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VARIABILITY REGIONS FOR A FAMILY OF UNIVALENT MAPPINGS SATISFYING A CERTAIN INEQUALITY

WASIM UL-HAQ

Abstract. In this article, regions of variability for a family of analytic univalent mappings satisfying a certain differential inequality are explicitly determined. The geometric view of our main result is also shown by using Mathematica.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and consider $\mathcal{A}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of E. Also, let $\mathcal{B}$ denote the class of analytic functions w on E such that $|w(z)| < 1$ and $w(0) = 0$. A complex valued function f is said to be convex in E if it is univalent and if the image domain $D = f(E)$ is convex. That is $\omega_1, \omega_2 \in D$ $(0 \le t \le 1) \Longrightarrow (1-t)\omega_1 + t\omega_2 \in D$. Similarly, a complex valued function f is said to be starlike in E if it is univalent and if the image domain $D = f(E)$ is starshaped with respect to 0. Let C and S^* denote the classes of functions $f \in \mathcal{A}$ which are convex and starlike, respectively. Now, let γ be a complex number with $\Re\gamma > -1$ $(\gamma \ne -1)$ and μ be a non-negative real number and say that a function $f \in \mathcal{A}$ is in the class $R(\gamma, \mu)$ if the following inequality is satisfied

$$\left| zf''(z) + \gamma(f'(z) - 1) \right| \le \mu, \quad z \in E. \tag{2}$$

It is known [1] that $R(\gamma,\mu) \subsetneq S^*$, if $0 \le \mu \le \frac{1+\Re\gamma}{1+|\gamma|+\Re\gamma}$, and $R(\gamma,\mu) \subsetneq C$, if $0 \le 2\mu \le \frac{1+\Re\gamma}{1+|\gamma|+\Re\gamma}$. In a recent work, Ponnusamy et al. [2] studied the variability regions for a certain family of univalent mappings satisfying (2) with $\gamma = 0$. For a related study, see [3].

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In this article, we are interested in determining the variability regions, when f ranges over a certain family of analytic and univalent mappings satisfying a certain inequality.

2. THE CLASS $R_\mu(\alpha,\beta,\gamma)$

Let $\alpha,\beta,\gamma\in\mathbb{C}$ be such that $\Re\gamma>-1,\ 0<\mu\le|\alpha|(\Re\gamma+1)$ and $|\beta|\le 1$. Let $R_\mu(\alpha,\beta,\gamma)$ denote the family of functions f analytic and univalent in E, with $f(0)=0,\ f'(0)=\alpha\neq 0$ and $f''(0)=\frac{\mu\beta}{\gamma+1}$ satisfying the inequality

$$\left|zf''(z)+\gamma(f'(z)-\alpha)\right|\le\mu,\quad z\in E. \tag{3}$$

For $\gamma=0$, this class was introduced and discussed by Ponnusamy et al. [2]. If $f\in R_\mu(\alpha,\beta,\gamma)$, then it may be written as

$$zf''(z)+\gamma(f'(z)-\alpha)=\mu w(z),$$

for some $w\in B$. From this, we have the following integral representation

$$f'(z)=\alpha+\mu\int_0^1 t^{\gamma-1}w(tz)\mathrm{d}t. \tag{4}$$

From the Schwarz lemma, we have

$$\left|f'(z)-\alpha\right|<\frac{\mu}{\Re\gamma+1}.$$

This shows that the functions in $R_\mu(\alpha,\beta,\gamma)$ are univalent in E, if $\mu\le|\alpha|(\Re\gamma+1)$.

Since $f\in R_\mu(\alpha,\beta,\gamma)$, the function

$$w_f(z)=\frac{z(f''(z)-\mu\beta)+\gamma\left(f'(z)-\alpha\right)}{z(\mu-\overline{\beta}f''(z))-\overline{\beta}\gamma\left(f'(z)-\alpha\right)},\ z\in E, \tag{5}$$

is in the class $\mathcal{B}$. Applying the Schwarz lemma, it can be shown that $f\in R_\mu(\alpha,\beta,\gamma)$ implies a restriction on $f'''(0)$. In particular,

$$\left|f'''(0)\right|=\frac{2\mu\left(1-|\beta|^2\right)}{\gamma+2}\left|w_f'(0)\right|\le\frac{2\mu\left(1-|\beta|^2\right)}{|\gamma+2|}.$$

For $\lambda\in\overline{E}=\{z\in\mathbb{C}:|z|\le 1\}$ and $z_0\in E$, set

$$R_{\lambda,\mu}(\alpha,\beta,\gamma)=\left\{f\in R_\mu(\alpha,\beta,\gamma):f'''(0)=\frac{2\mu\left(1-|\beta|^2\right)}{(\gamma+2)}\lambda\right\},$$

$$V(z_0,\lambda)=\left\{f'(z_0):f\in R_{\lambda,\mu}(\alpha,\beta,\gamma)\right\}.$$

The aim of this paper is to investigate explicitly the region of variability $V(z_0,\lambda)$ for the class $R_{\lambda,\mu}(\alpha,\beta,\gamma)$. Some general properties of the set $V(z_0,\lambda)$ are given in the following proposition.

