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MIXED BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. The purpose of this paper is to show a well-posedness result for a Dirichlet-Neumann boundary value problem for the Stokes system on compact Riemannian manifolds. Using layer potential techniques, we derive an equivalent boundary integral system for the Stokes system and prove the invertibility of the related matrix integral operator.

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Key words. Stokes system, mixed problem, layer potential analysis, compact Riemannian manifolds.

1. INTRODUCTION

The study of fluid flow on a compact smooth Riemannian manifold plays an important role in the analysis of the fundamental equations of meteorology and oceanography, as pointed out in [30, 22] (see also [29, 5]). Also, other types of flow equations, e.g. Stokes system or Darcy-Forchheimer-Brinkman, can be considered over compact surfaces (e.g. on the sphere S^2), which model the flow of water or other viscous fluids, passing through porous rocks or porous soil (see [14]).

This article is devoted to mixed boundary value problems of Dirichlet-Neumann type on compact Riemannian manifolds, which could resemble a mathematical model of a shallow ocean.

Boundary value problems for elliptic operators on smooth and even Lipschitz domains have a long history (see e.g. [2, 10, 27, 19]). One of the many valuable contributions has been provided by Fabes, Kening and Verchota in [6], for the study of the Dirichlet problem for the Stokes system on Lipschitz domains in the Euclidean settings. Their results have been extended by Mitrea and Taylor [27] to arbitrary Lipschitz domains on compact Riemannian manifolds, by using boundary integral methods (see also [26]). Also, the Brinkman system on Lipschitz domains on compact Riemannian manifolds has received great attention. We mention the work of Kohr, Pintea and Wendland [12], which defined the pseudodifferential Brinkman operator on compact Riemannian manifolds (see also [18, 8, 7]).

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Dindoš and Mitrea [4] obtained well-posedness results for the stationary Navier-Stokes system on non-smooth manifolds for C^1 domains and, more generally, for Lipschitz domains. Recently, the well-posedness of the nonlinear Darcy-Forchheimer-Brinkman system has been obtained by Kohr, Mikhailov and Wendland [14] (see also [11, 15], for the case of Euclidean settings).

The main outline of this article is as follows. After introducing the main operators and spaces for which the problem is considered, we begin by analyzing the linear Stokes system on a bounded Lipschitz domain $\mathfrak{D}$

$$L\mathbf{u} + \mathrm{d}p = 0, \quad \delta\mathbf{u} = 0, \text{ in } \mathfrak{D}, \tag{1}$$

where $(\mathbf{u}, p)$ is the velocity and pressure field of the fluid flow in a Lipschitz domain $\mathfrak{D} \subset M$. Moreover, we assume that the boundary $\Gamma = \partial\mathfrak{D}$ is decomposed into two open and adjoint parts, Γ_D and Γ_N, over which we consider the Dirichlet and Neumann boundary conditions

$$\mathrm{Tr}^+\mathbf{u} = \mathbf{f} \text{ on } \Gamma_D, \quad \partial_\nu^+(\mathbf{u}, p) = \mathbf{g} \text{ on } \Gamma_N, \tag{2}$$

where Tr^+ denotes the trace operator and ∂_ν^+ denotes the conormal derivative operator defined in Lemma 2.1 and 2.2, respectively.

2. PRELIMINARIES

We begin by introducing the main notions needed, for compact smooth Riemannian manifolds of dimension $m \geq 2$ without boundary, denoted by $(M, \langle\cdot,\cdot\rangle)$. The Riemannian metric is given by $g = \sum_{i,j} g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ and is transferred on the tangent space T_pM, $p \in M$, determining an inner product $\langle X, Y\rangle = g(X, Y)$, for $X, Y \in T_pM$. Due to this inner product, the tangent space T_pM can be naturally identified with the cotangent space T_p^*M and also the tangent bundle $TM := \cup_{p\in M} T_pM$ with the cotangent bundle $T^*M := \cup_{p\in M}(T_pM)^*$. Consequently, the space of vector functions $\mathfrak{X}(M)$ can be identified with the space of one forms Λ^1TM. This leads to the identification of the gradient operator with the exterior derivative operator, i.e.,

$$\mathrm{grad} : C^\infty(M) \to \mathfrak{X}(M) \text{ with } \mathrm{d} : C^\infty(M) \to C^\infty(M, \Lambda^1TM) \tag{3}$$

and the divergence operator with the exterior coderivative operator

$$-\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M) \text{ with } \delta : C^\infty(M, \Lambda^1TM) \to C^\infty(M). \tag{4}$$

Note that these operators are related by $\delta = \mathrm{d}^*$.

• **L^2-based Sobolev spaces**. For $s \in \mathbb{R}$, we consider the Sobolev (Bessel-potential) space $H^s(M)$ obtained by lifting the Sobolev space $H^s(\mathbb{R}^m) := \{(\mathbb{I} - \triangle)^{-s/2} f : f \in L^2(\mathbb{R}^m)\}$ via a partition of unity on M and pullback on corresponding local charts. The corresponding Sobolev space of one forms is defined as $H^s(M, \Lambda^1TM) := H^s(M) \otimes \Lambda^1TM$ and by duality $H^{-s}(M, \Lambda^1TM) = (H^s(M, \Lambda^1TM))^*$.

