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A TOPOLOGY VIA ω-LOCAL FUNCTIONS IN IDEAL SPACES

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Abstract. The class of ω-closed subsets of a space (X, τ) was defined to introduce ω-closed functions. The purpose of this paper to introduce the notion of ω-local functions and to give some of its basic properties in an ideal topological space. Moreover, we define and investigate the ω-compatible spaces.

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1. INTRODUCTION AND PRELIMINARIES

A point $x \in X$ is called a condensation point of A, if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed - see [12] - if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if, for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, which can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively. Several characterizations of ω-closed subsets and ideal spaces were provided in [1, 2, 3, 5, 6, 7, 9, 12, 13].

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ imply that $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [11, 14]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$, when there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is known - see [11] - that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*.

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Recall that A is said to be $*$-dense in itself (resp. τ^*-closed, $*$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X, τ^*), respectively. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the topology τ is compatible with the ideal $\mathcal{I}$ and denote $\tau \sim \mathcal{I}$, if, for every $A \subseteq X$, we have that if, for every $x \in A$, there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ - see [11].

2. ω-LOCAL FUNCTIONS

DEFINITION 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset A of X, we define the set $A_\omega(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau_\omega(x)\}$, where $\tau_\omega(x) = \{U \in \tau_w : x \in U\}$. When there is no confusion, $A_\omega(\mathcal{I}, \tau)$ is briefly denoted by A_ω and called the ω-local function of A with respect to $\mathcal{I}$ and τ.

LEMMA 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $A_\omega(\mathcal{I}, \tau) \subseteq A^*(\mathcal{I}, \tau)$, for every subset A of X.*

Proof. Let $x \in A_\omega(\mathcal{I}, \tau)$. Then $A \cap U \notin \mathcal{I}$, for every ω-open set U containing x. Since every open set is ω-open, $x \in A^*(\mathcal{I}, \tau)$. □

EXAMPLE 2.3. Let X be an uncountable set and let A, B, C be subsets of X such that each of them is an uncountable set and the family $\{A, B, C, D\}$ is a partation of X. We defined the topology $\tau = \{\emptyset, \{A\}, \{A, B\}, \{A, B, C\}, X\}$ with $\mathcal{I} = \{\emptyset, \{A\}, \{B\}, \{A, B\}\}$. Let $H = \{A, D\}$, then $H^*(I) = \{C, D\} = Cl(H^*)$ and $H_\omega(I) = \emptyset = Cl_\omega(H_\omega)$.

LEMMA 2.4. *Let (X, τ) be an ideal topological space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X, and let A and B be subsets of X. Then the following properties hold.*

(1) *If $A \subseteq B$, then $A_\omega \subseteq B_\omega$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $A_\omega(\mathcal{I}) \supseteq A_\omega(\mathcal{J})$.*
(3) *$A_\omega = Cl_\omega(A_\omega) \subseteq Cl_\omega(A)$ and A_ω is ω-closed in (X, τ).*
(4) *If $A \subseteq A_\omega$, then $A_\omega = Cl_\omega(A_\omega) = Cl_\omega(A)$.*
(5) *If $A \in \mathcal{I}$, then $A_\omega = \emptyset$.*

Proof. (1) Suppose that $x \notin B_\omega$. Then there exists $U \in \tau_\omega(x)$ such that $U \cap B \in \mathcal{I}$. Since $U \cap A \subseteq U \cap B$, $U \cap A \in \mathcal{I}$. Hence $x \notin A_\omega$. Thus $X \setminus B_\omega \subseteq X \setminus A_\omega$ or $A_\omega \subseteq B_\omega$.

(2) Suppose that $x \notin A_\omega(\mathcal{I})$. There exists $U \in \tau_\omega(x)$ such that $U \cap A \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $U \cap A \in \mathcal{J}$ and $x \notin A_\omega(\mathcal{J})$. Therefore $A_\omega(\mathcal{J}) \subseteq A_\omega(\mathcal{I})$.