Proposition 2.1. *We have:*

(i) *$V(z_0,\lambda)$ is a compact set.*

(ii) $V(z_0,\lambda)$ *is convex.*

(iii) *If* $|\lambda|=1$ *or* $z_0=0$*, then*

$$V(z_0,\lambda)=\begin{cases}\alpha+\frac{\mu z_0}{\overline{\beta}(\gamma+1)}-\frac{\mu z_0}{\overline{\beta}(\gamma+1)}(1-|\beta|^2)_2F_1\left(1,\gamma+1,\gamma+2,-\overline{\beta}\lambda z_0\right), & \beta\neq 0\\ \alpha+\frac{\mu\lambda}{\gamma+2}z_0^2, & \beta=0\end{cases}$$

and if $|\lambda|<1$ *and* $z_0\neq 0$*, then* $\alpha+\frac{\mu}{z_0^\gamma}\int_0^{z_0}\zeta^\gamma\frac{\lambda\zeta+\beta}{1+\overline{\beta}\lambda\zeta}\mathrm{d}\zeta$ *is an interior point of the set* $V(z_0,\lambda)$*, where* ${}_2F_1(a,b,c;z)$ *is the well known Gauss Hypergeometric function.*

Proof. The proof of (i) and (ii) follow immediately from the compactness and convexity of the class $R_{\lambda,\mu}(\alpha,\beta,\gamma)$.

Now we prove (iii). Since $|\lambda|=|w_f'(0)|=1$, from the Schwarz lemma, we obtain $w_f(z)=\lambda z$, which yields

$$\frac{zf''(z)+\gamma\left(f'(z)-\alpha\right)}{\mu}=\frac{[\lambda z+\beta]\,z}{1++\overline{\beta}\lambda z}.$$

Integrating the above expression from 0 to z_0 , we have

$$\begin{aligned}f'(z_0) &= \alpha+\frac{\mu}{z_0^\gamma}\int_0^{z_0}\zeta^\gamma\frac{\lambda\zeta+\beta}{1+\overline{\beta}\lambda\zeta}\mathrm{d}\zeta\\ &= \alpha+\tfrac{\mu}{\overline{\beta}z_0^\gamma}\int_0^{z_0}\zeta^\gamma\left[\left(1-\tfrac{1}{1+\overline{\beta}\lambda\zeta}\right)+\tfrac{\beta}{\lambda}\left(\tfrac{\lambda\overline{\beta}}{1+\overline{\beta}\lambda\zeta}\right)\right]\mathrm{d}\zeta,\end{aligned}$$

and simple computations yield, for $\beta\neq 0$,

$$f'(z_0)=\alpha+\tfrac{\mu z_0}{\overline{\beta}(\gamma+1)}-\tfrac{\mu z_0}{\overline{\beta}(\gamma+1)}(1-|\beta|^2)\ {}_2F_1\left(1,\gamma+1,\gamma+2,-\overline{\beta}\lambda z_0\right)$$

and, for $\beta=0$,

$$f'(z_0)=\alpha+\frac{\mu\lambda}{\gamma+2}z_0^2.$$

So, for $\beta\neq 0$,

$$V(z_0,\lambda)=\left\{\alpha+\tfrac{\mu z_0}{\overline{\beta}(\gamma+1)}\left(1-(1-|\beta|^2)\ {}_2F_1\left(1,\gamma+1,\gamma+2,-\overline{\beta}\lambda z_0\right)\right)\right\}$$

and, for $\beta=0$,

$$V(z_0,\lambda)=\alpha+\frac{\mu\lambda}{\gamma+2}z_0^2.$$

This is trivially true when $z_0=0$.

For $\lambda\in E$ and $a\in\overline{E}$, set

$$\delta(z,\lambda)=\frac{z+\lambda}{1+\overline{\lambda}z},$$

$$H_{a,\lambda}(z)=\alpha z+\int_0^z\left[\int_0^{\zeta_2}\frac{\mu\zeta_1^\gamma}{\zeta_2^\gamma}\frac{[\delta(a\zeta_1,\lambda)\zeta_1+\beta]}{1+\overline{\beta}\delta(a\zeta_1,\lambda)\zeta_1}\mathrm{d}\zeta_1\right]\mathrm{d}\zeta_2,\ \ z\in E.$$