Let $\mathfrak{D} := \mathfrak{D}_+ \subset M$ be a Lipschitz domain (see e.g. [4, 27]). The Sobolev spaces of functions on $\mathfrak{D}$ are given by

$$H^s(\mathfrak{D}) := \{f|_{\mathfrak{D}} : f \in H^s(M)\},\ \widetilde{H}^s(\mathfrak{D}) := \{f \in H^s(M) : \mathrm{supp} f \subseteq \overline{\mathfrak{D}}\}, \tag{5}$$

and, similarly, the spaces of one forms on $\mathfrak{D}$ are defines as

$$H^s(\mathfrak{D}, \Lambda^1 TM) := H^s(\mathfrak{D}) \otimes \Lambda^1 TM|_{\mathfrak{D}},\ \widetilde{H}^s(\mathfrak{D}, \Lambda^1 TM) := \widetilde{H}^s(\mathfrak{D}) \otimes \Lambda^1 TM. \tag{6}$$

For any $s \in \mathbb{R}$, the Sobolev spaces are linked, by duality (see [10, Proposition 2.9], [26, (4.14)]) as $\big(H^s(\mathfrak{D}, \Lambda^1 TM)\big)^* = \widetilde{H}^{-s}(\mathfrak{D}, \Lambda^1 TM)$, and $H^{-s}(\mathfrak{D}, \Lambda^1 TM) = \big(\widetilde{H}^s(\mathfrak{D}, \Lambda^1 TM)\big)^*$. The boundary Sobolev spaces $H^s(\Gamma)$ and $H^s(\Gamma, \Lambda^1 TM)$, for $s \in [-1, 1]$, are defined similarly to the Euclidean model $H^s(\mathbb{R}^{m-1})$, via partition of unity and pullback. For a more detailed explanation, we refer the reader to [31], [26], [32, Chapter 8].

• **The trace operators on Lipschitz domains.** The following trace lemma is an important result for the boundary problems which are analyzed in the sequel (see e.g. [2], [28, Theorem 2.5.2], [25, Theorem 2.3], [27]).

Lemma 2.1. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain with boundary Γ. The restrictions to the boundary $C^\infty(\overline{\mathfrak{D}}_\pm) \ni u \mapsto u|_\Gamma$ extend to some linear and bounded operators $\mathrm{Tr}^\pm : H^1(\mathfrak{D}_\pm) \to H^{\frac{1}{2}}(\Gamma)$ which are onto and have bounded, non-unique right inverse functions*

$$\mathscr{Z}^\pm : H^{\frac{1}{2}}(\Gamma) \to H^1(\mathfrak{D}_\pm), \quad \mathrm{Tr}^\pm(\mathscr{Z}^\pm \varphi) = \varphi, \quad \forall\, \varphi \in H^{\frac{1}{2}}(\Gamma). \tag{7}$$

The result in Lemma 2.1 holds also for the Gagliardo trace operators acting on Sobolev spaces of the one forms $\mathrm{Tr}^\pm : H^1(\mathfrak{D}_\pm, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$. Such operators are well defined, linear, bounded and onto (see [3, 10, 26]).

•**The conormal derivative operator.** In order to introduce the conormal derivative operator, we consider the following special Sobolev space (similar to e.g. [13, Lemma 2.3]).

$$\begin{aligned}\mathfrak{H}^1(\mathfrak{D}) := \big\{(\mathbf{u}, p, \mathcal{F}) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}) \times \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM) :\\ L\mathbf{u} + \mathrm{d}p = \mathcal{F}|_{\mathfrak{D}} \text{ and } \delta\mathbf{u} = 0 \text{ in } \mathfrak{D}\big\}.\end{aligned} \tag{8}$$

Let $\mathrm{d}\sigma$ denote the surface measure on Γ and let ν be the outward unit conormal defined a.e. on Γ, with respect to $\mathrm{d}\sigma$. Also, by $\langle \cdot, \cdot \rangle$ we denote the dual pairing of two dual spaces defined on the a $X \subset M$.

The next result defines the notion of conormal derivative for the Stokes system on compact Riemannian manifolds (see e.g. [25] in the case of second order elliptic differential operators with variable coefficients in Euclidean setting, [28, Theorem 10.4.1] in the case of the Stokes system on Lipschitz domains in $\mathbb{R}^n$, [18, Lemma 2.2] in the case of the matrix type operator (17) on Lipschitz domains in Riemannian manifolds.

LEMMA 2.2. *Let M be a compact Riemannian manifold and $\mathfrak{D} \subset M$ be a Lipschitz domain. Then the conormal derivative operator $\partial_\nu^+ : \mathfrak{H}^1(\mathfrak{D}) \to H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ given by*

$$\langle \partial_\nu^+(\mathbf{u}, p)_{\mathcal{F}}, \Phi\rangle := 2\langle \mathrm{Def}\mathbf{u}, \mathrm{Def}(\mathcal{Z}^+\Phi)\rangle + \big\langle p, \delta(\mathcal{Z}^+\Phi)\big\rangle$$
$$(9) \qquad - \langle \mathcal{F}, \mathcal{Z}^+\Phi\rangle, \quad \forall\ \Phi \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$$

is well-defined, linear and bounded, and does not depend on the choice of the right inverse $\mathcal{Z}^+ : H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) \to H^1(\mathfrak{D}, \Lambda^1 TM)$ of the trace operator. Also, for all $(\mathbf{u}, p, \mathcal{F}) \in \mathfrak{H}^1(\mathfrak{D})$ and $\mathbf{w} \in H^1(\mathfrak{D}, \Lambda^1 TM)$ the following Green formula holds

$$(10) \qquad \langle \partial_\nu^+(\mathbf{u}, p)_{\mathcal{F}}, \mathrm{Tr}^+\mathbf{w}\rangle = 2\langle \mathrm{Def}\mathbf{u}, \mathrm{Def}\mathbf{w}\rangle + \langle p, \delta\mathbf{w}\rangle - \langle \mathcal{F}, \mathbf{w}\rangle.$$

REMARK 2.3. The conormal derivative associated to the Stokes system on the domain $\mathfrak{D}_- := M \setminus \overline{\mathfrak{D}}$ is denoted by ∂_ν^- and is defined as in (9), except the negative sign in its left hand side.