(3) We have $A_\omega \subseteq Cl_\omega(A_\omega)$, in general. Let $x \in Cl_\omega(A_\omega)$. Then $A_\omega \cap U \neq \emptyset$, for every $U \in \tau_\omega(x)$. Therefore there exist $y \in A_\omega \cap U$ and $U \in \tau_\omega(y)$. Since $y \in A_\omega$, $A \cap U \notin \mathcal{I}$ and hence $x \in A_\omega$. Hence we have $Cl_\omega(A_\omega) \subseteq A_\omega$ and thus $A_\omega = Cl_\omega(A_\omega)$. Again, let $x \in Cl_\omega(A_\omega) = A_\omega$. Then $U \cap A \notin \mathcal{I}$ for every $U \in \tau_\omega(x)$. This implies $U \cap A \neq \emptyset$, for every $U \in \tau_\omega(x)$. Therefore $x \in Cl_\omega(A)$. This shows that $A_\omega(\mathcal{I}) = Cl_\omega(A_\omega) \subseteq Cl_\omega(A)$.

(4) For any subset A of X, by (3), we have $A_\omega = Cl_\omega(A_\omega) \subseteq Cl_\omega(A)$. Since $A \subseteq A_\omega$, $Cl_\omega(A) \subseteq Cl_\omega(A_\omega)$ and hence $A_\omega = Cl_\omega(A_\omega) = Cl_\omega(A)$.

(5) Suppose that $x \in A_\omega$. Then, for any $U \in \tau_\omega(x)$, $U \cap A \notin \mathcal{I}$. But, since $A \in \mathcal{I}$, $U \cap A \in \mathcal{I}$ for some $U \in \tau_\omega(x)$. This is a contradiction. So $A_\omega = \emptyset$. □

LEMMA 2.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $U \in \tau$, then $U \cap A_\omega = U \cap (U \cap A)_\omega \subseteq (U \cap A)_\omega$, for any closed set A of X.*

Proof. Suppose that U is open set and $x \in U \cap A_\omega$. Then $x \in U$ and $x \in A_\omega$. Let V be any ω-open set containing x. Then $V \cap U \in \tau_\omega(x)$ and $V \cap (U \cap A) = (V \cap U) \cap A \notin \mathcal{I}$. This shows that $x \in (U \cap A)_\omega$ and hence we obtain $U \cap A_\omega \subseteq (U \cap A)_\omega$. Moreover, $U \cap A_\omega \subseteq U \cap (U \cap A)_\omega$ and, by Lemma 2.4, $(U \cap A)_\omega \subseteq A_\omega$ and $U \cap (U \cap A)_\omega \subseteq U \cap A_\omega$. Therefore $U \cap A_\omega = U \cap (U \cap A)_\omega$. □

3. A TOPOLOGY ASSOCIATED WITH ω-LOCAL FUNCTIONS

THEOREM 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be any subsets of X. Then the following properties hold:*

(1) $(\emptyset)_\omega = \emptyset$.
(2) $(A_\omega)_\omega \subseteq A_\omega$.
(3) $A_\omega \cup B_\omega = (A \cup B)_\omega$.

Proof. (1) The proof is obvious.

(2) Let $x \in (A_\omega)_\omega$. Then, for every $U \in \tau_\omega(x)$, $U \cap A_\omega \notin \mathcal{I}$ and hence $U \cap A_\omega \neq \emptyset$. Let $y \in U \cap A_\omega$. Then $U \in \tau_\omega(y)$ and $y \in A_\omega$. Hence we have $U \cap A \notin \mathcal{I}$ and $x \in A_\omega$. This shows that $(A_\omega)_\omega \subseteq A_\omega$.

(3) It follows from Lemma 2.4 that $(A \cup B)_\omega \supseteq A_\omega \cup B_\omega$. To prove the reverse inclusion, let $x \notin A_\omega \cup B_\omega$. Then x belongs neither to A_ω nor to B_ω. Therefore there exist U_x, $V_x \in \tau_\omega(x)$ such that $U_x \cap A \in \mathcal{I}$ and $V_x \cap B \in \mathcal{I}$. Since $\mathcal{I}$ is additive, $(U_x \cap A) \cup (V_x \cap B) \in \mathcal{I}$. Since $\mathcal{I}$ is hereditary and

$$\begin{aligned}(U_x \cap A) \cup (V_x \cap B) &= [(U_x \cap A) \cup V_x] \cap [(U_x \cap A) \cup B] \\ &= (U_x \cup V_x) \cap (A \cup V_x) \cap (U_x \cup B) \cap (A \cup B) \\ &\supseteq (U_x \cap V_x) \cap (A \cup B),\end{aligned}$$

