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NEW CHARACTERIZATION FOR AN INEQUALITY

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Abstract. A new characterization for the Hardy inequality for the sum of two Hardy-type integral operators is obtained between suitable weighted Lebesgue spaces, for certain ranges of indices.

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1. INTRODUCTION

The Hardy inequality for the sum of the two Hardy-type integral operators between suitable weighted Lebesgue spaces has been characterized in dimension one in [4] (see also [1, Remark 2.4]) and in dimension two in [2]. Motivated by this, in Section 2, we give a new condition for the $L^p_v - L^q_u$ boundedness for the sum of the two Hardy-type operators, for the case $1 < p \leq q < \infty$, which is different from the corresponding results given in [1, 4].

Throughout the paper, u and v are weight functions, that is measurable, positive almost everywhere, in an appropriate interval, functions, $\chi_{(n,n+1)}$ is the characteristic function defined on $(n, n+1)$, f is a measurable function, $1 < p \leq q < \infty$, $p' = p/(p-1)$ is the conjugate to p and the same is true for the other indices and L^p_v is a weighted Lebesgue space.

2. MAIN RESULT

Consider the operator

$$(Sf)(x) = \phi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t) \mathrm{d}t + \phi_2(x) \int_{x}^{\infty} \psi_2(t) f(t) \mathrm{d}t,$$

where ϕ_i, ψ_i. $i = 1, 2$, are non-zero measurable functions not necessarily non-negative and f is a measurable function. The boundedness of S between weighted Lebesgue spaces in dimension one has been considered in [4], for the case $1 < p, q < \infty$, and, in [1, Remark 2.4], for the case $1 < p < \infty$, $0 < q < \infty$. Corresponding results in dimension two have been given in [2]. The characterizations given in [1, 4] are of Muckenhoupt type.

Motivated by these results, in this section, we give a new condition for the $L^p_v - L^q_u$ boundedness of S, for the case $1 < p \leq q < \infty$, which is of non-Muckenhoupt type and is different from those given in [1, 4].

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More precisely, we prove the following.

THEOREM 2.1. *The inequality*

$$\|Sf\|_{q,u} \leq C\|f\|_{p,v} \tag{1}$$

exists for a constant $C > 0$, $f \in L_v^p$ *and* $1 < p \leq q < \infty$ *if and only if* $\max(A, B) < \infty$, *where*

$$A = \sup_{-\infty<x<\infty} \left(\int_{-\infty}^{x} (v(t)|\psi_1(t)|^{-p})^{1-p'}\mathrm{d}t\right)^{-1/p} \times \left(\int_{-\infty}^{x} u(t)|\phi_1(t)|^q \left(\int_{-\infty}^{t} (v(s)|\psi_1(s)|^{-p})^{1-p'}\mathrm{d}s\right)^q \mathrm{d}t\right)^{1/q}$$

$$B = \sup_{-\infty<x<\infty} \left(\int_{-\infty}^{x} u(t)|\psi_2(t)|^q\mathrm{d}t\right)^{-1/q'} \times \left(\int_{-\infty}^{x} (v(t)|\phi_2(t)|^{-p})^{1-p'} \left(\int_{-\infty}^{t} u(s)|\psi_2(s)|^q\mathrm{d}s\right)^{p'} \mathrm{d}t\right)^{1/p'}$$

and the norm in (1) is of the weighted Lebesgue space.

Proof. The sufficiency:
Define

$$(S_1f)(x) = \phi_1(x)\int_{-\infty}^{x} \psi_1(t)f(t)\mathrm{d}t, (S_2f)(x) = \phi_2(x)\int_{x}^{\infty} \psi_2(t)f(t)\mathrm{d}t.$$

Then $S = S_1 + S_2$.

For S_1 and S_2 defined as above, the inequality

$$\|Sf\|_{q,u} \leq \|S_1f\|_{q,u} + \|S_2f\|_{q,u} \tag{2}$$

holds.

Now, consider the following.

LEMMA 2.2. *Suppose* $1 < p \leq q < \infty$. *The inequalities (1) exist for* $S = S_1$ *if and only if* $A < \infty$.

Lemma 2.2 has been proved for $\phi_1 \equiv \psi_1 \equiv 1$ in [1, Theorem 1.1] (see also [3]). Proof of Lemma 2.2 is similar to that proof. We omit the details. By making duality arguments and applying suitable substitutions in Lemma 2.2, the following can be proved:

LEMMA 2.3. *Suppose* $1 < p \leq q < \infty$. *The inequality in* (1) *exists, for* $S = S_2$, *if and only if* $B < \infty$.