Let ∇ stand for the Levi-Civita connection on M, i.e. ∇ is an affine connection on (M, g), which is compatible with the Riemannian metric g and is torsion-free. An affine connection is torsion-free, if it satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$, for all $X, Y \in \mathfrak{X}(M)$, where $[\cdot, \cdot]$ stands for the Lie bracket (see e.g. [14, Section 2.2], [32]). Then, for $X \in \mathfrak{X}(M)$, the Levi-Civita connection ∇X is given by

$$(11) \qquad (\nabla X)(Y, Z) = \langle \nabla_Y X, Z\rangle, \quad \forall\ Y, Z \in \mathfrak{X}(M).$$

The deformation operator Def is the symmetric part of ∇X, i.e.

$$(12) \qquad (\mathrm{Def}\ X)(Y, Z) = \frac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\ Y, Z \in \mathfrak{X}(M).$$

Let S^2T^*M be the set of symmetric tensor fields of type $(0, 2)$. Then the deformation operator $\mathrm{Def} : \mathfrak{X}(M) \to C^\infty(M, S^2T^*M)$ extends to a linear and continuous operator

$$(13) \qquad \mathrm{Def} : H^1(M, \Lambda^1 TM) \to H^{-1}(M, S^2T^*M).$$

Note that, the adjoint of Def is given by $\mathrm{Def}^* w = -\mathrm{div} w$, $w \in S^2T^*M$.

We introduce the second-order elliptic differential operator

$$(14) \qquad L : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad L := 2\mathrm{Def}^*\mathrm{Def} = -\triangle + \mathrm{d}\delta - 2\mathrm{Ric},$$

where $\triangle := -(\mathrm{d}\delta + \delta\mathrm{d})$ is the Hodge Laplacian and Ric is the Ricci tensor (see, e.g., [4, (2.6)]). Similarly to the deformation operator, the operator (14) extends to a bounded linear operator (see e.g. [21, p. 177])

$$(15) \qquad L = 2\mathrm{Def}^*\mathrm{Def} : H^1(M, \Lambda^1 TM) \to H^{-1}(M, \Lambda^1 TM).$$

DEFINITION 2.4. A vector field $X \in \mathfrak{X}(M)$ which satisfies the equation Def $X = 0$ on M is called a *Killing field.*

In the sequel, we assume that the only Killing vector field is the trivial one. This condition can be realized by altering M away from $\overline{\mathfrak{D}}$ (see e.g. [4]). Hence, we have that

$$\text{Def X} = 0 \Longleftrightarrow X = 0. \tag{16}$$

Now we are ready to introduce the *Stokes operator*, which is defined by

$$\begin{gathered} S_0 : H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}) \to H^{-1}(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}), \\ S_0 := (L\mathbf{u} + \mathrm{d}p, \delta\mathbf{u}). \end{gathered} \tag{17}$$

3. THE FUNDAMENTAL SOLUTION AND THE LAYER POTENTIAL THEORY FOR THE STOKES OPERATOR

Let $(\mathcal{G}, \Pi)$ stand for the fundamental solution for the Stokes system on compact Riemannian manifolds (see, e.g., [4]). For a given density with density $\mathbf{f} \in H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$, the single-layer potential $\mathbf{Vf}$ and its corresponding pressure potential $\mathcal{Q}^s\mathbf{f}$ for the Stokes system are defined by

$$(\mathbf{Vf})(x) := \langle \mathcal{G}(x, \cdot), \mathbf{f} \rangle_\Gamma,\ (\mathcal{Q}^s\mathbf{f})(x) := \langle \Pi(x, \cdot), \mathbf{f} \rangle_\Gamma,\ x \in M \setminus \Gamma. \tag{18}$$

The double-layer and pressure potentials $(\mathbf{Wg}, \mathcal{Q}^d\mathbf{g})$ for the Stokes system are defined for $\mathbf{g} \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ as (see, e.g., [4, Section 3.], [17])

$$(\mathbf{Wh})(x) := \Big\langle -2\text{Def } \mathcal{G}(x, \cdot)\nu + \Pi^\top(\cdot, x)\nu, \mathbf{g} \Big\rangle_\Gamma,\ x \in M \setminus \Gamma, \tag{19}$$

$$(\mathcal{Q}^d\mathbf{h})(x) := \langle -2\text{Def } \Pi(x, \cdot)\nu - \Xi(x, \cdot)\nu, \mathbf{g} \rangle_\Gamma,\ x \in M \setminus \Gamma, \tag{20}$$

where $\Pi^\top(\cdot, x)$ is the transpose of $\Pi(\cdot, x)$, and ν is the outward unit normal.

In view of the fundamental solution $(\mathcal{G}, \Pi)$, the single and double-layer potentials satisfy the relations

$$\begin{matrix} L(\mathbf{Vf}) + d(\mathcal{Q}^s\mathbf{f}) = 0,\ \delta\mathbf{Vf} = 0 \\ L(\mathbf{Wh}) + d(\mathcal{Q}^d\mathbf{h}) = 0,\ \delta\mathbf{Wh} = 0 \end{matrix} \quad \text{in } M \setminus \Gamma. \tag{21}$$

The principal value of $\mathbf{Wg}$ is denoted by $\mathbf{Kg}$ and is defined as the principal value as

$$\begin{aligned} (\mathbf{Kg})(x) &:= \int_\Gamma^{\text{p.v.}} \langle -2\big[(\text{Def}_y\ \mathcal{G}(x, \cdot))\nu\big](y) \\ &\qquad + \Pi^\top(y, x)\nu(y), \mathbf{g}(y) \rangle \mathrm{d}\sigma(y) \\ &= \lim_{\varepsilon \to 0} \int_{\{y \in \Gamma:\ r(x,y) > \varepsilon\}} \langle -2\big[(\text{Def}_y\ \mathcal{G}(x, \cdot))\nu\big](y) \\ &\qquad + (\Pi)^\top(y, x)\nu(y), \mathbf{g}(y) \rangle \mathrm{d}\sigma(y), \end{aligned} \tag{22}$$

where $r(x, y)$ is the geodesic distance between $x, y \in M$.

For the following formula, for the layer potentials for the Stokes system in the case of compact Riemannian manifolds, we refer the reader to [14, Theorem A.2], [4, Theorem 2.1, (3.5), Proposition 3.5], [27, Theorems 3.1, 6.1].