$(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I}$. Since $(U_x \cap V_x) \in \tau_\omega(x)$, $x \notin (A \cup B)_\omega$. Hence $(X \setminus A_\omega) \cap (X \setminus B_\omega) \subseteq X \setminus (A \cup B)_\omega$ or $(A \cup B)_\omega \subseteq A_\omega \cup B_\omega$. Hence we obtain $A_\omega \cup B_\omega = (A \cup B)_\omega$. □

THEOREM 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, $Cl^*_\omega(A) = A_\omega \cup A$ and A, B be subsets of X. Then:*

(1) $Cl^*_\omega(\emptyset) = \emptyset$.
(2) $A \subseteq Cl^*_\omega(A)$.
(3) $Cl^*_\omega(A \cup B) = Cl^*_\omega(A) \cup Cl^*_\omega(B)$.
(4) $Cl^*_\omega(A) = Cl^*_\omega(Cl^*_\omega(A))$.
(5) *If $A \subseteq B$, then $Cl^*_\omega(A) \subseteq Cl^*_\omega(B)$.*

Proof. By Theorem 3.1, we obtain:

(1) $Cl^*_\omega(\emptyset) = (\emptyset)_\omega \cup \emptyset = \emptyset$.

(2) $A \subseteq A \cup A_\omega = Cl^*_\omega(A)$.
(3) $Cl^*_\omega(A \cup B) = (A \cup B)_* \cup (A \cup B) = (A_\omega \cup B_\omega) \cup (A \cup B) = Cl^*_\omega(A) \cup Cl^*_\omega(B)$.
(4) $Cl^*_\omega(Cl^*_\omega(A)) = Cl^*_\omega(A_\omega \cup A) = (A_\omega \cup A)_\omega \cup (A_\omega \cup A) = ((A_\omega)_\omega \cup A_\omega) \cup (A_\omega \cup A) = A_\omega \cup A = Cl^*_\omega(A)$.
(5) Since $A \subseteq B$, we have $Cl^*_\omega(A) = A_\omega \cup A \subseteq B_\omega \cup B = Cl^*_\omega(B)$. □

By Theorem 3.2, we obtain that $Cl^*_\omega(A) = A \cup A_\omega$ is a Kuratowski closure operator. We will denote by τ^*_ω the topology generated by Cl^*_ω, that is $\tau^*_\omega = \{U \subseteq X : Cl^*_\omega(X - U) = X - U\}$.

Lemma 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then $A_\omega - B_\omega = (A - B)_\omega - B_\omega$.*

Proof. By Theorem 3.1, we obtain $A_\omega = [(A - B) \cup (A \cap B)]_\omega = (A - B)_\omega \cup (A \cap B)_\omega \subseteq (A - B)_\omega \cup B_\omega$. Therefore $A_\omega - B_\omega \subseteq (A - B)_\omega - B_\omega$. We have, by Theorem 3.1, $(A - B)_\omega \subseteq A_\omega$ and hence $(A - B)_\omega - B_\omega \subseteq A_\omega - B_\omega$. Hence we obtain $A_\omega - B_\omega = (A - B)_\omega - B_\omega$. □

Corollary 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X with $B \in \mathcal{I}$. Then $(A \cup B)_\omega = A_\omega = (A - B)_\omega$.*

Proof. By Theorem 3.2 and since $B \in \mathcal{I}$, $B_\omega = \phi$. Therefore $A_\omega = (A - B)_\omega$, by Lemma 3.3. Hence, by Theorem 3.2, $(A \cup B)_\omega = A_\omega \cup B_\omega = A_\omega$. □

Lemma 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then:*
(1) *$Cl^*_\omega(A \cap B) \subseteq Cl^*_\omega(A) \cap Cl^*_\omega(B)$.*
(2) *If $U \in \tau_\omega$, then $U \cap Cl^*_\omega(A) \subseteq Cl^*_\omega(U \cap A)$.*

Proof. (1) This is obvious by Theorem 3.2.
(2) Since $U \in \tau_\omega$, we have by Theorem 3.2, $U \cap Cl^*_\omega(A) = U \cap (A \cup A_\omega) = (U \cap A) \cup (U \cap A_\omega) \subseteq (U \cap A) \cup (U \cap A)_\omega = Cl^*_\omega(U \cap A)$. □

Corollary 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be subsets of X. If $A \subseteq A_\omega$, then $Cl_\omega(A) = Cl^*_\omega(A)$.*

Proof. The proof follows from Theorem 3.2. □

Definition 3.7. Let (X, τ) be a topological space and $\mathcal{I}$ an ideal on X. A subset A of X is said to be τ^*_ω-closed if and only if $A_\omega \subseteq A$.

