right], \ t \in \mathbb{T}.$$

2. MAIN RESULTS

We will consider the Sturm-Liouville equation

$$l(y) := -\left[p(t)\,y^{\Delta}(t)\right]^{\nabla} + q(t)\,y(t) = \lambda y(t),\ t \in [a,\infty)_{\mathbb{T}}, \tag{1}$$

with the boundary conditions

$$y(a,\lambda)\cos\beta + p(a)\,y^{\Delta}(a,\lambda)\sin\beta = 0,\ \beta \in \mathbb{R}, \tag{2}$$

$$y(b,\lambda)\cos\alpha + p(b)\,y^{\Delta}(b,\lambda)\sin\alpha = 0,\ \alpha \in \mathbb{R},\ b \in (a,\infty)_{\mathbb{T}}, \tag{3}$$

where p, q are real-valued continuous functions on $\mathbb{T}$ and $p(t) \neq 0$, for all $t \in \mathbb{T}$.

Denote by $\varphi(t,\lambda)$ and $\theta(t,\lambda)$ the solutions of the system (1) subject to the initial conditions

$$\begin{array}{l} \varphi(a,\lambda) = \sin\beta,\ p(a)\,\varphi^{\Delta}(a,\lambda) = -\cos\beta, \\ \theta(a,\lambda) = \cos\beta,\ p(a)\,\theta^{\Delta}(a,\lambda) = \sin\beta. \end{array} \tag{4}$$

We will denote by $\theta(x,\lambda) + m_b(\lambda)\,\varphi(x,\lambda)$ the solution of the equation (1) which satisfy the boundary condition

$$(\theta(b,\lambda) + m_b(\lambda)\,\varphi(b,\lambda))\cos\alpha + (\theta(b,\lambda) + m_b(\lambda)\,\varphi(b,\lambda))\sin\alpha = 0.$$

Then $m_b(\lambda)$ satisfies the relation

$$m_b(\lambda) = -\frac{\theta(b,\lambda)\cot\alpha + \theta(b,\lambda)}{\varphi(b,\lambda)\cot\alpha + \varphi(b,\lambda)}.$$

It is clear that $m_b(\lambda)$ is a meromorphic function of λ, since $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ are entire functions of λ. Furthermore, since the eigenvalues of the regular problem are real, all poles of $m_b(\lambda)$ are real and simple. The function m_b is called the *Titchmarsh-Weyl function* of the regular problem (1)-(3). If $\cot\beta$ is replaced by a complex variable z, then we have

$$m_b(\lambda,z) = -\frac{\theta(b,\lambda)\,z + \theta(b,\lambda)}{\varphi(b,\lambda)\,z + \varphi(b,\lambda)}. \tag{5}$$

For every λ, the equality in (5) is a one-to-one conformal mapping in z, which follows from the theory of Möbius transformations [19]. Hence, if $\operatorname{Im}\lambda \neq 0$, then $m_b(\lambda,z)$ varies on a circle $C_b(\lambda)$ with a finite radius in the m_b-plane as z varies over the real axis of the z-plane.

Using this notation we now state the result from [17].

Theorem 2.1. *Let $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ be two linearly independent solutions of equation (1) satisfying the initial conditions (4). Then the solution*

$$\omega(x,\lambda) = \theta(x,\lambda) + m_b(\lambda)\,\varphi(x,\lambda)$$

satisfies the boundary condition

$$(\theta(b,\lambda) + m_b(\lambda)\,\varphi(b,\lambda))\cos\alpha + (\theta(b,\lambda) + m_b(\lambda)\,\varphi(b,\lambda))\sin\alpha = 0.$$

if and only if $m_b(\lambda)$ *is on* C_b *with*

$$\lim_{b\to\infty} W(\omega,\overline{\omega})(b,\lambda)=0.$$

If $b\to\infty$, *then* C_b *tends either to the limit-circle* C_∞ *or to the limit-point* m_∞. *In the first case, all solutions of the equation (1) are in the space* $L^2_\nabla[a,\infty)_{\mathbb{T}}$. *In the second case, if* $\operatorname{Im}\lambda\neq 0$, *one linearly independent solution is in the space* $L^2_\nabla[a,\infty)_{\mathbb{T}}$. *In the limit-circle case, a point is on* C_∞ *if and only if*

$$\lim_{b\to\infty} W(\omega,\overline{\omega})(b,\lambda)=0.$$

The function $m(\lambda):=\lim_{b\to\infty} m_b(\lambda)$ is called the *Titchmars-Weyl function*, and $\chi(x,\lambda):=\theta(x,\lambda)+m(\lambda)\varphi(x,\lambda)$ is called the *Weyl solution* of the singular equation $l(y)=\lambda y$ satisfying (2).

Let us define

$$\chi_b(t,\lambda)=\theta(t,\lambda)+l(\lambda,b)\varphi(t,\lambda)\in L^2_\nabla[a,b]_{\mathbb{T}},\ b\in(a,\infty)_{\mathbb{T}}.$$

Then we have the following lemma.