THEOREM 3.1. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain. Assume that $\mathbf{f} \in H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ and $\mathbf{g} \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ are given. Then the following formulas hold almost everywhere on Γ:*

$$\text{Tr}^+(\mathbf{Vf}) = \text{Tr}^-(\mathbf{Vf}) := \mathcal{V}\mathbf{f}, \quad \partial_\nu^\pm(\mathbf{Vf}, \mathcal{Q}^s\mathbf{f}) = \Big(\pm \frac{1}{2}\mathbb{I} + \mathbf{K}^*\Big)\mathbf{f}, \tag{23}$$

$$\text{Tr}^\pm(\mathbf{Wg}) = \Big(\mp \frac{1}{2}\mathbb{I} + \mathbf{K}\Big)\mathbf{g}, \ \mathbf{D}^\pm\mathbf{g} := -\partial_\nu^\pm(\mathbf{Wg}, \mathcal{Q}^d\mathbf{g}), \ \mathbf{D}^+\mathbf{g} - \mathbf{D}^-\mathbf{g} \in \mathbb{R}\nu, \tag{24}$$

where $\mathbf{K}^*$ is the formal transpose of $\mathbf{K}$.

Also, we refer the reader to [16, Theorems 4.3, 4.9, 4.11, (131), (132), (137)] in the case of the Brinkman operator on compact Riemannian manifolds.

In the sequel, we are working with the closed subspace $H_\nu^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ of $H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ and the quotient space $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\nu$ of $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$, defined by

$$H_\nu^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) := \left\{\mathbf{f} \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) : \langle \nu, \mathbf{f}\rangle_\Gamma = 0\right\}, \tag{25}$$

$$H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\nu := \left\{[\mathbf{g}] = \mathbf{g} + \mathbb{R}\nu : \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)\right\}. \tag{26}$$

Note that $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\nu = (H_\nu^{\frac{1}{2}}(\Gamma, \Lambda^1 TM))^*$, (cf., e.g., [28, 5.118]), where $\mathbb{R}\nu = \{c\nu : c \in \mathbb{R}\}$.

Having these spaces, we state the following properties of the single-layer integral operators, which can be consulted in e.g. [27, Theorem 6.1].

THEOREM 3.2. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain. Then, the following operator is well-defined, linear and bounded and moreover an isomorphism:*

$$\mathcal{V} : H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\nu \to H_\nu^{\frac{1}{2}}(\Gamma, \Lambda^1 TM). \tag{27}$$

4. FORMULATION OF THE MIXED BOUNDARY VALUE PROBLEM

Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$, which is decomposed into two open, adjacent, nonoverlapping parts Γ_D, Γ_N with the following properties

$$\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \partial\Gamma_D = \partial\Gamma_N = \overline{\Gamma}_D \cap \overline{\Gamma}_N, \text{and meas } \Gamma_D > 0, \ \text{meas } \Gamma_N > 0. \tag{28}$$

Note that, the positive measure of both parts is critical to our case as will be explained in the sequel. Then, we consider the mixed problem with Dirichlet and Neumann boundary conditions for the Stokes system

$$\begin{cases} L\mathbf{u} + \mathrm{d}p = 0, \ \delta\mathbf{u} = 0 \text{ in } \mathfrak{D}, \\ (\text{Tr } \mathbf{u})|_{\Gamma_D} = \mathbf{f} \in H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \\ [\partial_\nu(\mathbf{u}, p)]\,|_{\Gamma_N} = [\mathbf{g}] \in H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)/\mathbb{R}\nu, \end{cases} \tag{29}$$

where $\cdot|_{\Gamma_D}, \cdot|_{\Gamma_N}$ denote the restrictions from $H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ to $H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$ and $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ to $H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$, respectively. Let us make some remarks regarding the restriction operators. The boundary space $H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$ is defined as follows

$$H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) = \{\mathbf{f} \in H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) : \langle \mathbf{f}, \nu|_{\Gamma_D} \rangle_{\Gamma_D} = 0\}. \tag{30}$$

By the restriction to Γ_N of a class of the conormal derivative operator we mean

$$[\partial_\nu(\mathbf{u}, p)]\,|_{\Gamma_N} = \partial_\nu(\mathbf{u}, p)|_{\Gamma_N} + \mathbb{R}\nu|_{\Gamma_N}, \tag{31}$$

i.e. the class of distributions $H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$ factorized by $\mathbb{R}\nu_N \equiv \mathbb{R}\nu|_{\Gamma_N}$. For simplicity we omit the subscript for the normal derivative, since its restriction should be clear from the context.

The main theorem of this paper is the following.

THEOREM 4.1. *Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$ which is decomposed into two parts Γ_D and Γ_N as in* (28). *Then the mixed Dirichlet-Neumann boundary value problem for the Stokes system* (29) *has a unique solution* $(\mathbf{u}, p)$*, up to a constant for the pressure, which can be represented by layer potentials and satisfies for some $C = C(\Gamma_D, \Gamma_N)$ the estimate*

$$\|\boldsymbol{u}\|_{H^1(\mathfrak{D}, \Lambda^1 TM)} + \|p\|_{L^2(\mathfrak{D})} \le C \|(\mathbf{f}, \mathbf{g})\|_{H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \times H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)/\mathbb{R}\nu}. \tag{32}$$

In order to prove the well-posedness of the boundary value problem (29), we will reformulate the problem as a system of boundary integral equations (BIE's), inspired by the main ideas in [1], for the Laplace equation, and [20], for the Stokes system (see also [24, Theorem 7.9], for a general elliptic system). We start with the Green representation formula of a weak solution of the Stokes system (see e.g. [4, (3.7)], [9, Chapter 5] and [28])

$$\mathbf{u}(x) = \mathbf{V}(\partial_\nu(\mathbf{u}, p)) - \mathbf{W}(\mathrm{Tr}\ \mathbf{u}), \tag{33}$$

$$p(x) = Q^s(\partial_\nu(\mathbf{u}, p)) - Q^d(\mathrm{Tr}\ \mathbf{u}). \tag{34}$$

