MATHEMATICA, 60 (83), N° 2, 2018, pp. 166–176

NEW INTEGRAL INEQUALITIES FOR (r,α)-FRACTIONAL MOMENTS OF CONTINUOUS RANDOM VARIABLES

MOHAMED HOUAS, ZOUBIR DAHMANI, and MEHMET ZEKI SARIKAYA

Abstract. In this paper, we establish new integral inequalities for continuous random variables. By introducing new concepts on fractional moments of continuous random variables, we generalize some interesting results of P. Kumar. Other fractional integral results are also presented.

MSC 2010. 26D15, 26A33, 60E15.

Key words. Integral inequalities, Riemann-Liouville integral, random variable, fractional dispersion, fractional variance, fractional moment.

1. INTRODUCTION

It is well known that the integral inequalities theory plays an important role in applied sciences. Recently, significant development in this theory has been achieved; for details, we refer to [1, 4, 7, 12, 13, 18, 21], [23]-[27] and the references therein. Moreover, the study of fractional type inequalities is also of great importance. We refer the reader to [2, 3, 6], [8]-[12], [14, 19, 22], for further information and applications.

Let us present the results that inspired this paper. We begin by [5], where N.S. Barnett et al. established some new integral inequalities for the expectation $E(X)$ and the variance $\sigma^2(X)$ of a continuous random variable X having a probability density function $p.d.f.$ $f:[a,b]\to\mathbb{R}^+$. In [15, 16], P. Kumar presented new results for the moments of continuous random variables. Then, Y. Miao and G. Yang [17] gave some upper bounds for the standard deviation and for the L_p absolute deviation of X. Recently, M. Niezgoda [20] proposed new generalizations of the results of P. Kumar [16], by applying Ostrowski-Gruss type inequalities. In a very recent work, Z. Dahmani [8, 12] introduced new concepts on fractional random variables. Then, he established several integral inequalities for the fractional dispersion and the fractional variance functions of continuous random variables with probability density functions $p.d.f.$

In succession to our earlier work in [8, 12], we introduce new fractional concepts on (r,α)- moments for continuous random variables. Then, we obtain new results for the (r,α)-fractional functions. We also present new fractional integral results for the (r,α)-fractional moments. Some classical integral inequalities of P. Kumar [16] can be deduced as some special cases of our results.

DOI: 10.24193/mathcluj.2018.2.08

2. PRELIMINARIES

In this section, we present some definitions and properties that will be used throughout this paper (cf. [8, 12, 14]).

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$, is defined as

$$J_a^\alpha [f(t)] := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \ \alpha > 0, a < t \leq b, \tag{1}$$

$$J_a^0 [f(t)] = f(t),$$

where $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-u} u^{\alpha-1} \mathrm{d}u$.

For, $\alpha \geq 0, \beta \geq 0$, the following properties hold:

$$J_a^\alpha J_a^\beta [f(t)] = J_a^{\alpha+\beta} [f(t)], \tag{2}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]. \tag{3}$$

We give the following definitions (see [8, 12]).

Definition 2.2. The fractional expectation function of order $\alpha > 0$, for a random variable X with a positive $p.d.f.$ f defined on $[a, b]$, is defined as

$$E_{X,\alpha}(t) := J_a^\alpha [t f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau f(\tau)\,\mathrm{d}\tau, \ \alpha \geq 0, a < t \leq b. \tag{4}$$

Definition 2.3. The fractional expectation function of order $\alpha > 0$, for the random variable $X - E(X)$, is defined as

$$E_{X-E(X),\alpha}(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X)) f(\tau)\,\mathrm{d}\tau, \ a < t \leq b, \tag{5}$$

where $f : [a, b] \to \mathbb{R}^+$ is a $p.d.f.$ of X and $E(X) := \int_a^b \tau f(\tau)\,\mathrm{d}\tau$ is the classical expectation of X.