Lemma 2.2. *For each nonreal* λ, *we have*

$$\chi_b(t,\lambda)\to\chi(t,\lambda),$$

$$\int_a^b |\chi_b(t,\lambda)|^2\nabla t\to\int_a^\infty |\chi(t,\lambda)|^2\nabla t,\ b\to\infty.$$

Proof. It is clear that

$$\chi(t,\lambda)=\theta(t,\lambda)+m(\lambda)\varphi(t,\lambda)\in L^2_\nabla[a,\infty)_{\mathbb{T}}.$$

In the limit-circle case, $l(\lambda,b)\to m(\lambda)$, so $\chi_b(t,\lambda)\to\chi(t,\lambda)$. Since $\varphi(t,\lambda)\in L^2_\nabla[a,\infty)_{\mathbb{T}}$, we have

$$\int_a^b |\chi_b(t,\lambda)|^2\nabla t\to\int_a^\infty |\chi(t,\lambda)|^2\nabla t,\ b\to\infty.$$

In the limit-point case, according to (16) in [17],

$$|l(\lambda,b)-m(\lambda)|\le r_b(\lambda)=\left(2|v|\int_a^b|\varphi(t,\lambda)|^2\nabla t\right)^{-1},\ \operatorname{Im}\lambda=v\neq 0.$$

Hence, as $r_b(\lambda)\to 0,\ \chi_b(t,\lambda)\to\chi(t,\lambda)$. Furthermore, we have

$$\begin{aligned}\int_a^b |\{l(\lambda,b)-m(\lambda)\}\varphi(t,\lambda)|^2\nabla t&=|l(\lambda,b)-m(\lambda)|^2\int_a^b|\varphi(t,\lambda)|^2\nabla t\\&\le\left(4|v|^2\int_a^b|\varphi(t,\lambda)|^2\nabla t\right)^{-1}.\end{aligned}$$

Therefore, we get

$$\int_a^b |\chi_b(t,\lambda)|^2\nabla t\to\int_a^\infty |\chi(t,\lambda)|^2\nabla t,\ b\to\infty.$$

□

Putting

$$G_b(t,u,\lambda)=\begin{cases}\chi_b(t,\lambda)\,\varphi(u,\lambda), & u\leq t\\ \varphi(t,\lambda)\,\chi_b(u,\lambda), & u>t,\end{cases}$$

$$(R_bf)(t,\lambda)=\int_a^b G_b(t,u,\lambda)\,f(u)\,\nabla u,\ \ \lambda\in\mathbb{C}. \tag{6}$$

Now, we shall show that (6) satisfies the equation $l(y)=\lambda y+f$ and the boundary condition (2). From (6), we get

$$\Psi(t,\lambda)=\chi_b(t,\lambda)\int_a^t\varphi(u,\lambda)\,f(u)\,\nabla u+\varphi(t,\lambda)\int_t^b\chi_b(u,\lambda)\,f(u)\,\nabla u. \tag{7}$$

From (7), it follows that

$$\Psi^{\Delta}(t,\lambda)=\chi_b^{\Delta}(t,\lambda)\int_a^t\varphi(u,\lambda)\,f(u)\,\nabla u+\varphi^{\Delta}(u,\lambda)\int_t^b\chi_b(u,\lambda)\,f(u)\,\nabla u,$$

and

$$\begin{aligned}\left(p(t)\,\Psi^{\Delta}(t,\lambda)\right)^{\nabla}&=\left(p(t)\,\chi_b^{\Delta}(t,\lambda)\right)^{\nabla}\int_a^t\varphi(u,\lambda)\,f(u)\,\nabla u\\&\quad+\left(p(t)\,\varphi^{\Delta}(u,\lambda)\right)^{\nabla}\int_t^b\chi_b(u,\lambda)\,f(u)\,\nabla u-W(\varphi,\chi_b)\,f(t).\end{aligned}$$

Since $W(\varphi,\chi_b)=1$, we have

$$\begin{aligned}-\left(p(t)\,\Psi^{\Delta}(t,\lambda)\right)^{\nabla}&=(\lambda-q(t))\,\chi_b(t,\lambda)\int_a^t\varphi(u,\lambda)\,f(u)\,\nabla u\\&\quad+(\lambda-q(t))\,\varphi(t,\lambda)\int_t^b\chi_b(u,\lambda)\,f(u)\,\nabla u+f(t)\\&=(\lambda-q(t))\,\Psi(t,\lambda)+f(t).\end{aligned}$$

Moreover,

$$\Psi(a,\lambda)=\varphi(a,\lambda)\int_a^b\chi_b(u,\lambda)\,f(u)\,\nabla u=\sin\beta\int_a^b\chi_b(u,\lambda)\,f(u)\,\nabla u,$$

$$\Psi^{\Delta}(a,\lambda)=\varphi^{\Delta}(a,\lambda)\int_a^b\chi_b(u,\lambda)\,f(u)\,\nabla u=-\cos\beta\int_a^b\chi_b(u,\lambda)\,f(u)\,\nabla u,$$

so that $\Psi(t,\lambda)$ satisfies condition (2). Similarly, one can show that $\Psi(t,\lambda)$ satisfies condition (3).

Let $\lambda_{m,b}$ and $\varphi_{m,b}$ $(n\in\mathbb{N}:=\{1,2,3,...\})$ be the eigenvalues and the eigenfunctions of problem (1)-(3) and

$$\alpha_{m,b}^2=\int_a^b\varphi_{m,b}^2(t)\,\nabla t.$$

Now, let us define the nondecreasing step function ϱ_b on $(-\infty, \infty)$ by

$$\varrho_b(\lambda) = \begin{cases} -\sum_{\lambda<\lambda_{m,b}<0} \frac{1}{\alpha_{m,b}^2}, & \text{for } \lambda \leq 0 \\ \sum_{0\leq\lambda_{m,b}<\lambda} \frac{1}{\alpha_{m,b}^2}, & \text{for } \lambda > 0. \end{cases}$$