Letting $x \to \Gamma$ from the inside of $\mathfrak{D}$ and following the jump relations given in Theorem 3.1, we obtain the following integral equations on Γ

$$\mathcal{V}(\partial_\nu(\mathbf{u}, p)) - \left(\frac{1}{2}\mathbb{I} + \mathbf{K}\right) \mathrm{Tr}\ \mathbf{u} = 0 \tag{35}$$

$$\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}^*\right)(\partial_\nu(\mathbf{u}, p)) + \mathbf{D}\mathrm{Tr}\ \mathbf{u} = 0. \tag{36}$$

In order to match the system composed of (35) and (36) to the mixed boundary problem for the Stokes system (29), let us denote by $\tilde{\mathbf{f}} \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$, $\tilde{\mathbf{g}} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\nu$ the arbitrary extensions to the entire Γ of the corresponding boundary data $\mathbf{f}$ and a representative $\mathbf{g}$ of the class $[\mathbf{g}]$ such that the boundary data is given by

$$(\mathrm{Tr}\ \mathbf{u})\,|_\Gamma = \varphi_N + \tilde{\mathbf{f}}, \quad [\partial_\nu(\mathbf{u}, p)]\,|_\Gamma = [\psi_D] + \tilde{\mathbf{g}}. \tag{37}$$

Obviously $\varphi_N \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$ and $[\psi_D] \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu$ since $\varphi_N = 0$ on Γ_D and $\langle [\psi_D], v\rangle = 0$, for every $v \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$.

By restricting (35) to Γ_D and (36) to Γ_N we obtain the following system of integral equations with the unknowns $\psi_D := [\psi_D] \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu$ and $\varphi_N \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$

$$\begin{cases} \mathcal{V}_D\psi_D - \mathbf{K}_{ND}\varphi_N = f_1, & x \in \Gamma_D \\ \mathbf{K}^*_{DN}\psi_D + \mathbf{D}_N\varphi_N = f_2, & x \in \Gamma_N \end{cases} \tag{38}$$

where

$$f_1 = \frac{1}{2}\tilde{\mathbf{f}} + \mathbf{K}\tilde{\mathbf{f}} - \mathcal{V}\tilde{\mathbf{g}}, \quad f_2 = -\mathbf{D}\tilde{\mathbf{f}} + \frac{1}{2}\tilde{\mathbf{g}} - \mathbf{K}^*\tilde{\mathbf{g}}. \tag{39}$$

The above system of equations can be written in a matrix form:

$$\mathcal{A}\begin{bmatrix} \psi_D \\ \varphi_N \end{bmatrix} = \begin{bmatrix} \mathcal{V}_D & -\mathbf{K}_{ND} \\ \mathbf{K}^*_{DN} & \mathbf{D}_N \end{bmatrix}\begin{bmatrix} \psi_D \\ \varphi_N \end{bmatrix} = \mathbf{f}, \tag{40}$$

where $\mathbf{f} = [f_1, f_2]^T$. Let us decompose the matrix operator $\mathcal{A}$ as

$$\mathcal{A} = \begin{bmatrix} \mathcal{V}_D & -\mathbf{K}_{ND} \\ \mathbf{K}^*_{DN} & \mathbf{D}_N \end{bmatrix} = \begin{bmatrix} \mathcal{V}_D & 0 \\ 0 & \mathbf{D}_N \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{K}_{ND} \\ \mathbf{K}^*_{DN} & 0 \end{bmatrix} = \mathfrak{B} + \mathfrak{C}, \tag{41}$$

where $\mathfrak{B}$ is some invertible matrix operator and $\mathfrak{C}$ is some compact matrix operator, which will imply that the operator $\mathcal{A}$ is a Fredholm operator of index zero.

4.1. Invertibility properties of the related layer potential operators.

In this subsection we prove that the single-layer integral operator $\mathcal{V}_D$ and the hypersingular integral operator $\mathbf{D}_N$ defined for distributions and functions, respectively, with support over a part of the boundary.

THEOREM 4.2. *Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary Γ as in* (28)*. Then the following operators are invertible:*

(i) *the hypersingular integral operator*

$$\mathbf{D}_N : \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM) \to H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)/\mathbb{R}\nu, \quad \mathbf{D}_N\varphi = \mathbf{D}\varphi|_{\Gamma_N}. \tag{42}$$

(ii) *the single-layer integral operator*

$$\mathcal{V}_D : \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu \to H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \quad \mathcal{V}_D\psi = \mathcal{V}\psi|_{\Gamma_D}. \tag{43}$$

Proof. We prove the invertibility of the hypersingular operator $\mathbf{D}_N$, following the main ideas in [1, Lemma 5.1, 5.2]. For any $\varphi \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$, let us consider the double-layer potentials $\mathbf{u}_0 = \mathbf{W}\varphi$, $p_0 = Q^s\varphi$, for which $(\mathbf{u}_0, p_0) \in \mathfrak{H}^1(\mathfrak{D})$. Moreover, we have the jump relations for the single-layer potential operator (see, e.g., [4, Theorem 2.1], [13, Lemma 3.1])

$$\mathrm{Tr}^-\big(\mathbf{W}\varphi\big) - \mathrm{Tr}^+\big(\mathbf{W}\varphi\big) = \varphi, \tag{44}$$

We omit $[\cdot]$ in order to simplify the expressions.

$$\partial_\nu^+ (\mathbf{W}\varphi, \mathcal{Q}^s\varphi) - \partial_\nu^- (\mathbf{W}\varphi, \mathcal{Q}^s\varphi) = -\mathbf{D}_N^+\varphi + \mathbf{D}_N^-\varphi = \mathbb{R}\nu. \tag{45}$$

Since the manifold M under consideration is boundaryless, we obtain the following relations based of the Green identity (10) for the interior and the exterior domain $\mathfrak{D}_\pm$

$$\langle \partial_\nu^+(\mathbf{u}_0, p_0), \mathrm{Tr}^+\mathbf{u}_0\rangle = \langle \mathbf{D}_N^+\varphi, \left(\frac{1}{2}\mathbb{I} + \mathbf{K}\right)\varphi\rangle = 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_+}, \tag{46}$$

$$-\langle \partial_\nu^-(\mathbf{u}_0, p_0), \mathrm{Tr}^-\mathbf{u}_0\rangle = \langle \mathbf{D}_N^-\varphi, \left(\frac{1}{2}\mathbb{I} - \mathbf{K}\right)\varphi\rangle = 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_-}, \tag{47}$$

where $2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_\pm} = 2\|\mathrm{Def}\mathbf{u}_0\|^2_{L^2(\mathfrak{D}_\pm, S^2T^*M)}$.