It is well known that if $U \in \tau^*_\omega$ if and only if $X - U$ is τ^*_ω-closed, then $U \subseteq X - (X - U)_\omega$. Thus, if $x \in U$, $x \notin (X - U)_\omega$, i.e there exists a ω-open set V such that $V \cap (X - U) \in \mathcal{I}$. Hence $I_0 = V \cap (X - U)$ and we have $x \in V - I_0 \subseteq U$, where V is ω-open set and $I_0 \in \mathcal{I}$.
Theorem 3.8. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. Then β is a basis, where $\beta(\mathcal{I}, \tau) = \{V - I_0 : V \in \tau_\omega, I_0 \in \mathcal{I}\}$.*

Proof. Since $\phi \in \mathcal{I}$, then $\tau_\omega \subseteq \beta$ and hence $X = \cup\beta$. Also, for every $\beta_1, \beta_2 \in \beta$, we have $\beta_1 = V_1 - I_1$ and $\beta_2 = V_2 - I_2$, where $V_1, V_2 \in \tau_\omega$ and

$I_1, I_2 \in \mathcal{I}$. Then $\beta_1 \cap \beta_2 = (V_1 - I_1) \cap (V_2 - I_2) = (V_1 \cap -(X - I_1)) \cap (V_2 \cap (X - I_2)) = (V_1 \cap V_2) - (I_1 \cup I_2) \in \beta$, where $V_1 \cap V_2 \in \tau_\omega, I_1 \cup I_2 \in \mathcal{I}$. □

Remark 3.9. The topology τ^*_ω finer than τ_ω. See the following example.

Example 3.10. Let $X = \mathbb{R}$ be the set of all real numbers with the topology $\tau = \{\emptyset, X, \{1\}\}$ and let Q be the set of all rational numbers, $\mathcal{I} = \{\mathcal{P}(Q)\}$. Put $A = Q$. Then $A \in \tau^*_\omega$, but A is not ω-open, since $Cl_\omega(A) \nsubseteq A$

Remark 3.11. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$.

(1) If $\mathcal{I} = \{\phi\}$, then $A_\omega = Cl_\omega(A) = Cl^*_\omega(A)$.
(2) If $\mathcal{I} = \mathcal{P}(X)$, then $A^* = A_\omega = \{\phi\}$ and $Cl^*(A) = Cl^*_\omega(A) = A$.

In view of our remarks, the following implications hold:

$$\begin{array}{ccc} \text{open} & \longrightarrow & \tau^*\text{-open} \\ \downarrow & & \downarrow \\ \omega\text{-open} & \longrightarrow & \tau^*_\omega\text{-open} \end{array}$$

Example 3.12. Let $(X, \tau, \mathcal{I})$ be an ideal space, with $X\{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $\{b\}$ is ω-open, but not τ^*-open.

Example 3.13. Let $X = R$ with the usual topology τ and let Q be the set of all rational numbers, $\mathcal{I} = \{\mathcal{P}(Q^c)\}$. Let $A = Q$. Then A is τ^*-open, but it is not an ω-open set.

4. ω-COMPATIBLE IN IDEAL TOPOLOGICAL SPACE

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the topology τ is ω-compatible with the ideal $\mathcal{I}$ and denote $\tau \sim_\omega \mathcal{I}$, if the following holds for every $A \subseteq X$: if, for every $x \in A$, there exists $U \in \tau_\omega(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Remark 4.2. A compatible space is ω-compatible, but not conversely.

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent:*

(1) $\tau \sim_\omega \mathcal{I}$;
(2) *if a subset A of X has a cover of ω-open sets, each of whose intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$;*
(3) *for every $A \subseteq X$, $A \cap A_\omega = \phi$ implies that $A \in \mathcal{I}$;*
(4) *for every $A \subseteq X$, $A - A_\omega \in \mathcal{I}$;*
(5) *for every $A \subseteq X$, if A contains no non-empty subset B with $B \subseteq B_\omega$, then $A \in \mathcal{I}$.*

Proof. (1) ⇒ (2) The proof is obvious.