Let $f(.)$ be an arbitrary function on $L^2_\nabla[a,b]_{\mathbb{T}}$ and

$$\alpha_{m,b}^2 = \int_a^b \varphi_{m,b}^2(t)\,\nabla t \;\; (m \in \mathbb{N}).$$

Then we have

$$\int_a^b |f(t)|^2\,\nabla t = \sum_{m=1}^{\infty} \frac{1}{\alpha_{m,b}^2} \left| \int_a^b f(x)\,\varphi_{m,b}(t)\,\nabla t \right|^2,$$

which is called the *Parseval equality.*

A function f defined on an interval $[c,d]$ is said to be of *bounded variation* if there is a constant $C > 0$ such that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq C$$

for every partition

$$c = x_0 < x_1 < \ldots < x_n = d$$

of $[c,d]$ by points of the subdivision $x_0, x_1, \ldots, x_n$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[c,d]$, denoted by $\overset{d}{\underset{c}{V}}(f)$, we mean the quantity

$$\overset{d}{\underset{c}{V}}(f) := \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions of the interval $[c,d]$ (see [20]).

Lemma 2.3. *For any positive κ, there is a positive constant $\Upsilon = \Upsilon(\kappa)$ not depending on b such that*

$$\overset{\kappa}{\underset{-\kappa}{V}}\{\varrho_b(\lambda)\} = \sum_{-\kappa\leq\lambda_{m,b}<\kappa} \frac{1}{\alpha_{m,b}^2} = \varrho_b(\kappa) - \varrho_b(-\kappa) < \Upsilon. \tag{8}$$

Proof. Let $\sin\beta \neq 0$. Since $\varphi(t,\lambda)$ is continuous on the region

$$\{(t,\lambda) : -\kappa \leq \lambda \leq \kappa, \; t \in [a,\infty)_{\mathbb{T}}\},$$

by condition $\varphi(a,\lambda) = \sin\beta$, there is a positive number h close to a such that

$$\left(\frac{1}{h}\int_a^h y(t,\lambda)\,\nabla t\right)^2 > \frac{1}{2}\sin^2\beta. \tag{9}$$

Let us define $f_h(x)$ by

$$f_h(x) = \begin{cases} \frac{1}{h}, & a \leq t \leq h \\ 0, & t > h. \end{cases}$$

From (8) and (9), we get

$$\int_a^h f_h^2(x)\,\nabla t = \frac{1}{h} = \int_{-\infty}^{\infty} \left(\frac{1}{h} \int_0^h y(t,\lambda)\,\nabla t \right)^2 \mathrm{d}\varrho_b(\lambda)$$

$$\geq \int_{-\kappa}^{\kappa} \left(\frac{1}{h} \int_a^h y(t,\lambda)\,\nabla t \right)^2 \mathrm{d}\varrho_b(\lambda)$$

$$> \frac{1}{2}\sin^2\beta \left\{ \varrho_b(\kappa) - \varrho_b(-\kappa) \right\},$$

which proves inequality (8).

If $\sin\beta = 0$, then we define the function $f_h(x)$ by the formula

$$f_h(x) = \begin{cases} \frac{1}{h^2}, & a \leq t \leq h \\ 0, & t > h. \end{cases}$$

□

Now, we will obtain an expansion into a Fourier series of resolvent, if one knows the expansion of the function $f(t)$. By integration by parts, we find

$$\begin{aligned} &\int_a^b \left[-\left[p(t)\,y^{\Delta}(t,\lambda)\right]^{\nabla} + q(t)\,y(t,\lambda) \right] \varphi_{m,b}(t)\,\nabla t \\ &= \int_a^b \left[-\left[p(t)\,\varphi_{m,b}^{\Delta}(t)\right]^{\nabla} + q(t)\,\varphi_{m,b}(t) \right] y(t,\lambda)\,\nabla t \\ &= -\lambda_{m,b} \int_a^b y(t,\lambda)\,\varphi_{m,b}(t)\,\nabla t = -\lambda_{m,b}\gamma_m(\lambda). \end{aligned} \tag{10}$$

Set

$$y(t,\lambda) = \sum_{m=1}^{\infty} \gamma_m(\lambda)\,\varphi_{m,b}(t),\ c_m = \int_a^b f(t)\,\varphi_{m,b}(t)\,\nabla t.$$

Since $y(t,\lambda)$ satisfies the equation

$$-\left[p(t)\,y^{\nabla}(t,\lambda)\right]^{\Delta} + (q(t) - \lambda)\,y(t,\lambda) = f(t),$$

we get

$$\begin{aligned} c_m &= \int_a^b \left[-\left[p(t)\,y^{\Delta}(t,\lambda)\right]^{\nabla} + (q(t) - \lambda)\,y(t,\lambda) \right] \varphi_{m,b}(t)\,\nabla t \\ &= -\lambda_{m,b}\gamma_m(\lambda) + \lambda\gamma_m(\lambda). \end{aligned}$$

Then we obtain

$$\gamma_m(\lambda) = \frac{c_m}{\lambda - \lambda_{m,b}},$$

and

$$y(t,\lambda) = \int_a^b G_b(t,u,\lambda) f(u) \nabla u = \sum_{m=1}^{\infty} \frac{c_m}{\lambda - \lambda_{m,b}} \varphi_{m,b}(t).$$