Definition 2.4. The fractional expectation of order $\alpha > 0$, for a random variable X with a positive $p.d.f.$ f defined on $[a, b]$, is defined as

$$E_{X,\alpha} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau f(\tau)\,\mathrm{d}\tau. \tag{6}$$

Definition 2.5. The fractional variance function of order $\alpha > 0$, for a random variable X having a $p.d.f.$ $f : [a, b] \to \mathbb{R}^+$, is defined as

$$\begin{aligned} \sigma_{X,\alpha}^2(t) &:= J_a^\alpha \left[(t - E(X))^2 f(t) \right] \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X))^2 f(\tau)\,\mathrm{d}\tau, \ a < t \leq b, \end{aligned} \tag{7}$$

where $E(X) := \int_a^b \tau f(\tau)\,\mathrm{d}\tau$ is the classical expectation of X.

DEFINITION 2.6. The fractional variance of order $\alpha > 0$, for a random variable X with a *p.d.f.* $f : [a,b] \to \mathbb{R}^+$, is defined as

$$\sigma^2_{X,\alpha} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} (\tau - E(X))^2 f(\tau)\, d\tau. \tag{8}$$

We give the following important properties (see [8, 12]).

(P_1) If we take $\alpha = 1$ in Definition 2.4, we obtain the classical expectation: $E_{X,1} = E(X)$.

(P_2) If we take $\alpha = 1$ in Definition 2.6, we obtain the classical variance: $\sigma^2_{X,\alpha} = \sigma^2(X) = \int_a^b (\tau - E(X))^2 f(\tau)\, d\tau$.

(P_3) For $\alpha > 0$, the *p.d.f.* f satisfies $J_a^\alpha [f(t)] \le \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$.

We also introduce the following definitions.

DEFINITION 2.7. The fractional moment function of orders $(r > 0, \alpha > 0)$, for a continuous random variable X with a positive *p.d.f.* f defined on $[a,b]$, is defined as:

$$M_{r,\alpha}(t) := J_a^\alpha [t^r f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau^r f(\tau)\, d\tau, \alpha > 0, a < t \le b. \tag{9}$$

For $t = b$, we introduce the fractional moment of X.

DEFINITION 2.8. The fractional moment of orders $(r > 0, \alpha > 0)$, for a continuous random variable X, is defined as

$$M_{r,\alpha} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau^r f(\tau)\, d\tau, \alpha > 0. \tag{10}$$

We note that:

(P_4) if we take $\alpha = 1$ in Definition 2.8, we obtain the classical moment of order r, given by $M_r = \int_a^b \tau^r f(\tau)\, d\tau$.

3. MAIN RESULTS

In this section, we present new results for the (r,α)-fractional moments of continuous random variables. The first main result is the following theorem.

THEOREM 3.1. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$. Then we have:*

(1) *For $a < t \le b$, $\alpha > 0$,*

$$\begin{aligned} &J_a^\alpha [f(t)]\, J_a^\alpha \left[t^{r-1} (t - E(X)) f(t)\right] \\ &\quad - \left(J_a^\alpha [(t - E(X)) f(t)]\right) M_{r-1,\alpha}(t) \\ &\le \|f\|_\infty^2 \left[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha [t^r] - J_a^\alpha [t]\, J_a^\alpha \left[t^{r-1}\right]\right], \end{aligned} \tag{11}$$

where $f \in L_\infty [a,b]$.

