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ON THE DIOPHANTINE EQUATION $x^5 + ky^3 = z^5 + kw^3$

N. YOUSEFNEJAD, H. SHABANI-SOLT, and A. S. JANFADA

Abstract. In this article we consider the symmetric Diophantine equation $x^m + ky^n = z^m + kw^n$, where k is a rational number and prove that, for any rational number k, the equation $x^5+ky^3 = z^5+kw^3$ has infinitely many rational nontrivial solutions. The strategy is to use the elliptic fibration method.

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1. INTRODUCTION

By a symmetric Diophantine equation in two variables we mean an equation of the form

$$f(x, y) = f(z, w),$$

where f is a polynomial with integer coefficients. Some authors have studied the above symmetric Diophantine equation in low degrees, see [1, 2, 3]. An attractive case is the equation

$$x^m + ky^n = z^m + kw^n,$$

where k is a rational number. Shorey studied this equation by analytic number theory methods [6]. As a matter of fact, in the case where m, n are coprime, the above equation is reduced to a simple form.

LEMMA 1.1. *Let m, n be coprime integers. Then the rational solutions of*

$$x^m + ky^n = z^m + kw^n \tag{1}$$

are equivalent to the rational solutions of $x^m + y^n = z^m + w^n$.

The proof is straightforward. The coprime relation $\gcd(m, n) = 1$ implies the existence of the positive integers α, β such that $\alpha n - \beta m = 1$. Multiplying both sides of (1) by $k^{\beta m}$ gives the result.

One of the effective methods to solve a given Diophantine equation is to find an elliptic fibration and use the specialization process to obtain an elliptic curve. Now, to check that the Diophantine equation has (infinitely many) nontrivial solutions is equivalent to check that the resulted elliptic curve has positive rank. We call a solution(x, y, z, w) a trivial solution if $x = z$, and $y = w$. Throughout this article, by a solution, we mean a nontrivial solution.

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Using the above method, the authors of [5] studied the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$, where k is a rational number, and showed by computer calculations that, for all integers $1 \leq k \leq 500$, this equation has infinitely many integral solutions. Then, using this results and some other evidences, they conjectured that, for all rational numbers k, the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many integral solutions.

Now consider the Diophantine equation

$$x^5 + ky^3 = z^5 + kw^3, \ k \in \mathbb{Q}. \tag{2}$$

We use a similar method to show that, for all k, the equation (2) has infinitely many rational solutions. In general, it is not easy to find a single elliptic fibration that works for all or finitely many k. In this article, we manage to do this. In fact, in Theorem 1.2 we exhibit an elliptic fibration over (2) and then deduce the existence of infinitely many rational solutions.

In this article we consider the equation (2) as a three-fold and prove the following main result.

THEOREM 1.2. *Consider the algebraic singular three-fold*

$$x^5 + ky^3 = z^5 + kw^3, k \in \mathbb{Q}. \tag{3}$$

(I) *The three-fold* (3) *is an elliptic three-fold having an elliptic fibration defined over* $\mathbb{Q}$.

(II) *The three-fold* (3) *has infinitely many rational solutions.*

Recall that an elliptic three-fold is an algebraic projective (resp. affine) three-fold C together with a morphism $\pi : C \to \mathbb{P}^2$ (resp. $\pi : C \to \mathbb{Q}^2$) such that for all but finitely many $t \in \mathbb{P}^2$ (resp. $t \in \mathbb{Q}^2$), $\pi^{-1}(t)$ is an elliptic curve. Here, we consider the affine case. We cite [4] for more details on three-folds and [8] for elliptic three-folds.

2. PRELIMINARIES

Consider the following equation defined over the field K.

$$v^2 = au^4 + bu^3 + cu^2 + du + e, \quad a \neq 0.$$

Take a point $(u, v) = (p, q)$ on this curve. We may assume $p = 0$. Then $e = q^2$ and we get the curve

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \quad a \neq 0.$$

Suppose $q = 0$. If $d = 0$, then the curve has a singularity at $(u, v) = (0, 0)$. Therefore, assume $d \neq 0$. Then,

$$\left(\frac{v}{u^2}\right)^2 = d\left(\frac{1}{u}\right)^3 + c\left(\frac{1}{u}\right)^2 + b\left(\frac{1}{u}\right) + a,$$

and, putting $X = 1/u$ and $Y = 1/u^2$, we obtain the elliptic curve $Y^2 = dX^3 + cX^2 + bX + a$. The more complicated case is when $q \neq 0$, for which we have the following result [7].

Theorem 2.1. *Let K be a field of characteristic not 2. Consider the quartic equation*

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2$$

with $a, b, c, d, q, \in K$ and $q \neq 0$. Let

$$X = \frac{2q(v+q)+du}{u^2}, \quad Y = \frac{4q^2(v+q)+2q(du+cu^2)-(d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q,\ a_2 = c - (d^2/4q^2),\ a_3 = 2qb,\ a_4 = -4q^2 a,\ a_6 = a_2 a_4.$$

Then

$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6.$$

The inverse transformation is

$$u = \frac{2q(X+c)-(d^2/2q)}{Y},\ v = -q + \frac{u(uX-d)}{2q}.$$

The point $(u,v) = (0,q)$ corresponds to the point $(X,Y) = \infty$ and $(u,v) = (0,-q)$ corresponds to $(X,Y) = (-a_2, a_1 a_2 - a_3)$.