(2) ⇒ (3) Let $A \subseteq X$ and $x \in A$. Then $x \notin A_\omega$ and there exists $V_x \in \tau_\omega(x)$ such that $V_x \cap A \in \mathcal{I}$. Therefore we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in \tau_\omega(x)$ and, by (2), $A \in \mathcal{I}$.

(3) $\Rightarrow$ (4) For any $A \subseteq X$, $A - A_\omega \subseteq A$ and $(A - A_\omega) \cap (A - A_\omega)_\omega \subseteq (A - A_\omega) \cap A_\omega = \phi$. By (3), $A - A_\omega \in \mathcal{I}$.

(4) $\Rightarrow$ (5) By (4), for every $A \subseteq X$, $A - A_\omega \in \mathcal{I}$. Let $A - A_\omega = J \in \mathcal{I}$. Then $A = J \cup (A \cap A_\omega)$. By Lemma 2.4, $A_\omega = J_\omega \cup (A \cap A_\omega)_\omega = (A \cap A_\omega)_\omega$. Therefore we have $A \cap A_\omega = A \cap (A \cap A_\omega)_\omega \subseteq (A \cap A_\omega)_\omega$ and $(A \cap A_\omega) \subseteq A$. By the assumption $A \cap A_\omega = \phi$, $A = A - A_\omega \in \mathcal{I}$.

(5) $\Rightarrow$ (1) Let $A \subseteq X$ and assume that, for every $x \in A$, there exists $U \in \tau_\omega(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A_\omega = \phi$. Since $(A - A_\omega) \cap (A - A_\omega)_\omega \subseteq (A - A_\omega) \cap A_\omega = \phi$, $A - A_\omega$ contains no nonempty subset B with $B \subseteq B_\omega$. By (5), $A - A_\omega \in \mathcal{I}$ and we have $A = A \cap (X - A_\omega) = A - A_\omega \in \mathcal{I}$. □

THEOREM 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) $\tau \sim_\omega \mathcal{I}$;

(2) *for every τ^*_ω-closed subset A, $A - A_\omega \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2) It follows by Theorem 4.3.

(2) $\Rightarrow$ (1) Let $A \subseteq X$ and suppose that, for every $x \in A$, there exists an ω-open set U containing x such that $U \cap A \in \mathcal{I}$. Then $A \cap A_\omega = \emptyset$. Since $Cl^*_\omega(A) = A \cup A_\omega$ is τ^*_ω-closed, we have $(A \cup A_\omega) - (A \cup A_\omega)_\omega \in \mathcal{I}$. Moreover, $(A \cup A_\omega) - (A \cup A_\omega)_\omega = (A \cup A_\omega) - (A_\omega \cup (A_\omega)_\omega) = (A \cup A_\omega) - A_\omega = A$. Therefore $A \in \mathcal{I}$. □

THEOREM 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and τ be ω-compatible with $\mathcal{I}$. A set is closed with respect to the τ^*_ω-topology if and only if it is the union of a set which is ω-closed with respect to τ and a set in $\mathcal{I}$.*

Proof. Let A be τ^*_ω-closed. Then $A_\omega \subseteq A$ implies that $A = (A - A_\omega) \cup A_\omega$. We have, by Theorem 4.4, $A - A_\omega \in \mathcal{I}$ and, by Theorem 3.1, A_ω is ω-closed with respect to τ.

Conversely, if $A = B \cup \mathcal{I}$, where B is ω-closed with respect to τ and $I \in \mathcal{I}$, then, by Theorem 3.1 and Lemma 2.4, we have $A_\omega = B_\omega \cup I_\omega = B_\omega \subseteq Cl_\omega(B) = B \subseteq A$. Thus $A_\omega \subseteq A$ and A is τ^*_ω-closed. □

COROLLARY 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is ω-compatible with $\mathcal{I}$, then $\beta(\tau, \mathcal{I}) = \tau^*_\omega$.*