(2) *For all $a < t \leq b$, $\alpha > 0$,*

$$J_a^\alpha [f(t)] J_a^\alpha \left[t^{r-1}(t - E(X)) f(t)\right] - \left(J_a^\alpha [(t - E(X)) f(t)]\right) M_{r-1,\alpha}(t) \leq \frac{1}{2}(t-a)\left(t^{r-1} - a^{r-1}\right)\left(J_a^\alpha [f(t)]\right)^2. \tag{12}$$

Proof. Let us consider the quantity

$$H(\tau, \rho) = (g(\tau) - g(\rho))(h(\tau) - h(\rho)), \quad \tau, \rho \in (a, t), a < t \leq b. \tag{13}$$

Taking a function $p : [a, b] \to \mathbb{R}^+$, multiplying (13) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} p(\tau), \tau \in (a, t)$ and then integrating the resulting identity with respect to τ over (a, t), we can state that

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} p(\tau) H(\tau, \rho) \mathrm{d}\tau = J_a^\alpha [pgh(t)] - g(\rho) J_a^\alpha [ph(t)] - h(\rho) J_a^\alpha [pg(t)] + g(\rho) h(\rho) J_a^\alpha [p(t)]. \tag{14}$$

Multiplying (14) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} p(\rho), \rho \in (a, t)$, and integrating the resulting identity with respect to ρ from a to t, we can write

$$\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} p(\tau) p(\rho) H(\tau, \rho) \mathrm{d}\tau \mathrm{d}\rho = 2 J_a^\alpha [p(t)] J_a^\alpha [pgh(t)] - 2 J_a^\alpha [pg(t)] J_a^\alpha [ph(t)]. \tag{15}$$

In (15), taking $p(t) = f(t), g(t) = t - E(X), h(t) = t^{r-1}, t \in (a, b)$, we get

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} \\ &\times (\tau - \rho)\left(\tau^{r-1} - \rho^{r-1}\right) f(\tau) f(\rho) \mathrm{d}\tau \mathrm{d}\rho \\ &= 2 J_a^\alpha [f(t)] J_a^\alpha \left[t^{r-1}(t - E(X)) f(t)\right] \\ &- 2 J_a^\alpha [(t - E(X)) f(t)] J_a^\alpha \left[t^{r-1} f(t)\right] \\ &= 2 J_a^\alpha [f(t)] J_a^\alpha \left[t^{r-1}(t - E(X)) f(t)\right] \\ &\quad - 2 J_a^\alpha [(t - E(X)) f(t)] M_{r-1,\alpha}(t). \end{aligned} \tag{16}$$

On the other hand, we have:

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} \\ &\quad \times (\tau - \rho)\left(\tau^{r-1} - \rho^{r-1}\right) f(\tau) f(\rho) \mathrm{d}\tau \mathrm{d}\rho \\ &\leq \|f\|_\infty^2 \frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} \\ &\quad \times (\tau - \rho)\left(\tau^{r-1} - \rho^{r-1}\right) \mathrm{d}\tau \mathrm{d}\rho \\ &\leq \|f\|_\infty^2 \left[2 \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha [t^r] - 2 J_a^\alpha [t] J_a^\alpha \left[t^{r-1}\right]\right]. \end{aligned} \tag{17}$$

Thanks to (16) and (17), we obtain (11).

To prove (12), we remark that

$$\begin{aligned}\frac{1}{\Gamma^{2}(\alpha)}\int_{a}^{t}\int_{a}^{t}&(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\\ &\times(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,\mathrm{d}\tau\mathrm{d}\rho\\ \leq&\sup_{\tau,\rho\in[a,t]}|(\tau-\rho)|\left|\left(\tau^{r-1}-\rho^{r-1}\right)\right|J_{a}^{\alpha}[f(t)]^{2}\\ &=(t-a)\left(t^{r-1}-a^{r-1}\right)J_{a}^{\alpha}[f(t)]^{2}.\end{aligned}\tag{18}$$

Therefore, by (16) and (18), we get the desired inequality. This ends the proof of Theorem 3.1. □

We also prove the following theorem.