The assumption that the density φ belongs to the space $\widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$ and the second relation in (24) yield the equality (as equivalence of classes) between the interior and exterior hypersingular potentials $[\mathbf{D}_N^+\varphi] = [\mathbf{D}_N^-\varphi] =: \mathbf{D}_N\varphi$. Hence, adding (46) and (47), we obtain the relation

$$\langle \mathbf{D}_N\varphi, \varphi\rangle = 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_+} + 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_-}. \tag{48}$$

• **The operator $\mathbf{D}_N$ is one-on-one.** Assume that $\varphi \in \widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$, such that

$$\mathbf{D}_N\varphi = [0], \text{ i.e., } \exists c_1, c_2 \in \mathbb{R} \text{ such that } \mathbf{D}_N^\pm\varphi = c_{1,2}\nu. \tag{49}$$

In view of equation (48), we have $\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_\pm} = 0$. Consider the double-layer potential $\mathbf{u}_0 = \mathbf{W}\varphi$ and restrict the jump relation given in (44) to Γ_D. Since φ has the support on Γ_N, we obtain that $\mathrm{Tr}^+\mathbf{u}_0 = \mathrm{Tr}^-\mathbf{u}_0 = 0$ on Γ_D. This and the second Korn inequality (see e.g. [23, Theorem 6]) $\|\mathbf{u}_0\|_{H^1(\mathfrak{D}_\pm, \Lambda^1 TM)} \leq c\|\mathrm{Def}\mathbf{u}_0\|_{L^2(\mathfrak{D}_\pm, \Lambda^1 TM)}$ imply that $\mathbf{u}_0 = 0$ in $\mathfrak{D}_\pm$. Consequently, the jump relation $\varphi = \mathrm{Tr}^-\mathbf{u}_0 - \mathrm{Tr}^+\mathbf{u}_0$ implies that $\varphi = 0$ (see e.g. [1, Lemma 5.1]).

Note that the assumption meas $\Gamma_D > 0$ is critical for the above idea.

• **The operator D is onto.** In order to show that the double-layer operator is onto, we show that it is bounded below and has dense range. Applying Korn's inequality (see e.g. [9, Lemma 5.4.4]) to the right hand side of (48), we obtain

$$\langle \mathbf{D}_N\varphi, \varphi\rangle \geq c_0\|\mathbf{u}_0\|^2_{H^1(\mathfrak{D}_\pm, \Lambda^1 TM)} - \|\mathbf{u}_0\|^2_{L^2(\mathfrak{D}_\pm, \Lambda^1 TM)}. \tag{50}$$

The adjoint double-layer potential $\mathbf{W}^*_{\mathfrak{D}_+} : H^1(\mathfrak{D}_+, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ on Hilbert spaces and the compactness of the embedding $H^1(\mathfrak{D}_+, \Lambda^1 TM) \hookrightarrow L^2(\mathfrak{D}_+, \Lambda^1 TM)$ yield the relation

$$\langle \mathbf{W}_{\mathfrak{D}_+}u, \mathbf{W}_{\mathfrak{D}_+}\varphi\rangle = \langle u, \mathbf{W}^*_{\mathfrak{D}_+}\mathbf{W}_{\mathfrak{D}_+}\varphi\rangle. \tag{51}$$

Applying the trace theorem to (50) yields, for the given compact operator $C_D : \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM) \to H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$,

$$(52)\quad \langle \mathcal{C}_D u, \varphi\rangle = \langle u, \mathbf{W}^*_{\mathfrak{D}_+}\mathbf{W}_{\mathfrak{D}_+}\varphi\rangle + \langle u, \mathbf{W}^*_{\mathfrak{D}_-}\mathbf{W}_{\mathfrak{D}_-}\varphi\rangle,\ \forall \varphi \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM),$$

the Garding inequality

$$(53)\qquad \langle(\mathbf{D}_N + \mathcal{C}_D)\varphi, \varphi\rangle \geq c_1\|\varphi\|^2_{\widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N,\Lambda^1 TM)}.$$

The injectivity of the operator $\mathbf{D}_N$ implies that

$$(54)\qquad \langle \mathbf{D}_N\varphi, \varphi\rangle \geq c_1\|\varphi\|^2_{\widetilde{H}_\nu^{\frac{1}{2}}(\Gamma_N,\Lambda^1 TM)}.$$

Hence, the Lax-Milgram theorem (see e.g. [9, Theorem 5.2.3]) implies that the operator $\mathbf{D}_N$ is an isomorphism.

Next, following the main ideas as in the proof of the invertibility of the hypersingular operator, we show that the single-layer potential is invertible. For $\psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu$, i.e. a chosen representative of the class $[\psi]$, we consider the single-layer potential $\mathbf{u}_0 = \mathbf{V}_D\psi$ and $p_0 = Q^s\psi$. Applying the Green formula for the interior and the exterior domain and adding the two equations, we obtain

$$(55)\qquad \langle \mathcal{V}_D\psi, \psi\rangle = 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_+} + 2\langle \mathrm{Def}\mathbf{u}_0, \mathrm{Def}\mathbf{u}_0\rangle_{\mathfrak{D}_-}.$$