Hence, the expansion of the resolvent is

$$(R_b f)(t,z) = \sum_{m=1}^{\infty} \frac{\varphi_{m,b}(t) \int_a^b f(u) \varphi_{m,b}(u) \nabla u}{\alpha_{m,b}^2 (z - \lambda_{m,b})} \tag{11}$$

$$= \int_{-\infty}^{\infty} \frac{\varphi(t,\lambda)}{z-\lambda} \left\{ \int_a^b f(u) \varphi(u,\lambda) \nabla u \right\} \mathrm{d}\varrho_b(\lambda). \tag{12}$$

Lemma 2.4. *Let z be a non real number and t be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(t,\lambda)}{z-\lambda} \right|^2 \mathrm{d}\varrho_b(\lambda) < K. \tag{13}$$

Proof. Putting $f(u) = \frac{\varphi_{m,b}(u)}{\alpha_{m,b}}$ in (11), we get

$$\frac{1}{\alpha_{m,b}} \int_a^b G_b(t,u,z) \varphi_{m,b}(u) \nabla u = \frac{\varphi_{m,b}(t)}{\alpha_{m,b}(z - \lambda_{m,b})}, \tag{14}$$

since the eigenfunctions $\varphi_{m,b}(t)$ are orthogonal. Using (14), if we apply the Parseval equality to $G_b(t,u,z)$, we have

$$\int_a^b |G_b(t,u,z)|^2 \nabla u = \sum_{m=1}^{\infty} \frac{|\varphi_{m,b}(t)|^2}{\alpha_{m,b}^2 |z - \lambda_{m,b}|^2} = \int_{-\infty}^{\infty} \left| \frac{\varphi(t,\lambda)}{z-\lambda} \right|^2 \mathrm{d}\varrho_b(\lambda).$$

Since the last integral is convergent by Lemma 2.2, the statement of the lemma follows. □

Now, we recall the following well-known theorems of Helly.

Theorem 2.5 ([20]). *Let $(w_n)_{n\in\mathbb{N}}$ be an uniformly bounded sequence of real nondecreasing functions on a finite interval $c \le \lambda \le d$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \ c \le \lambda \le d.$$

Theorem 2.6 ([20]). *Assume $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing functions on a finite interval $c \le \lambda \le d$, and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda), \ c \le \lambda \le d.$$

If f is any continuous function on $c \le \lambda \le d$, then

$$\lim_{n\to\infty} \int_c^d f(\lambda) dw_n(\lambda) = \int_c^d f(\lambda) dw(\lambda).$$

By Lemma 2.3, the set$\{\varrho_b(\lambda)\}$ is bounded. Using Theorems 2.5 and 2.6, we can find a sequence $\{b_k\}$ such that the function $\varrho_{b_k}(\lambda)$ $(b_k\to\infty)$ converges to a monotone function $\varrho(\lambda)$. Then we have the next lemma.

LEMMA 2.7. *Let z be a non real number and x be a fixed number. Then we have*

$$\int_{-\infty}^{\infty}\left|\frac{\varphi(t,\lambda)}{z-\lambda}\right|^2 \mathrm{d}\varrho(\lambda)\le K. \tag{15}$$

Proof. By the inequality (13), for arbitrary $\eta>0$, we have

$$\int_{-\eta}^{\eta}\left|\frac{\varphi(t,\lambda)}{z-\lambda}\right|^2 \mathrm{d}\varrho_b(\lambda)< K.$$

Letting $\eta\to\infty$ and $b\to\infty$, we get the desired result. □

LEMMA 2.8. *For arbitrary $\eta>0$, we have the following inequalities.*

$$\int_{-\infty}^{-\eta}\frac{\mathrm{d}\varrho(\lambda)}{|z-\lambda|^2}<\infty,\ \int_{\eta}^{\infty}\frac{\mathrm{d}\varrho(\lambda)}{|z-\lambda|^2}<\infty. \tag{16}$$

Proof. Let $\sin\beta\neq 0$. Then, if we put $t=a$ in (15), we get

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\varrho(\lambda)}{|z-\lambda|^2}<\infty.$$

Let $\sin\beta=0$, then we have

$$\frac{1}{\alpha_{m,b}}\int_a^b \Delta_t G_b(t,u,z)\,\varphi_{m,b}(u)\,\nabla u=\frac{\Delta_t\varphi_{m,b}(t)}{\alpha_{m,b}(z-\lambda_{m,b})}.$$

By the Parseval equality,

$$\int_a^b |\Delta_t G_b(t,u,z)|^2\,\nabla u=\int_{-\infty}^{\infty}\left|\frac{\Delta_t\varphi(t,\lambda)}{z-\lambda}\right|^2 \mathrm{d}\varrho_b(\lambda).$$

Proceeding similarly, we have the desired result. □

LEMMA 2.9. *Let $f(.)\in L^2_\nabla[a,\infty)_{\mathbb{T}}$, and let*

$$(Rf)(t,z)=\int_a^\infty G(t,u,z)\,f(u)\,\nabla u,$$

where

$$G(t,u,z)=\begin{cases}\chi(t,z)\,\varphi(u,z), & u\le t\\ \varphi(t,z)\,\chi(u,z), & u>x.\end{cases}$$

Then

$$\int_a^\infty |(Rf)(t,z)|^2\,\nabla t\le\frac{1}{s^2}\int_a^\infty |f(t)|^2\,\nabla t,\ z=x+is.$$

Proof. For each $b > a$, it follows from (11) and the Parseval equality that

$$\int_a^b |(R_b f)(t,z)|^2 \nabla t = \sum_{m=1}^{\infty} \frac{1}{\alpha_{m,b}^2 |z-\lambda_{m,b}|^2} \left\{ \int_a^b f(u) \varphi_{m,b}(u) \nabla u \right\}^2$$

$$\le \frac{1}{s^2} \sum_{m=1}^{\infty} \frac{1}{\alpha_{m,b}^2} \left\{ \int_a^b f(u) \varphi_{m,b}(u) \nabla u \right\}^2 = \frac{1}{s^2} \int_a^b |f(u)|^2 \nabla u.$$

Letting $b \to \infty$, we get the desired result. □

Now, we will obtain the integral representations for the resolvent.