THEOREM 3.2. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^{+}$.*

(1*) *For any $a<t\leq b,\alpha>0,\beta>0$, we have:*

$$\begin{aligned}&J_{a}^{\alpha}[f(t)]J_{a}^{\beta}\left[f(t)t^{r-1}(t-E(x))\right]\\ &\quad+J_{a}^{\beta}[f(t)]J_{a}^{\alpha}\left[f(t)t^{r-1}(t-E(x))\right]\\ &\quad-J_{a}^{\alpha}[f(t)(t-E(x))]M_{r-1,\beta}\\ &\quad-J_{a}^{\beta}[f(t)(t-E(x))]M_{r-1,\alpha}\\ &\leq\|f\|_{\infty}^{2}\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_{a}^{\beta}[t^{r}]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_{a}^{\alpha}[t^{r}]\\ &\quad-J_{a}^{\alpha}[t]J_{a}^{\beta}\left[t^{r-1}\right]-J_{a}^{\beta}[t]J_{a}^{\alpha}\left[t^{r-1}\right],\end{aligned}\tag{19}$$

where $f\in L_{\infty}[a,b]$.

(2*) *We also have:*

$$\begin{aligned}&J_{a}^{\alpha}[f(t)]J_{a}^{\beta}\left[f(t)t^{r-1}(t-E(x))\right]\\ &\quad+J_{a}^{\beta}[f(t)]J_{a}^{\alpha}\left[f(t)t^{r-1}(t-E(x))\right]\\ &-J_{a}^{\alpha}[f(t)(t-E(x))]M_{r-1,\beta}-J_{a}^{\beta}[f(t)(t-E(x))]M_{r-1,\alpha}\\ &\quad\leq(t-a)\left(t^{r-1}-a^{r-1}\right)J_{a}^{\alpha}[f(t)]J_{a}^{\beta}[f(t)],\end{aligned}\tag{20}$$

for any $a<t\leq b$, $\alpha>0$, $\beta>0$.

Proof. Multiplying (14) by $\frac{(b-\rho)^{\beta-1}}{\Gamma(\beta)}p(\rho),\rho\in(a,t)$, and integrating the resulting identity with respect to ρ over (a,t), we can write

$$\begin{aligned}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{t}\int_{a}^{t}&(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)H(\tau,\rho)\,\mathrm{d}\tau\mathrm{d}\rho\\ =&J_{a}^{\alpha}[p(t)]J_{a}^{\beta}[pgh(t)]+J_{a}^{\beta}[p(t)]J_{a}^{\alpha}[pgh(t)]\\ &-J_{a}^{\alpha}[ph(t)]J_{a}^{\beta}[pg(t)]-J_{a}^{\beta}[ph(t)]J_{a}^{\alpha}[pg(t)].\end{aligned}\tag{21}$$

Taking $p(t) = f(t), g(t) = t - E(X), h(t) = t^{r-1}$, $t \in (a,b)$ in the above identity, yields

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1} \\
&\quad\times(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,\mathrm{d}\tau\mathrm{d}\rho \\
&\quad= J_a^{\alpha}[f(t)]\,J_a^{\beta}\left[t^{r-1}(t-E(x))f(t)\right] \\
&\quad+ J_a^{\beta}[f(t)]\,J_a^{\alpha}\left[t^{r-1}(t-E(x))f(t)\right] \\
&- M_{r-1,\alpha}\,J_a^{\beta}[f(t)(t-E(x))] - M_{r-1\ \beta}J_a^{\alpha}[f(t)(t-E(x))]\,.
\end{aligned}\tag{22}$$

We have also

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho) \\
&\quad\times\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,\mathrm{d}\tau\mathrm{d}\rho \\
&\le \|f\|_{\infty}^{2}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho) \\
&\quad\times\left(\tau^{r-1}-\rho^{r-1}\right)\mathrm{d}\tau\mathrm{d}\rho \\
&= \|f\|_{\infty}^{2}\Big[\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}[t^{r}] + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}[t^{r}] \\
&\quad - J_a^{\alpha}[t]\,J_a^{\beta}\left[t^{r-1}\right] - J_a^{\beta}[t]\,J^{\alpha}\left[t^{r-1}\right]\Big].
\end{aligned}\tag{23}$$

Thanks to (22) and (23), we obtain (1*).