- **The operator $\mathcal{V}_D$ is one-to-one.** Let us consider a density $\psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu$ such that $\mathcal{V}_D\psi = 0$. By equation (55), we deduce that $\mathrm{Def}\mathbf{u}_0 = 0$ on $\mathfrak{D}_\pm$. This and the jump relation for the single-layer potential operator $\mathrm{Tr}^+\mathbf{u}_0 = \mathrm{Tr}^-\mathbf{u}_0 = 0$ imply that $\mathbf{u}_0 = 0$ on $\mathfrak{D}_\pm$ and, moreover, that $\psi = 0$. Hence the operator $\mathcal{V}_D$ is one-to-one.
- **The operator $\mathcal{V}_D$ is onto.** Let us now prove that the operator is bounded from bellow. To this end, we apply the Korn inequality to equation (55) and obtain

$$(56)\qquad \langle \mathcal{V}_D\psi, \psi\rangle \geq c_0\|\mathbf{u}_0\|^2_{H^1(\mathfrak{D}_\pm,\Lambda^1 TM)} - \|\mathbf{u}_0\|^2_{L^2(\mathfrak{D}_\pm,\Lambda^1 TM)}.$$

Similar arguments to those for the hypersingular operator yield the following Garding inequality

$$(57)\qquad \langle(\mathcal{V}_D + C_V)\psi, \psi\rangle \geq c_0\|\psi\|^2_{H^{-\frac{1}{2}}(\Gamma_D,\Lambda^1 TM)/\mathbb{R}\nu},$$

where the compact operator $C_V : H^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)/\mathbb{R}\nu \to H_\nu^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$ is given by $\langle C_V u, \psi\rangle = \langle u, \mathbf{V}^*_{\mathfrak{D}_+}\mathbf{V}_{\mathfrak{D}_+}\psi\rangle + \langle u, \mathbf{V}^*_{\mathfrak{D}_-}\mathbf{V}_{\mathfrak{D}_-}\psi\rangle$. Equation (57) together with the property that $\mathcal{V}_D = \mathcal{V}^*_D$ imply that the operator is onto. □

4.2. Compactness of the related operators. This subsection is devoted to the compactness of the operators $\mathbf{K}_{ND}$ and $\mathbf{K}^*_{DN}$. Note that the condition that the operators are defined for functions/distributions with support on one part of the boundary with values on the other part is essential for our purpose.

THEOREM 4.3. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain with boundary as in* (28). *Then the following operators are compact:*

$$\boldsymbol{K}_{ND} : \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \qquad \boldsymbol{K}_{ND}\varphi = \boldsymbol{K}\varphi|_{\Gamma_D}, \tag{58}$$

$$\boldsymbol{K}^*_{DN} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \to H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \quad \boldsymbol{K}^*_{DN}\psi = \boldsymbol{K}^*\psi|_{\Gamma_N}. \tag{59}$$

Proof. We prove the compactness of the adjoint operator $\mathbf{K}^*_{DN}$, which implies that $\mathbf{K}_{ND}$ is also compact. The compactness of the operator given in (59) is provided by the regularity of the kernel $\mathbf{S}(x, y)\nu(x) = -2[(\mathrm{Def}_x \mathcal{G}(\cdot, y))\nu](x) -\Pi^\top(x, y)\nu(x)$. Working in local coordinates (cf. [4, Prop 3.1]), we can express the kernel in the following way (see also [16, Section 5.2])

$$S_{ksl}(x, y)\nu_k(x) = \nu_j(x) g^{jk}(x)(\partial_{x_k}\mathcal{G}_{ls}(x, y) + \partial_{x_l}\mathcal{G}_{ks}(x, y)) - \nu_l(x)\Pi_s(x, y). \tag{60}$$

For clarity, we prove first the compactness for the particular case $\mathbf{K}^*_{DN} : \widetilde{L}^2(\Gamma_D, \Lambda^1 TM) \to L^2(\Gamma_N, \Lambda^1 TM)$. Then a density argument implies the compactness for the operator in (59). For a $\psi \in \widetilde{L}^2(\Gamma_D, \Lambda^1 TM)$, the operator $\mathbf{K}^*_{DN}$ can be rewritten as

$$\begin{aligned}(\mathbf{K}^*_{DN}\psi)_s(x) &= \int_\Gamma^{\text{p.v.}} \big\langle -2\big[(\mathrm{Def}_x\, \mathcal{G}(\cdot, y))\nu\big](x) \\ &\qquad + \Pi^\top(x, y)\nu(x), \psi(y)\big\rangle_s \mathrm{d}\sigma(y) \\ &= \nu_k(x) \int_{\Gamma_D}^{\text{p.v.}} S_{ksl}(x, y)\psi_l(y)\mathrm{d}\sigma(y), \quad \forall x \in \Gamma_N.\end{aligned} \tag{61}$$

Taking into account that $\Gamma_D \cap \Gamma_N = \emptyset$, the kernel of the operator in equation (60) is smooth $S_{ksl}(x, y) \in C^\infty(\Gamma_N \times \Gamma_D)$, because the fundamental solution for the Stokes system has no singularity if $x \neq y$. Hence the operator $\mathbf{K}^*_{DN} : \widetilde{L}^2(\Gamma_D, \Lambda^1 TM) \to L^2(\Gamma_N, \Lambda^1 TM)$ is compact. Finally, by the density of the embedding $\widetilde{L}^2(\Gamma_D, \Lambda^1 TM) \hookrightarrow \widetilde{H}^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$, we conclude that $\mathbf{K}^*_{DN}$ given in (59) is compact. □

4.3. Proof of the main theorem. Having the above results, we now give the proof of Theorem 4.1.

Proof of Theorem 4.1. The solvability of the matrix equation (41) is equivalent to the well-posedness of the mixed Dirichlet-Neumann problem for the Stokes system. By rewriting the equation under the form of (41), we obtain an invertible matrix

$$\mathfrak{B}^{-1} = \begin{bmatrix} \mathcal{V}_D^{-1} & 0 \\ 0 & \mathbf{D}_N^{-1} \end{bmatrix}, \tag{62}$$

in view of the invertibility of the two operators in Theorem 3.1 and the compact matrix operator $\mathfrak{C}$ given by Theorem 4.2. Thus the operator $\mathcal{A}$ is a Fredholm operator with index zero and, by the Fredholm Alternative, the injectivity is equivalent to the invertibility.