Proof. Let $U \in \tau^*_\omega$. Then, by Theorem 4.5, $X - U = F \cup B$, where F is ω-closed and $B \in \mathcal{I}$. Then $U = X - (F \cup B) = (X - F) \cap (X - B) = (X - F) - B = V - B$, where $V = X - F$ is ω-open set of X. Thus every τ^*_ω-open set is of the from $V - B$, where V is ω-open and $B \in \mathcal{I}$. It follows from Theorem 3.8 that $\beta(\tau, \mathcal{I}) = \tau^*_\omega$. □

THEOREM 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) $\tau_\omega \cap \mathcal{I} = \emptyset$;

(2) *if $I \in \mathcal{I}$, then $Int_\omega(I) = \emptyset$;*

(3) *for every $G \in \tau_\omega$, $G \subseteq G_\omega$;*

(4) $X = X_\omega$.

Proof. (1) $\Rightarrow$ (2) Let $\tau_\omega \cap \mathcal{I} = \emptyset$ and $I \in \mathcal{I}$. Suppose that $x \in Int_\omega(I)$. Then there exists $U \in \tau_\omega$ such that $x \in U \subseteq I$. Since $I \in \mathcal{I}$, $\emptyset \neq \{x\} \subseteq U \in \tau_\omega \cap \mathcal{I}$. This implies $\tau_\omega \cap \mathcal{I} = \emptyset$.Therefore $Int_\omega(I) = \emptyset$.

(2) $\Rightarrow$ (3) Let $x \in G$. Assume that $x \notin G_\omega$. Then there exists $U_x \in \tau_\omega(x)$ such that $G \cap U_x \in \mathcal{I}$. By (2), $x \in G \cap U_x = Int_\omega(G \cap U_x) = \emptyset$. Hence $x \in G_\omega$ and $G \subseteq G_\omega$.

(3) $\Rightarrow$ (4) Since X is ω-open, $X = X_\omega$.

(4) $\Rightarrow$ (1) $X = X_\omega = \{x \in X : U \cap X = U \notin \mathcal{I}\}$, for each ω-open set U containing x. Hence $\tau_\omega \cap \mathcal{I} = \emptyset$. □

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, τ be ω-compatible with $\mathcal{I}$ and $\tau_\omega \cap \mathcal{I} = \emptyset$. Let G be a τ^*_ω-open set such that $G = U - A$, where $U \in \tau_\omega$ and $A \in \mathcal{I}$. Then $Cl_\omega(G_\omega) = Cl_\omega(G) = G_\omega = U_\omega = Cl_\omega(U) = Cl_\omega(U_\omega)$.*

Proof. Let $G = U - A$, where $U \in \tau_\omega$ and $A \in \mathcal{I}$. Since $\tau_\omega \cap \mathcal{I} = \emptyset$, by Theorem 4.7, we have $U \subseteq U_\omega$. Hence, by Lemma 2.4, $U_\omega = Cl_\omega(U_\omega) = Cl_\omega(U)$.

Now, we prove that $G \subseteq G_\omega$, by using $G \in \tau^*_\omega$. Since $Cl^*_\omega(X - G) = X - G$, $(X - G)_\omega \subseteq X - G$ and, by Lemma 3.3, $X_\omega - G_\omega \subseteq X - G$. Since $\tau_\omega \cap \mathcal{I} = \emptyset$, by Theorem 4.7, $X - G_\omega \subseteq X - G$ and hence we have $G \subseteq G_\omega$. Hence, by Lemma 2.4, $G_\omega = Cl_\omega(G) = Cl_\omega(G_\omega)$.

Now, since $G \subseteq U$, $G_\omega \subseteq U_\omega$. By Lemma 3.3, $G_\omega = (U - A)_\omega \supseteq U_\omega - A_\omega = U_\omega$, since $A \in \mathcal{I}$. Thus $U_\omega = G_\omega$. Hence we obtain the desired result. □

Definition 4.9. An ideal $\mathcal{I}$ is called a σ-ideal - see [11] - if it is countably additive, that is if $I_n \in \mathcal{I}$, for each $n \in N$, then $\cup\{I_n : n \in N\} \in \mathcal{I}$.

Definition 4.10. A space (X, τ) is said to satisfy the C_1 condition - see [10] - if every infinite subset of X has non-empty interior.

