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# A NOTE ON NUMERICAL RADIUS AND THE KREĬN-LIN INEQUALITY

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**Abstract.** In this note we show that the Kreĭn-Lin triangle inequality can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on complex Hilbert spaces, due to C. Pearcy.

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**Key words.** Triangle inequality, operator norm, numerical radius.

## 1. INTRODUCTION

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x$, $y \in H$ two nonzero vectors. One can define the *angle* between the vectors $x$, $y$ either by the standard formula $\cos \Phi_{x,y} = \dfrac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|}$ or by $\cos \Psi_{x,y} = \dfrac{|\langle x, y \rangle|}{\|x\| \|y\|}$. The function $\Psi_{x,y}$ is a natural metric on the complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y}, \tag{1}$$

for any $x$, $y$, $z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y} \tag{2}$$

and Kreĭn's inequality (1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{z,y}, \tag{3}$$

for any $x$, $y$, $z \in H \setminus \{0\}$.

In this note we show that the Kreĭn-Lin triangle inequality (3) can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on a complex Hilbert space, due to C. Pearcy [7].

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## 2. A REVERSE INEQUALITY

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator $T$ is the subset of the complex numbers $\mathbb{C}$ given by [3, p. 1]:

$$W(T)=\{\langle Tx,x\rangle\,,\ x\in H,\ \|x\|=1\}\,.$$

The *numerical radius* $w(T)$ of an operator $T$ on $H$ is given by [3, p. 8]:

$$w(T)=\sup\{|\lambda|\,,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|\,,\|x\|=1\}\,. \tag{4}$$

It is well known that $w(\cdot)$ is a *norm* on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [3, p. 9]:

$$w(T)\le\|T\|\le 2w(T)\,, \tag{5}$$

for any $T\in B(H)$

For other results on numerical radii, see [4, Chapter 11], [3] and the recent monograph [2].

The following result is well known in the literature [7]:

$$w(T^n)\le w^n(T)\,, \tag{6}$$

for each positive integer $n$ and any operator $T\in B(H)$.

The following elegant reverse inequality for $n=2$ can be derived from the Kreĭn-Lin triangle inequality (3).

THEOREM 2.1. *For any $T\in B(H)$, we have*

$$w^2(T)\le w\left(T^2\right)+\inf_{\lambda\in\mathbb{C}}\|T-\lambda I\|^2. \tag{7}$$

*Proof.* The inequality (3) is equivalent to

$$\cos\Psi_{x,y}\ge\cos(\Psi_{x,z}+\Psi_{y,z})=\cos\Psi_{x,z}\cos\Psi_{y,z}-\sin\Psi_{x,z}\sin\Psi_{y,z} \tag{8}$$

or to

$$\frac{|\langle x,y\rangle|}{\|x\|\|y\|}+\sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}}\sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}}\ge\frac{|\langle x,z\rangle|}{\|x\|\|z\|}\frac{|\langle y,z\rangle|}{\|y\|\|z\|}, \tag{9}$$

for any $x,y,z\in H\setminus\{0\}$.

If we multiply (10) by $\|x\|\|z\|^2\|y\|>0$, then we get

$$\begin{aligned}|\langle x,y\rangle|\,\|z\|^2+\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2}\\ \ge|\langle x,z\rangle|\,|\langle y,z\rangle|\,.\end{aligned} \tag{10}$$

We notice that the inequality (10) remains true, becoming equality, if either $x=0$ or $y=0$ or $z=0$.

We know that, for any $u$, $e\in H$ with $\|e\|=1$, we have the representation (see for instance [1, Lemma 2.4])

$$\|u\|^2-|\langle u,e\rangle|^2=\|u-\langle u,e\rangle e\|^2=\inf_{\lambda\in\mathbb{C}}\|u-\lambda e\|^2.$$

Then, by (10), we have, for any $x$, $y$, $z \in H$ with $\|z\| = 1$, that

$$|\langle x, y\rangle| + \inf_{\lambda\in\mathbb{C}} \|x - \lambda z\| \inf_{\mu\in\mathbb{C}} \|y - \mu z\| \geq |\langle x, z\rangle| \, |\langle y, z\rangle| \,. \tag{11}$$

By taking $x = Tz$ and $y = T^*z$ in (11), we get

$$\begin{aligned} |\langle Tz, z\rangle| \, |\langle T^*z, z\rangle| &\leq |\langle Tz, T^*z\rangle| + \inf_{\lambda\in\mathbb{C}} \|Tz - \lambda z\| \inf_{\mu\in\mathbb{C}} \|T^*z - \mu z\| \\ &\leq |\langle Tz, T^*z\rangle| + \|Tz - \lambda z\| \|T^*z - \mu z\|, \end{aligned}$$

for any $z \in H$ with $\|z\| = 1$ and $\lambda$, $\mu \in \mathbb{C}$.

Therefore

$$|\langle Tz, z\rangle|^2 \leq \left|\left\langle T^2 z, z\right\rangle\right| + \|Tz - \lambda z\| \|T^*z - \mu z\|,$$

for any $z \in H$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$.

By taking the supremum over $z \in H$ with $\|z\| = 1$, we deduce

$$w^2(T) \leq w\left(T^2\right) + \|T - \lambda I\| \|T^* - \mu I\|, \tag{12}$$

for any $\lambda$, $\mu \in \mathbb{C}$.

Finally, by taking the infimum in (12) over $\lambda$, $\mu \in \mathbb{C}$ and since

$$\inf_{\mu\in\mathbb{C}} \|T^* - \mu I\| = \inf_{\mu\in\mathbb{C}} \|T - \overline{\mu} I\| = \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|,$$

we deduce the desired result (7). □

COROLLARY 2.2. *Let $T \in B(H)$. If there exist $\omega \in \mathbb{C}$ and $r > 0$ such that $\|T - \omega I\| \leq r$, then $w^2(T) \leq w\left(T^2\right) + r^2$.*

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