For (2*), we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1} \\
&\quad\times(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,\mathrm{d}\tau\mathrm{d}\rho \\
&\le \sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho|\left|\tau^{r-1}-\rho^{r-1}\right|\right]J_a^{\alpha}[f(t)]\,J_a^{\beta}[f(t)] \\
&\quad= (t-a)\left(t^{r-1}-a^{r-1}\right)J_a^{\alpha}[f(t)]\,J_a^{\beta}[f(t)]\,.
\end{aligned}\tag{24}$$

And by (22) and (24), we get (20). □

Remark 3.3. ($R1$) Applying Theorem 3.2 for $\alpha = \beta$, we obtain Theorem 3.1.

($R2$) Taking $\alpha = \beta = 1$ in (1*) of Theorem 3.2, we obtain the last inequality of Theorem 1 in [16].

($R3$) Taking $\alpha = \beta = 1$ in (2*) of Theorem 3.2, we obtain the first inequality of Theorem 1 in [16].

We give also the following result.

Theorem 3.4. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$. Then we have:*

$$\begin{aligned} & J_a^\alpha [f(t)] M_{2r,\alpha}(t) - M_{r,\alpha}^2(t) \\ & \leq \frac{1}{4} (J_a^\alpha [f(t)])^2 (b^r - a^r)^2, \alpha > 0, t \in (a,b). \end{aligned} \tag{25}$$

Proof. Using Theorem 3.1 of [9], we can write

$$\left| J_a^\alpha [p(t)] J_a^\alpha \left[pg^2(t)\right] - (J_a^\alpha [pg(t)])^2 \right| \leq \frac{1}{4} (J_a^\alpha [p(t)])^2 (M-m)^2. \tag{26}$$

Taking $p(t) = f(t)$ and $g(t) = t^r, t \in [a,b]$, we obtain $m = a^r$, $M = b^r$. Hence, the inequality (26) allows us to obtain

$$0 \leq J_a^\alpha [f(t)] J_a^\alpha \left[t^{2r} f(t)\right] - (J_a^\alpha [t^r f(t)])^2 \leq \frac{1}{4} (J_a^\alpha [f(t)])^2 (b^r - a^r)^2. \tag{27}$$

This implies that

$$J_a^\alpha [f(t)] M_{2r,\alpha}(t) - M_{r,\alpha}^2(t) \leq \frac{1}{4} (J_a^\alpha [f(t)])^2 (b^r - a^r)^2. \tag{28}$$

Theorem 3 .3 is thus proved. □

Using two fractional parameters, we consider the following generalization ofthe above results.

Theorem 3.5. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$. Then, for all $a < t \leq b$, $\alpha > 0$, $\beta > 0$, we have:*

$$\begin{aligned} & J_a^\alpha [f(t)] M_{2r,\beta}(t) + J_a^\beta [f(t)] M_{2r,\alpha}(t) + 2a^r b^r J_a^\alpha [f(t)] J_a^\beta [f(t)] \\ & \leq (a^r + b^r) \left[J_a^\alpha [f(t)] M_{r,\beta}(t) + J_a^\beta [f(t)] M_{r,\alpha}(t) \right]. \end{aligned} \tag{29}$$

Proof. Thanks to Theorem 3.4 of [9], we can state that

$$\begin{aligned} & \Big[J_a^\alpha [p(t)] J_a^\beta \left[pg^2(t)\right] + J_a^\beta [p(t)] J_a^\alpha \left[pg^2(t)\right] \\ & \qquad - 2 J_a^\alpha [pg(t)] J_a^\beta [pg(t)] \Big]^2 \\ & \leq \Big[(M J_a^\alpha [p(t)] - J_a^\alpha [pg(t)]) \left(J_a^\beta [pg(t)] - m J_a^\beta [p(t)] \right) \\ & + (J_a^\alpha [pg(t)] - m J_a^\alpha [p(t)]) \left(M J_a^\beta [p(t)] - J_a^\beta [pg(t)] \right) \Big]^2. \end{aligned} \tag{30}$$