Proposition 4.11 ([10]). *If a space (X, τ) satisfies the condition C_1, then $A - Int(A)$ is finite, for any $A \subseteq X$.*

Definition 4.12 ([12]). A space (X, τ) is said to be ω-Lindelöf if and only if every cover of X by ω-open sets of X has a countable subcover. A space (X, τ) is said to have the hereditary ω-Lindelöf property if every subspace has the ω-Lindelöf property.

Lemma 4.13 ([8]). *If U is an ω-open subset of a space (X, τ), then $U - C$ is ω-open, for every countable subsets C of X.*

Theorem 4.14. *Let (X, τ) be a hereditary ω-Lindelöf space satisfying condition C_1 and let $\mathcal{I}$ be a σ-ideal on X. Then $\tau \sim_\omega \mathcal{I}$.*

Proof. Let $A \subseteq X$ and assume that, for every $x \in A$, there exists an ω-open set U such that $U \cap A \in \mathcal{I}$. This implies $U \cap Int(A) \in \mathcal{I}$. Now, $\{(U_x - C) \cap Int(A) : x \in A\}$ is a cover of $Int(A)$ by ω-open sets and a countable subset C of X. By the assumption that (X, τ) is hereditarily ω-Lindelöf, this cover has a countable subcover $\{(U_{x(n)} - C) \cap Int(A) : n \in N\}$.

Since $\mathcal{I}$ is a σ-ideal, $Int(A) = \cup\{(U_{x(n)} - C) \cap Int(A) : n \in N\} \in \mathcal{I}$. If A is an open subset of X, then the proof is complete. If A is not open, then, by Proposition 4.11, $A - Int(A)$ is finite. For every $x \in A - Int(A)$, there exists an ω-open set U_x such that $U_x \cap A \in \mathcal{I}$, hence $U_x \cap (A - Int(A)) \in \mathcal{I}$. By the finite additivity of $\mathcal{I}$, we have $A - Int(A) = \cup\{U_x \cap (A - Int(A))\} \in \mathcal{I}$. This means that $A = Int(A) \cup (A - Int(A)) \in \mathcal{I}$. Hence $\tau \sim_\omega \mathcal{I}$. □

REFERENCES

[1] A. Al-Omari and M.S.M. Noorani, *Regular generalized ω-closed sets*, Int. J. Math. Math. Sci., **2007** (2007), Article 16292, 1–11.

[2] A. Al-Omari and M.S.M. Noorani, *Contra-ω-continuous and almost contra-ω-continuous*, Int. J. Math. Math. Sci., **2007** (2007), Article 040469, 1–13.

[3] A. Al-Omari and T. Noiri, *Local closure functions in ideal topological spaces*, Novi Sad J. Math., **43** (2013), 139–149.

[4] A. Al-Omari, T. Noiri, M.S. Noorani, *Weak and strong forms of sT-continuous functions*, Commun. Korean Math. Soc., **30** (2015), 493–504.

[5] A. Al-Omari, S. Modak and T. Noiri, *On θ-modifications of generalized topologies via hereditary classes*, Commun. Korean Math. Soc., **31** (2016), 857–868.

[6] A. Al-Omari, T. Noiri and S. Modak, *Paracompact spaces with m-structures*, An. Univ. Oradea Fasc. Mat., **24** (2017), 155–162.

[7] H. Al-Saadi and A.Al-Omari, *Some operators in ideal topological spaces*, Missouri J. Math. Sci., **30** (2018), 1–13.

[8] K.Y. Al-Zoubi, *On generalized ω-closed sets*, Int. J. Math. Math. Sci., **13** (2005), 2011–2021.

[9] E. Ekici, S. Jafari and S.P. Moshokoa, *On a weaker form of ω-continuity*, An. Univ. Craiova Ser. Mat. Inform., **37** (2010), 38–46.

[10] M. Ganster, *Some remarks on strongly compact spaces and semi compact spaces*, Bull. Malays. Math. Sci. Soc., **10** (1987), 67–81.

[11] D. Jankovic and T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 295–310.

[12] H.Z. Hdeib, *ω-closed mapping*, Rev. Colombiana Mat., **16** (1982), 65–78.

[13] H.Z. Hdeib, *ω-continuous functions*, Dirasat Journal, **16** (1989), 136–142.

[14] K. Kuratowski, *Topology I*, Warszawa, 1933.

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