THEOREM 2.10. *For every nonreal z and for each $f(.) \in L_\nabla^2[a,\infty)_{\mathbb{T}}$, one has the following equality*

$$(Rf)(t,z) = \int_{-\infty}^{\infty} \frac{\varphi(t,\lambda)}{z-\lambda} F(\lambda)\, \mathrm{d}\varrho(\lambda), \tag{17}$$

where $F(\lambda) = \lim_{\xi\to\infty} \int_a^\xi f(t)\varphi(t,\lambda)\nabla t$.

Proof. Let the function $f_\xi(t)$ vanish outside the interval $[a,\xi]_{\mathbb{T}},\ \xi < b$, and satisfies the boundary condition (2) and let ς be an arbitrary positive number. Set

$$F_\xi(\lambda) = \int_a^\xi f(t)\varphi(t,\lambda)\nabla t.$$

From (12), we get

$$\begin{aligned} (R_b f_\xi)(t,z) &= \int_{-\infty}^{\infty} \frac{\varphi(t,\lambda)}{z-\lambda} F_\xi(\lambda)\, \mathrm{d}\varrho_b(\lambda) = \int_{-\infty}^{-\varsigma} \frac{\varphi(t,\lambda)}{z-\lambda} F_\xi(\lambda)\, \mathrm{d}\varrho_b(\lambda) \\ &+ \int_{-\varsigma}^{\varsigma} \frac{\varphi(t,\lambda)}{z-\lambda} F_\xi(\lambda)\, \mathrm{d}\varrho_b(\lambda) + \int_{\varsigma}^{\infty} \frac{\varphi(t,\lambda)}{z-\lambda} F_\xi(\lambda)\, \mathrm{d}\varrho_b(\lambda) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{18}$$

Now, we will estimate I_1. By (11), we get

$$\begin{aligned} I_1 &= \int_{-\infty}^{-\varsigma} \frac{\varphi(t,\lambda)}{z-\lambda} F_\xi(\lambda)\, \mathrm{d}\varrho_b(\lambda) \\ &= \sum_{\lambda_{k,b}<-\varsigma} \frac{\varphi_{k,b}(t) \int_a^\xi f_\xi(u)\varphi_{k,b}(u)\nabla u}{\alpha_{k,b}^2 (z-\lambda_{k,b})} \\ &\le \left(\sum_{\lambda_{k,b}<-\varsigma} \frac{\varphi_{k,b}^2(t)}{\alpha_{k,b}^2 |z-\lambda_{k,b}|^2} \right)^{1/2} \\ &\times \left(\sum_{\lambda_{k,b}<-\varsigma} \frac{1}{\alpha_{k,b}^2} \left[\int_a^\xi f_\xi(t)\varphi_{k,b}(t)\nabla t \right]^2 \right)^{1/2}. \end{aligned} \tag{19}$$

By integration by parts, we find

$$\begin{aligned}
&\int_a^{\xi} f_{\xi}(t)\,\varphi_{k,b}(t)\,\nabla t \\
&= -\frac{1}{\lambda_{k,b}}\int_a^{\xi} f_{\xi}(t)\left\{-\left[p(t)\,\varphi_{k,b}^{\Delta}(t)\right]^{\nabla}+q(t)\,\varphi_{k,b}(t)\right\}\nabla t \\
&= -\frac{1}{\lambda_{k,b}}\int_a^{\xi}\left\{-\left[p(t)\,f_{\xi}^{\Delta}(t)\right]^{\nabla}+q(t)\,f_{\xi}(t)\right\}\varphi_{k,b}(t)\,\nabla t.
\end{aligned} \tag{20}$$

By Lemma 2.4, we have

$$I_1 \leq \frac{K^{1/2}}{\varsigma}\left(\sum_{\lambda_{k,b}<-\varsigma}\frac{1}{\alpha_{k,b}^2}\left[\int_a^{\xi}\left\{\begin{array}{c}-\left[p(t)\,f_{\xi}^{\Delta}(t)\right]^{\nabla}\\ +q(t)\,f_{\xi}(t)\end{array}\right\}\varphi_{k,b}(t)\,\nabla t\right]^2\right)^{1/2}.$$

Using Bessel inequality, we get

$$I_1 \leq \frac{K^{1/2}}{\varsigma}\left[\int_a^{\xi}\left\{-\left[p(t)\,f_{\xi}^{\Delta}(t)\right]^{\nabla}+q(t)\,f_{\xi}(t)\right\}^2\nabla t\right]^{1/2}=\frac{C}{\varsigma}.$$

By a similar method, one can prove that $I_3 \leq \frac{C}{\varsigma}$. Then I_1 and I_3 tend to zero as $\varsigma \to \infty$, uniformly in b. Using Theorems 2.5 and 2.6 in (18), we obtain

$$(Rf_{\xi})(t,z)=\int_{-\infty}^{\infty}\frac{\varphi(t,\lambda)}{z-\lambda}F_{\xi}(\lambda)\,\mathrm{d}\varrho(\lambda). \tag{21}$$