In (30), we take $p(t) = f(t), g(t) = t^r, t \in [a,b]$. So, we get

$$\begin{aligned} & \Big[J_a^\alpha [f(t)] J_a^\beta \left[t^{2r} f(t)\right] + J_a^\beta [f(t)] J_a^\alpha \left[t^{2r} f(t)\right] \\ & \qquad - 2 J_a^\alpha [t^r f(t)] J_a^\beta [t^r f(t)] \Big]^2 \\ & \leq \Big[(M J_a^\alpha [f(t)] - J_a^\alpha [t^r f(t)]) \left(J^\beta [t^r f(t)] - m J_a^\beta [f(t)] \right) \\ & + (J_a^\alpha [t^r f(t)] - m J_a^\alpha [f(t)]) \left(M J_a^\beta [f(t)] - J_a^\beta [t^r f(t)] \right) \Big]^2, \end{aligned} \tag{31}$$

which implies that

$$\begin{aligned} & J_a^\alpha [f(t)] M_{2r,\beta}(t) + J_a^\beta [f(t)] M_{2r,\alpha}(t) - 2M_{r,\alpha}(t) M_{r,\beta}(t) \\ & \le (M J_a^\alpha [f(t)] - M_{r,\alpha}(t)) \left(M_{r,\beta}(t) - m J_a^\beta [f(t)] \right) \\ & \quad + (M_{r,\alpha}(t) - m J_a^\alpha [f(t)]) \left(M J_a^\beta [f(t)] - M_{r,\beta}(t) \right). \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} & J_a^\alpha [f(t)] M_{2r,\beta}(t) + J_a^\beta [f(t)] M_{2r,\alpha}(t) \\ & \le M \left(J_a^\alpha [f(t)] M_{r,\beta}(t) + J_a^\beta [f(t)] M_{r,\alpha}(t) \right) \\ & + m \left(J_a^\alpha [f(t)] M_{r,\beta}(t) + J_a^\beta [f(t)] M_{r,\alpha}(t) \right) \\ & \quad - 2mM J_a^\alpha [f(t)] J_a^\beta [f(t)]. \end{aligned} \tag{33}$$

Substituting the values of m and M in (33), we obtain

$$\begin{aligned} & J_a^\alpha [f(t)] M_{2r,\beta}(t) + J_a^\beta [f(t)] M_{2r,\alpha}(t) \\ & \le (a^r + b^r) J_a^\alpha [f(t)] M_{r,\beta}(t) + (a^r + b^r) J_a^\beta [f(t)] M_{r,\alpha}(t) \\ & \quad - 2a^r b^r J_a^\alpha [f(t)] J_a^\beta [f(t)]. \end{aligned} \tag{34}$$

□

Another result is the following theorem.

THEOREM 3.6. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$ and $\alpha > 0$. Then we have:*

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} M_{r,\alpha}(t) - J_a^\alpha [f(t)] J_a^\alpha [t^r] \right| \\ & \le \frac{(t-a)^\alpha}{2\Gamma(\alpha+1)} (M-m) \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha \left[t^{2r}\right] - (J_a^\alpha [t^r])^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{35}$$

Proof. Using Theorem 3.1 and Lemma 3.2 of [11], we can write

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha [fg(t)] - J_a^\alpha [f(t)] J_a^\alpha [g(t)] \right| \\ & \le \frac{(t-a)^\alpha}{2\Gamma(\alpha+1)} (M-m) \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha \left[g^2(t)\right] - (J_a^\alpha [g(t)])^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{36}$$

Taking $g(t) = t^r, a < t \le b$, we obtain

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha [t^r f(t)] - J_a^\alpha [f(t)] J_a^\alpha [t^r] \right| \\ & \le \frac{(t-a)^\alpha}{2\Gamma(\alpha+1)} (M-m) \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha \left[t^{2r}\right] - (J_a^\alpha [t^r])^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{37}$$

□

Using two fractional parameters, we establish the following generalization.