In order to show that $\mathcal{A}$ is one-to-one, let us consider the homogeneous version of (38), i.e.

$$\begin{cases} \mathcal{V}_D\psi_D^0 - \mathbf{K}_{ND}\varphi_N^0 = 0, & x \in \Gamma_D \\ \mathbf{K}_{DN}^*\psi_D^0 + \mathbf{D}_N\varphi_N^0 = 0, & x \in \Gamma_N \end{cases}, \tag{63}$$

and let $(\mathbf{u}_0, p_0) := (\mathbf{V}\psi_D^0 - \mathbf{W}\varphi_N^0, Q^s\psi_D^0 - Q^d\varphi_N^0)$ be the fields determined by the densities (ψ_D^0, φ_N^0). The boundary conditions lead to

$$\mathrm{Tr}^{\pm}\mathbf{u}_0 = \mathcal{V}_D\psi_D^0 - \Big(\pm\frac{1}{2}\mathbb{I} + \mathbf{K}_{ND}\Big)\varphi_N^0, \quad \text{i.e.,} \quad \mathrm{Tr}^{\pm}\mathbf{u}_0|_{\Gamma_\mathrm{D}} = 0, \tag{64}$$

$$\partial_\nu^{\pm}(\mathbf{u}_0, p_0) = \Big(\mp\frac{1}{2}\mathbb{I} + \mathbf{K}_{DN}^*\Big)\psi_D^0 + \mathbf{D}_N\varphi_N^0, \ \text{i.e.,} \ [\partial_\nu^{\pm}(\mathbf{u}_0, p_0)]|_{\Gamma_N} = [0], \tag{65}$$

since $(\varphi_N^0)|_{\Gamma_D} = 0$ and $(\psi_D^0)|_{\Gamma_N} = 0$ in the distributional sense.

Applying the Green formula for $\mathfrak{D}_{\pm}$, we obtain

$$0 = \langle \partial_\nu^{\pm}(\mathbf{u}_0, p_0), \mathrm{Tr}^{\pm}\,\mathbf{u}_0\rangle_\Gamma = 2\langle \mathrm{Def}\,\mathbf{u}_0, \mathrm{Def}\,\mathbf{u}_0\rangle_{\mathfrak{D}_+} + 2\langle \mathrm{Def}\,\mathbf{u}_0, \mathrm{Def}\,\mathbf{u}_0\rangle_{\mathfrak{D}_-}, \tag{66}$$

which imply $\mathrm{Def}\,\mathbf{u}_0 = 0$ in $\mathfrak{D}_{\pm}$. This property together with relation (64) and the assumption that the manifold has no nontrivial Killing fields, imply, by the second Korn inequality ([23, Theorem 6]), that $\mathbf{u}_0 = 0$ in $\mathfrak{D}_{\pm}$. Hence, the jump relations yield that $0 = \mathrm{Tr}^{-}\mathbf{u}_0 - \mathrm{Tr}^{+}\mathbf{u}_0 = \varphi_\mathrm{N}^0$ and $[0] = [\partial_\nu^{-}(\mathbf{u}_0, p_0)] - [\partial_\nu^{+}(\mathbf{u}_0, p_0)] = [\psi_D^0]$, i.e. $([\psi_D^0], \varphi_N^0) = ([0], 0)$, which proves the injectivity of the operator $\mathcal{A}$ and completes the proof.

Finally, let $\mathcal{S} : H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1TM) \times H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1TM)/\mathbb{R}\nu \to H^1(\mathfrak{D}, \Lambda^1TM) \times L^2(\mathfrak{D})$ be the solution operator delivering the unique solution $(\mathbf{u}, p)$ (up to a constant of pressure) for the Stokes system (29), i.e.

$$\begin{aligned} &(\mathbf{u}, p) = \mathcal{S}(\mathbf{f}, \mathbf{g}) := (\mathcal{U}(\mathbf{f}, \mathbf{g}), \mathcal{P}(\mathbf{f}, \mathbf{g})), \ \text{where} \\ &\mathcal{U}(\mathbf{f}, \mathbf{g}) := \mathbf{V}([\tilde{\mathbf{g}}] + [\mathrm{pr}_1\mathcal{A}^{-1}(\mathbf{f}, \mathbf{g})]) - \mathbf{W}(\tilde{\mathbf{f}} + \mathrm{pr}_2\mathcal{A}^{-1}(\mathbf{f}, \mathbf{g})), \\ &\mathcal{P}(\mathbf{f}, \mathbf{g}) := Q^s([\tilde{\mathbf{g}}] + [\mathrm{pr}_1\mathcal{A}^{-1}(\mathbf{f}, \mathbf{g})]) - Q^d(\tilde{\mathbf{f}} + \mathrm{pr}_2\mathcal{A}^{-1}(\mathbf{f}, \mathbf{g})), \end{aligned} \tag{67}$$

where pr_1, (pr_2) is the canonical projection on the first (second) component of $\mathcal{A}^{-1}$. By the linearity of the involved operators, we obtain the estimate

$$\|\mathbf{u}\|_{H^1(\mathfrak{D}, \Lambda^1TM)} + \|p\|_{L^2(\mathfrak{D})} \leq C\|(\mathbf{f}, \mathbf{g})\|_{H_\nu^{\frac{1}{2}}(\Gamma_D, \Lambda^1TM) \times H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1TM)/\mathbb{R}\nu}. \tag{68}$$

□

5. CONCLUSIONS

A well-posed result for the mixed boundary value problem of Dirichlet and Neumann type for the Stokes system on Riemannian manifolds is obtained, by reducing the boundary problem to a system of boundary integral equations. The related matrix operator is written as the sum of an invertible operator and a compact one. Hence, the operator is Fredholm of index zero and its injectivity proves that the operator is an isomorphism.

Moreover, the paper shows the isomorphism properties for the single-layer potential operator and the hypersingular integral operator defined on one part of the boundary. Also, we prove the compactness of the double-layer potential operator and its adjoint defined from one part of the boundary to the other one. These results are helpful to show the Fredholm property of the matrix operator related to the boundary integral system.

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