The sufficiency now follows from Lemma 2.2, Lemma 2.3 and (2).

The necessity:

Consider first non-negative ϕ_i, ψ_i. Suppose that (1) holds and $f \geq 0$. Since $\|S_if\|_{q,u} \leq \|Sf\|_{q,u}$, $i = 1, 2$, (1) holds for both $S = S_1$ and $S = S_2$. This implies $A < \infty$, for $i = 1$ and $B < \infty$, for $i = 2$.

Consider now some general ϕ_i, ψ_i. Suppose (1) holds. For a given $\varepsilon > 0$, define

$$v_\varepsilon(x) = \max\{v(x), |\phi_1(x)|^p \varepsilon\}.$$

Since $\|f\|_{p,v} \leq \|f\|_{p,v_\varepsilon}$, (1) is equivalent to

$$\|Sf\|_{q,u} \leq C\|f\|_{p,v_\varepsilon}. \tag{3}$$

For $-\infty < \alpha < \beta < \gamma < \infty$, define

$$f(x) = \left(\frac{v_\varepsilon(x)}{\psi_1(x)}\right)^{1-p'} sgn(\psi_1(x))\chi_{(\alpha,\beta)}(x). \tag{4}$$

Clearly, the RHS of (3), for f defined as in (4), is dominated by

$$C\left(\int_\alpha^\gamma |\psi_1(x)|^{p'} v_\varepsilon^{1-p'}(x)\mathrm{d}x\right)^{1/p},$$

which is less than $C(\varepsilon^{1-p'}(\gamma-\alpha))^{1/p} < \infty$, whereas the LHS of (1) can be estimated as

$$\|Sf\|_{q,u} \geq \left(\int_\beta^\gamma u(x)|\phi_1(x)|^q \left(\int_{-\infty}^x |\psi_1(t)|^{p'} v_\varepsilon^{1-p'}(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q}.$$

Consequently,

$$\left(\int_\alpha^\gamma |\psi_1(x)|^{p'} v_\varepsilon^{1-p'}(x)\mathrm{d}x\right)^{-1/p} \left(\int_\beta^\gamma u(x)|\phi_1(x)|^q \left(\int_{-\infty}^x |\psi_1(t)|^{p'} v_\varepsilon^{1-p'}(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q} \leq C < \infty$$

holds for C independent of $\alpha, \beta, \varepsilon$ and v_ε. For $\alpha \to -\infty$, $\beta \to -\infty$ and $\varepsilon \to 0$ (via a subsequence), $v_\varepsilon \to v$ and, taking the supremum over γ satisfying $-\infty < \gamma < \infty$, we have $A < \infty$.

By using a dual argument, (1) is equivalent to

$$\|S^* f\|_{p',v^{1-p'}} \leq C\|f\|_{q',u^{1-q'}},$$

where S^* is the adjoint operator of S defined as

$$(S^* f)(x) = \psi_1(x)\int_x^\infty \phi_1(t)f(t)\mathrm{d}t + \psi_2(x)\int_{-\infty}^x \phi_2(t)f(t)\mathrm{d}t.$$

For a given $\varepsilon > 0$, define

$$u_\varepsilon(x) = \min\{u(x), |\phi_2(x)|^{-q}\varepsilon\}.$$

Since $\|f\|_{q',u^{1-q'}} \leq \|f\|_{q',u_\varepsilon^{1-q'}}$, (3) is equivalent to

$$\|S^* f\|_{q,u} \leq C\|f\|_{q',u_\varepsilon^{1-q'}}. \tag{5}$$

For $-\infty < \alpha < \beta < \gamma < \infty$, define

$$g(x) = u_\varepsilon(x)|\phi_2(x)|^{q-1} sgn(\phi_2(x))\chi_{(\alpha,\beta)}(x). \tag{6}$$

Necessity of $B < \infty$, now, can be established by making parallel arguments and using (5) for $f = g$. $\square$

REMARK 2.4. The corresponding result of Theorem 2.1 for the dual operator of S can be easily obtained by using a dual argument. We omit the details.

REFERENCES

[1] A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific, 2003.

[2] S. Kumar, *A Hardy-type inequality in two dimensions*, Indag. Math.(N.S.), **20** (2009), 247–260.

[3] L.-E. Persson and V.D. Stepanov, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl., **7** (2002), 727–746.

[4] P.A. Zharov, *On a two-weight inequality. Generalization of inequalities of Hardy and Poincaré* (in Russian), Trudy Mat. Inst. Steklov **194** (1992), 97–110, translation in Proc. Steklov Inst. Math., **194** (1993), 101–114.

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