THEOREM 3.7. *Let f be a p.d.f. of X on $[a,b]$. Then for all $a<t\leq b,\alpha>0,\beta>0$, we have*

$$\begin{aligned}
&\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}M_{r,\alpha}(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}M_{r,\beta}(t) \\
&\qquad -J_a^{\alpha}[f(t)]J_a^{\beta}[t^r]-J_a^{\beta}[f(t)]J_a^{\alpha}[t^r] \\
&\leq\Bigg[\left(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}[f(t)]\right)\left(J_a^{\beta}[f(t)]-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right) \\
&\quad+\left(J_a^{\alpha}[f(t)]-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-J_a^{\beta}[f(t)]\right)\Bigg] \\
&\quad\times\left[\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\left[t^{2r}\right]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}\left[t^{2r}\right]-2J_a^{\alpha}[t^r]J_a^{\beta}[t^r]\right]^{\frac{1}{2}}.
\end{aligned} \tag{38}$$

Proof. We use Theorem 3.3 and Lemma 3.4 of [11] to obtain

$$\begin{aligned}
&\Bigg|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}[fg(t)]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}[fg(t)] \\
&\qquad -J_a^{\alpha}[f(t)]J_a^{\beta}[g(t)]-J_a^{\beta}[f(t)]J_a^{\alpha}[g(t)]\Bigg| \\
&\leq\Bigg[\left(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}[f(t)]\right)\left(J_a^{\beta}[f(t)]-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right) \\
&\quad+\left(J_a^{\alpha}[f(t)]-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-J_a^{\beta}[f(t)]\right)\Bigg] \\
&\quad\times\Bigg(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\left[g^2(t)\right]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}\left[g^2(t)\right] \\
&\qquad -2J_a^{\alpha}[g(t)]J_a^{\beta}[g(t)]\Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{39}$$

Taking $g(t)=t^r$, $a<t\leq b$, we can write

$$\begin{aligned}
&\Bigg|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}[t^rf(t)]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}[t^rf(t)] \\
&\qquad -J_a^{\alpha}[f(t)]J_a^{\beta}[t^r]-J_a^{\beta}[f(t)]J_a^{\alpha}[t^r]\Bigg| \\
&\leq\Bigg[\left(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}[f(t)]\right)\left(J_a^{\beta}[f(t)]-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right) \\
&\quad+\left(J_a^{\alpha}[f(t)]-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-J_a^{\beta}[f(t)]\right)\Bigg]
\end{aligned} \tag{40}$$

$$\times\left(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\left[t^{2r}\right]+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}\left[t^{2r}\right]-2J_a^{\alpha}\left[t^{r}\right]J_a^{\beta}\left[t^{r}\right]\right)^{\frac{1}{2}}.$$

□

REMARK 3.8. Taking $\alpha=\beta$ in Theorem 3.6, we obtain Theorem 3.5.

REFERENCES

[1] A.M. Acu, F. Sofonea and C.V. Muraru, *Gruss and Ostrowski type inequalities and their applications generalization of Chebyshev inequality*, Sci. Stud. Res. Ser. Math. Inform., **23** (2013), 5–14.

[2] G.A. Anastassiou, M.R. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, *Montgomery identities for fractional integrals and related fractional inequalities*, JIPAM, **10** (2009), 1–6.

[3] G.A. Anastassiou, *Fractional differentiation inequalities*, Springer-Verlag, New York, 2009.

[4] N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis, *Some inequalities for the expectation and variance of a random variable whose pdf is n-time differentiable*, JIPAM, **1** (2000), 1–29.