As it is known, if $f(.) \in L^2_{\nabla}[a,\infty)_{\mathbb{T}}$, then one can find a sequence $\{f_{\xi}(t)\}$ which satisfies the previous conditions and tends to $f(t)$, as $\xi \to \infty$. From the Parseval equality, the sequence of Fourier transforms converges to the transform of $f(t)$. By Lemmas 2.4 and 2.9, we can pass to the limit $\xi \to \infty$ in (21). So, we obtain the assertion of the theorem. □

Remark 2.11. Using Theorem 2.10, we can obtain the following formula

$$\int_a^{\infty}(Rf)(t,z)\,g(t)\,\nabla t=\int_{-\infty}^{\infty}\frac{F(\lambda)\,G(\lambda)}{z-\lambda}\mathrm{d}\varrho(\lambda), \tag{22}$$

where

$$\begin{aligned}
F(\lambda) &= \lim_{\xi\to\infty}\int_a^{\xi} f(t)\,\varphi(t,\lambda)\,\nabla t, \\
G(\lambda) &= \lim_{\xi\to\infty}\int_0^{\xi} g(t)\,\varphi(t,\lambda)\,\nabla t.
\end{aligned}$$

Now, we will prove that the resolvent operator is compact. Hence, we need the following definition and theorems.

DEFINITION 2.12. A complex-valued function $M(t,u)$ of two variables with $c<t<d$ and $c<u<d$ is called the *Hilbert-Schmidt kernel on time scales* if

$$\int_c^d \int_c^d |M(t,u)|^2 \nabla t \nabla u < +\infty.$$

THEOREM 2.13 ([23]). *If*

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty \tag{23}$$

then the operator A defined by the formula $A\{x_i\} = \{y_i\}$, where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \ \ i \in \mathbb{N}, \tag{24}$$

is compact in the sequence space l^2.

THEOREM 2.14. *Let for equation (1) the limit-circle case hold. Then $G(t,u)$ defined by the formula*

$$G(t,u) = G(t,u,0) = \begin{cases} \chi(t)\varphi(u), & u \le t \\ \varphi(t)\chi(u), & t < u, \end{cases} \tag{25}$$

is a Hilbert-Schmidt kernel on time scales.

Proof. By the upper half of formula (25), we have

$$\int_a^\infty \nabla t \int_a^t |G(t,u)|^2 \nabla u < +\infty$$

and, by the lower half of (25), we have

$$\int_a^\infty \nabla t \int_t^\infty |G(t,u)|^2 \nabla u < +\infty,$$

since the inner integral exists and is a product $\varphi(t)\chi(s)$ and these products belong to $L^2_\nabla[a,\infty)_\mathbb{T} \times L^2_\nabla[a,\infty)_\mathbb{T}$, because each of the factors belongs to $L^2_\nabla[a,\infty)_\mathbb{T}$. Then we obtain

$$\int_a^\infty \int_a^\infty |G(t,u)|^2 \nabla t \nabla u < +\infty. \tag{26}$$

□

THEOREM 2.15. *Under the condition of Theorem 2.14, the operator R defined by the formula*

$$(Rf)(t) = \int_a^\infty G(t,u) f(u) \nabla u$$

is compact.

Proof. Let $\phi_i = \phi_i(u)$ $(i \in \mathbb{N})$ be a complete orthonormal basis of the space $L^2_\nabla[a,\infty)_\mathbb{T}$. Since $G(t,u)$ is a Hilbert-Schmidt kernel on time scales, we can define

$$x_i = (f,\phi_i) = \int_a^\infty f(u)\overline{\phi_i(u)}\nabla u,$$
$$y_i = (g,\phi_i) = \int_a^\infty g(u)\overline{\phi_i(u)}\nabla u,$$
$$a_{ik} = \int_a^\infty \int_a^\infty G(t,u)\overline{\phi_i(t)\phi_k(u)}\nabla t\nabla u,\ i,k \in \mathbb{N}.$$

Then $L^2_\nabla[a,\infty)_\mathbb{T}$ is mapped isometrically to l^2. Consequently, our integral operator becomes the operator defined by formula (24) in the space l^2, by this mapping and condition (26), which is translated into condition (23). By Theorem 2.15, this operator is compact. Therefore, the original operator is compact. □

Now, we will show that the Weyl-Titchmarsh function $m(\lambda)$ and the spectral function $\varrho(\lambda)$ are closely related.

THEOREM 2.16. *For all nonreal values of* λ*, there exists a solution*

$$\chi(t,\lambda) = \theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)$$

of (1) *such that* $\chi(t,\lambda) \in L^2_\nabla[a,\infty)_\mathbb{T}$.

Proof. See [17]. □

LEMMA 2.17. *For any fixed nonreal* λ *and* λ', *the equalities*

$$\lim_{t\to\infty} W\left(\chi(t,\lambda),\chi\left(t,\lambda'\right)\right) = 0, \tag{27}$$

$$\int_a^\infty \chi(t,\lambda)\chi\left(t,\lambda'\right)\nabla t = \frac{m(\lambda)-m(\lambda')}{\lambda'-\lambda} \tag{28}$$

hold.