3. PROOFS

Proof of Theorem 1.2. We prove Part (I). By Lemma 1.1, we prove the result for the three-fold

$$x^5 + y^3 = z^5 + w^3. \tag{4}$$

Intersecting the three-fold (4) with the hyperplanes

$$x - z - t = 0,\ w - y - s = 0$$

we get

$$\frac{5t}{3s}z^4 + \frac{10t^2}{3s} + \frac{10t^3}{3s}z^2 + \frac{5t^4}{3s}z + \frac{t^5}{3s} - \frac{s^2}{12} = \left(y + \frac{s}{2}\right)^2. \tag{5}$$

We want the expression $\frac{t^5}{3s} - \frac{s^2}{12}$ to be a square, say

$$\frac{t^5}{3s} - \frac{s^2}{12} = q^2.$$

Multiplying both sides by $\frac{3}{s^4}$, we get

$$\left(\frac{t}{s}\right)^5 - \left(\frac{1}{2s}\right)^2 = 3\left(\frac{q}{s^2}\right)^2.$$

Now, putting $x' = \frac{1}{2s},\ y' = \frac{q}{s^2},\ z' = \frac{t}{s}$, we have

$$x'^2 + 3y'^2 = z'^5, \tag{6}$$

where we may assume that x', y' are integers. Write (6) as

$$(x' + \sqrt{-3}y')(x' - \sqrt{-3}y') = z'^5.$$

We are looking for a parametric solution for (6) with rational coefficients such that $(x'+\sqrt{-3}y'), (x'-\sqrt{-3}y')$ are coprime in the integral ring of the number field $\mathbb{Q}(\sqrt{-3})$. Since this ring is a unique factorization domain, we have

$$x'+\sqrt{-3}y' = (g+\sqrt{-3}h)^5,\ x'-\sqrt{-3}y' = (g-\sqrt{-3}h)^5, \tag{7}$$

for some integers g, h. One can immediately deduce that

$$z' = g^2 + 3h^3.$$

Expanding the right hand side of one of the equations in (7) we get

$$x' = g^5 - 30g^3h^2 + 45gh^4,\ y' = 5g^4h - 30g^2h^3 + 9h^5.$$

Finally, we get

$$s = 1/2(g^5 - 30g^3h^2 + 45gh^4),\ t = s(g^2+3h^3),\ q = s^2(5g^4h - 30g^2h^3 + 9h^5).$$

Putting $v = y + s/2$, now, the equation (5) becomes

$$v^2 = \frac{5t}{3s}z^4 + \frac{10t^2}{3s} + \frac{10t^3}{3s}z^2 + \frac{5t^4}{3s}z + q^2, \tag{8}$$

which corresponds, by Theorem 2.1, to the elliptic curve

$$E_{g,h} : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6,$$

where

$$a_1 = \frac{5t^3}{3q},\ a_2 = \frac{10t^3}{3s} - \frac{25t^6}{36q^2},\ a_3 = \frac{20qt^2}{3s},\ a_4 = -\frac{20tq^2}{3s},\ a_6 = a_2a_4.$$

Since t, s, q are rational functions of g, h, $E_{g,h}$ is an elliptic curve over $\mathbb{Q}(g,h)$ and this proves Part (I).

To prove (II), we note that, by the specialization $g = g_0, h = h_0$, the elliptic curve E_{g_0,h_0} has positive rank with a non-torsion (X,Y) on it. Then, by Theorem 2.1, the point (X,Y) on the elliptic curve $E_{g,h}$ corresponds to the point (z,v) on the quartic curve (8), where

$$z = \frac{36q^2s^2 + 90q^2st^4 - 25t^8}{18qs^2Y},\ v = \frac{-6q^2s + 3sz^2X - 5zt^4}{6qs}.$$

Now, from the equations $x - z - t = 0,\ w - y - s = 0,\ v = y + s/2$ we get the rational point

$$(x,y,z,w) = (z+t, v-s/2, z, y+s)$$

on the three-fold (3) and therefore we get an integral point on it. Since the rank of E_{g_0,h_0} is assumed to be positive, we conclude that the three-fold (3) has infinitely many rational, and hence integral, solutions.

On the other hand, a straightforward search by the MWRANK software records the following positive-rank elliptic curves.

(g_0,h_0)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(1,2)	(2,2)
rank r	2	2	3	2	1	$1 \le r \le 3$	$1 \le r \le 3$

This proves Part (II) and completes the proof of Theorem 1.2. □

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Urmia University
Department of Mathematics
Urmia 57561-51818, I.R. Iran
E-mail: yusefnejadnazanin@yahoo.com
E-mail: h.shabani.solt@gmail.com
E-mail: asjanfada@gmail.com