[5] N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis, *Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval*, JIPAM, **2** (2001), 1–18.

[6] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, JIPAM, **10** (2009), 1–12.

[7] P.L. Chebyshev, *Sur les expressions approximatives des integrales definis par les autres prises entre les memes limite*, Proc. Math. Soc. Charkov, **2** (1882), 93–98.

[8] Z. Dahmani, *New applications of fractional calculus on probabilistic random variables*, Acta Math. Univ. Comenian. (N.S.), **86** (2017), 299–307.

[9] Z. Dahmani and L. Tabharit, *On weighted Gruss type inequalities via fractional integrals*, JARPM (IASR), **2** (2010), 31–38.

[10] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., **1** (2010), 51–58.

[11] Z. Dahmani, L. Tabharit and S. Taf, *New generalisations of Gruss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., **2** (2010), 93–99.

[12] Z. Dahmani, *Fractional integral inequalities for continuous random variables*, Malaya J. Mat., **2** (2014), 172–179.

[13] Z. Dahmani, A.E. Bouziane, M. Houas and M.Z. Sarikaya, *New W-weighted concepts for continuous random variables with applications*, Note Mat., **37** (2017), 23–40.

[14] S.S. Dragomir, *A generalization of Gruss inequality in inner product spaces and applications*, J. Math. Anal. Appl., **237** (1999), 74–82.

[15] P. Kumar, *Moment inequalities of a random variable defined over a finite interval*, JIPAM, **3** (2002), 1–24.

[16] P. Kumar, *Inequalities involving moments of a continuous random variable defined over a finite interval*, Computers and Mathematics with Applications, **48** (2004), 257–273.

[17] Y. Miao and G. Yang, *A note on the upper bounds for the dispersion*, JIPAM, **8** (2007), 1–13.

[18] D.C. Mitrinovic, J.E. Pecaric and A.M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[19] T.F. Mori, *Sharp inequalities between centered moments*, JIPAM, **10** (2009), 1–19.

[20] M. Niezgoda, *New bounds for moments of continuous random variables*, Computers and Mathematics with Applications, **60** (2004), 3130–3138.
[21] B.G. Pachpatte, *On multidimensional Gruss type integral inequalities*, JIPAM, **3** (2002), 1–15.
[22] R. Sharma, S. Devi, G. Kapoor, S. Ram and N.S. Barnett, *A brief note on some bounds connecting lower order moments for random variables defined on a finite interval*, International Journal of Theoretical & Applied Sciences, **1** (2009), 83–85.
[23] M.Z. Sarikaya and H. Yaldiz, *Note on the Ostrowski type inequalities for fractional integrals*, Vietnam J. Math., **41** (2013), 2–6.
[24] M.Z. Sarikaya and H. Yaldiz, *New generalization fractional inequalities of Ostrowski-Gruss type*, Lobachevskii J. Math., **34** (2013), 326–331.
[25] M.Z. Sarikaya, H. Yaldiz and N. Basak, *New fractional inequalities of Ostrowski-Grüss type*, Le Matematiche, **69** (2014), 227–235.
[26] M.Z. Sarikaya, N. Aktan and H. Yildirim, *On weighted Chebyshev-Gruss like inequalities on time scales*, J. Math. Inequal., **2** (2008), 185–195.
[27] M.Z. Sarikatya and H. Ogunmez, *New fractional inequalities of Ostrowski-Gruss type*, Abstr. Appl. Anal., **2012** (2012), Article 428983, 1–10.

Received December 16, 2017
Accepted March 24, 2018

University of Khemis Miliana
Laboratory FIMA, UDBKM
Algeria
E-mail: houasmed@yahoo.fr

University of Mostaganem
Laboratory LPAM, UMAB
Algeria
E-mail: zzdahmani@yahoo.fr

Düzce University
Department of Mathematics
Faculty of Science and Arts,
Düzce, Turkey
E-mail: sarikaya@aku.edu.tr