Proof. Since the function $\theta(t,\lambda) + l(\lambda)\varphi(t,\lambda)$ satisfies the boundary condition (3), we get

$$W_b\left\{\theta(t,\lambda) + l(\lambda)\varphi(t,\lambda), \theta\left(t,\lambda'\right) + l\left(\lambda'\right)\varphi\left(t,\lambda'\right)\right\} = 0.$$

Hence,

$$W_b\{\chi(t,\lambda) + (l(\lambda)-m(\lambda))\varphi(t,\lambda),$$
$$\chi\left(t,\lambda'\right) + \left(l\left(\lambda'\right)-m\left(\lambda'\right)\right)\varphi\left(t,\lambda'\right)\} = 0,$$

i.e.

$$\begin{aligned}&W_b\left\{\chi(t,\lambda),\chi\left(t,\lambda'\right)\right\} + (l(\lambda)-m(\lambda))W_b\left\{\varphi(t,\lambda),\chi\left(t,\lambda'\right)\right\}\\ &+\left(l\left(\lambda'\right)-m\left(\lambda'\right)\right)W_b\left\{\chi(t,\lambda),\varphi\left(t,\lambda'\right)\right\}\\ &+(l(\lambda)-m(\lambda))\left(l\left(\lambda'\right)-m\left(\lambda'\right)\right)W_b\left\{\varphi(t,\lambda),\varphi\left(t,\lambda'\right)\right\} = 0.\end{aligned} \tag{29}$$

On the other hand, we know that (see [17])

$$W_b\left\{\varphi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\}-W_a\left\{\varphi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\} \\ =\left(\lambda'-\lambda\right)\int_a^b\varphi\left(t,\lambda\right)\chi\left(t,\lambda'\right)\nabla t. \tag{30}$$

By Lemma 2.2, as $b\to\infty$,

$$W_b\left\{\varphi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\}=O\left(\int_a^b\left|\varphi\left(t,\lambda\right)\right|^2\nabla t\right)+O\left(1\right).$$

In the limit-point case,

$$\left|l\left(\lambda\right)-m\left(\lambda\right)\right|\leq r_b\left(\lambda\right)=\left(2\left|v\right|\int_a^b\left|\varphi\left(t,\lambda\right)\right|^2\nabla t\right)^{-1},\ \ \operatorname{Im}\lambda=v\neq0,$$

such that

$$\lim_{b\to\infty}\left|l\left(\lambda\right)-m\left(\lambda\right)\right|W_b\left\{\varphi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\}=0.$$

Since the integral $\int_a^b\left|\varphi\left(t,\lambda\right)\right|^2\nabla t$ remains bounded, this also occurs in the limit-circle case, if $l\left(\lambda\right)\to m\left(\lambda\right)$. In (27), the other addends are estimated similarly.

Now, we will prove equality (28). From (30), we have

$$W_a\left\{\chi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\}-W_b\left\{\chi\left(t,\lambda\right),\chi\left(t,\lambda'\right)\right\} \\ =\left(\lambda'-\lambda\right)\int_a^b\chi\left(t,\lambda\right)\chi\left(t,\lambda'\right)\nabla t. \tag{31}$$

By condition (4), we find that the first term on the left is equal to

$$\left\{\cos\beta+m\left(\lambda\right)\sin\beta\right\}\left\{\sin\beta-m\left(\lambda'\right)\cos\beta\right\} \\ -\left\{\cos\beta+m\left(\lambda'\right)\sin\beta\right\}\left\{\sin\beta-m\left(\lambda\right)\cos\beta\right\} \\ =m\left(\lambda\right)-m\left(\lambda'\right).$$

If we pass to the limit $b\to\infty$ in (31), we get the desired result. □

In particular, if we take $\lambda=u+iv$ and $\lambda'=\overline{\lambda}$ in (28), we get

$$\int_a^\infty\left|\chi\left(t,\lambda\right)\right|^2\nabla t=-\frac{\operatorname{Im}\left\{m\left(\lambda\right)\right\}}{v}. \tag{32}$$

Lemma 2.18. *For fixed u_1 and u_2, we have*

$$\int_{u_1}^{u_2}-\operatorname{Im}\left\{m\left(u+\mathrm{i}\delta\right)\right\}\mathrm{d}u=O\left(1\right),\ \ \textit{as }\delta\to0. \tag{33}$$

Proof. Let $\sin\beta\neq0$. Using the Parseval equality and (17) for $t=0$, we obtain

$$\int_a^\infty\left|\chi\left(y,z\right)\right|^2\nabla t=\int_{-\infty}^\infty\frac{\mathrm{d}\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+v^2},\ \ z=u+iv. \tag{34}$$

Let $\sin\beta = 0$. Then, we will prove that the Fourier transform of the function $\Delta_t G_b(t, y, z)$ is $\frac{\Delta_t \varphi(t,\lambda)}{z-\lambda}$. It follows from (14), if the equality is differentiated with respect to t and the limit is taken as $b \to \infty$. Therefore, formula (34) is obtained.

From (32) and (34), we get

$$-\operatorname{Im}\{m(u + \mathrm{i}\delta)\} = \delta \int_{-\infty}^{\infty} \frac{\mathrm{d}\varrho(\lambda)}{(u-\lambda)^2 + \delta^2}.$$

So, we have

$$\int_{u_1}^{u_2} -\operatorname{Im}\{m(u + \mathrm{i}\delta)\}\,\mathrm{d}u = \delta \int_{u_1}^{u_2} \mathrm{d}u \int_{-\infty}^{\infty} \frac{\mathrm{d}\varrho(\lambda)}{(u-\lambda)^2 + \delta^2}.$$

Let (c, d), $(c < u_1 < u_2 < d)$ be a finite interval. Then, by (16), we have

$$\delta \int_{u_1}^{u_2} \mathrm{d}u \int_{-\infty}^{c} \frac{\mathrm{d}\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = O(1),$$

$$\delta \int_{u_1}^{u_2} \mathrm{d}u \int_{d}^{\infty} \frac{\mathrm{d}\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = O(1).$$

Hence, we get

$$\delta \int_{u_1}^{u_2} \mathrm{d}u \int_{c}^{d} \frac{\mathrm{d}\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = \int_{c}^{d} \mathrm{d}\varrho(\lambda) \int_{\frac{u_1-\lambda}{\delta}}^{\frac{u_2-\lambda}{\delta}} \frac{dv}{1+v^2} = O(1).$$

□

Now, we recall the Stieltjes inversion formula. Let $\sigma(\lambda) = \sigma_1(\lambda) + i\sigma_2(\lambda)$ be a complex function of bounded variation on the entire line. We put

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{z-\lambda},\ \psi(\sigma,\tau) = \frac{\operatorname{sgn}\tau}{\pi}\frac{\varphi(z) - \varphi(\overline{z})}{2i}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|\, d\sigma(\lambda)}{(\lambda-\sigma)^2 + \tau^2},\ z = \sigma + i\tau.$$

Theorem 2.19 ([22]). *If the points c, d are points of continuity of $\sigma(\lambda)$, then we have*

$$\sigma(d) - \sigma(c) = -\lim_{\tau\to 0}\int_{c}^{d} \psi(\sigma,\tau)\, d\sigma.$$

Theorem 2.20. *Let the end points of the interval $\Lambda = (\lambda, \lambda + \Lambda)$ be the points of continuity of the function $\varrho(\lambda)$. Then we have*

$$\varrho(\lambda + \Lambda) - \varrho(\lambda) = -\frac{1}{\pi}\lim_{\delta\to 0}\int_{\Lambda} \operatorname{Im}\{m(u + \mathrm{i}\delta)\}\,\mathrm{d}u. \tag{35}$$

Proof. Let $f(.), g(.) \in L^2_\nabla[a,\infty)_{\mathbb{T}}$ vanish outside a finite interval. Using (22), we get

$$\Psi(\lambda) = \int_{a}^{\infty} (Rf)(t,z)\, g(t)\, \nabla t = \int_{-\infty}^{\infty} \frac{F(\lambda)\, G(\lambda)}{z-\lambda} \mathrm{d}\varrho(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(\lambda)}{z-\lambda},$$

where

$$\mu(\Lambda) = \int_{\Lambda} F(\lambda)\, G(\lambda)\, \mathrm{d}\varrho(\lambda).$$

By Theorem 2.19, we obtain

$$\mu(\Lambda) = -\frac{1}{\pi} \lim_{\delta \to 0} \int_{\Lambda} \operatorname{Im}\{\Psi(u + \mathrm{i}\delta)\}\, \mathrm{d}u. \tag{36}$$

Furthermore, we have

$$\begin{aligned} \operatorname{Im}\{\Psi(u+\mathrm{i}\delta)\} &= \int_a^{\infty} g(t)\, \nabla t \\ &\times \operatorname{Im}\{\int_a^t [\theta(t, u+\mathrm{i}\delta) + m(u+\mathrm{i}\delta)\, \varphi(t, u+\mathrm{i}\delta)]\, \varphi(s, u+\mathrm{i}\delta)\, f(s)\, \nabla s \\ &+ \int_t^{\infty} [\theta(s, u+\mathrm{i}\delta) + m(u+\mathrm{i}\delta)\, \varphi(s, u+\mathrm{i}\delta)]\, \varphi(t, u+\mathrm{i}\delta)\, f(s)\, \nabla s\}, \end{aligned}$$

where $\varphi(t,u)$, $\theta(t,u)$, $f(t)$ and $g(t)$ are real-valued functions. By Lemma 2.18 and relation (36), we get

$$\mu(\Lambda) = -\frac{1}{\pi} \lim_{\delta \to 0} \int_{\Lambda} \operatorname{Im}\{m(u+\mathrm{i}\delta)\}\, F(u)\, G(u)\, \mathrm{d}u. \tag{37}$$

If we choose $f(t)$ and $g(t)$ conveniently, we can make $F(u)$ and $G(u)$ differ from the unity in the fixed interval Λ. So, (35) follows from Lemma 2.18 and relation (37). □

Theorem 2.21. *For any nonreal z, we have the formula*

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{\mathrm{d}\varrho(\lambda)}{z - \lambda}. \tag{38}$$

Proof. Since $f(t)$ is arbitrary, from (17), we get

$$G(t, s, z) = \int_{-\infty}^{\infty} \frac{\varphi(t, \lambda)\, \varphi(s, \lambda)\, \mathrm{d}\varrho(\lambda)}{z - \lambda}. \tag{39}$$

But, by definition,

$$G(t,s,z) = \begin{cases} [\theta(t,z) + m(z)\, \varphi(t,z)]\, \varphi(s,z), & s \le t \\ [\theta(s,z) + m(z)\, \varphi(s,z)]\, \varphi(t,z), & s > t. \end{cases}$$

Then it follows from conditions (4) and (39) that

$$G(0,0,z) = \{\cos\beta + m(z)\sin\beta\}\sin\beta = \int_{-\infty}^{\infty} \frac{\sin^2\beta}{z - \lambda} \mathrm{d}\varrho(\lambda),$$

i.e.

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{\mathrm{d}\varrho(\lambda)}{z - \lambda}.$$

□

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Süleyman Demirel University
Department of Mathematics
32260 Isparta, Turkey
E-mail: bilenderpasaoglu@sdu.edu.tr

Mehmet Akif Ersoy University
Department of Mathematics
15030 Burdur, Turkey
E-mail: hustuna@gmail.com