Then $H_{a,\lambda} \in R_{\lambda,\mu}(\alpha,\beta,\gamma)$ and $w_{H_{a,\lambda}}(z) = z\delta(az,\lambda)$. For fixed $\lambda \in E$ and $z_0 \in E\backslash\{0\}$, the function

$$E \ni a \mapsto H'_{a,\lambda}(z_0) = \alpha + \frac{\mu}{z_0^{\gamma}} \int_0^{z_0} \zeta^{\gamma} \frac{[\delta(a\zeta,\lambda)\zeta + \beta]}{1 + \overline{\beta}\delta(a\zeta,\lambda)\zeta} \mathrm{d}\zeta$$

is a non-constant analytic function of $a \in E$ and therefore is an open mapping. Hence $H'_{0,\lambda}(z_0) = \alpha + \frac{\mu}{z_0^{\gamma}} \int_0^{z_0} \zeta^{\gamma} \frac{[\lambda\zeta+\beta]}{1+\overline{\beta}\lambda\zeta} \mathrm{d}\zeta$ is an interior point of

$$\left\{H'_{a,\lambda}(z_0) : a \in E\right\} \subset V(z_0,\lambda).$$

Keeping in view the above proposition, it is sufficient to find $V(z_0,\lambda)$ for $0 \le \lambda < 1$ and $z_0 \in E\backslash\{0\}$. For this we need the following lemma, stated below. □

Lemma 2.2 ([5]). *For $\theta \in \mathbb{R}$ and $|\lambda| < 1$, the function*

$$G(z) = \int_0^z \frac{\mathrm{e}^{\mathrm{i}\theta}\zeta^2}{\left(1 + \left(\mathrm{e}^{\mathrm{i}\theta}\overline{\lambda} + \overline{\beta}\lambda\right)\zeta + \mathrm{e}^{\mathrm{i}\theta}\overline{\beta}\zeta^2\right)^2} \mathrm{d}\zeta, \quad z \in E,$$

has a zero of order three at the origin and no zero elsewhere in E. Moreover, there exists a starlike normalized univalent function s in E such that $G(z) = 3^{-1}\mathrm{e}^{\mathrm{i}\theta}s^3(z)$.

3. SOME USEFUL RESULTS

In this section, we state and prove some results which are needed in the proof of our main theorems.

Proposition 3.1. *For $f \in R_{\lambda,\mu}(\alpha,\beta,\gamma)$, we have*

$$\left|f''(z) + \gamma\left(\frac{f'(z)-\alpha}{z}\right) - q(z,\lambda)\right| \le r(z,\lambda), \; z \in E, \lambda \in \overline{E}, \tag{6}$$

where

$$q(z,\lambda) = \frac{\mu\left(1-|z|^2\right)\left[\beta(1+|z|^2) + \beta^2\overline{\lambda}\overline{z} + \lambda z\right]}{1 - |\beta|^2|z|^4 - \left(1-|\beta|^2\right)|\lambda|^2|z|^2 + 2\left(1-|z|^2\right)\Re\left(\overline{\beta}\lambda z\right)},$$

$$r(z,\lambda) = \frac{\left(1-|\lambda|^2\right)\left(1-|\beta|^2\right)|z|^2}{1 - |\beta|^2|z|^4 - \left(1-|\beta|^2\right)|\lambda|^2|z|^2 + 2\mu\left(1-|z|^2\right)\Re\left(\overline{\beta}\lambda z\right)}.$$

The inequality is sharp for $z_0 \in E\backslash\{0\}$ if and only if $f(z) = H_{\mathrm{e}^{\mathrm{i}\theta},\lambda}(z)$ for some $\theta \in \mathbb{R}$.

Proof. Since, for $w_f \in \mathcal{B}$, $w'_f(0) = \lambda$, from the Schwarz lemma, it follows that

$$\left|\frac{f''(z) + \gamma\left(\frac{f'(z)-\alpha}{z}\right) - \frac{\mu[\lambda z+\beta]}{1+\overline{\beta}\lambda z}}{f''(z) + \gamma\left(\frac{f'(z)-\alpha}{z}\right) - \frac{\mu(z+\overline{\lambda}\beta)}{\overline{\beta}z+\overline{\lambda}}}\right| \le |z|\left|\frac{\overline{\beta}z+\overline{\lambda}}{1+\overline{\beta}\lambda z}\right|. \tag{7}$$

From (5) this can be written equivalently as

$$\left|\frac{f''(z)+\gamma\left(\frac{f'(z)-\alpha}{z}\right)-b(z,\lambda)}{zf''(z)+\gamma\left(\frac{f'(z)-\alpha}{z}\right)+c(z,\lambda)}\right|\leq |z|\,|\tau(z,\lambda)|\,, \tag{8}$$

where

$$\begin{cases} b(z,\lambda)=\frac{\mu[\lambda z+\beta]}{1+\overline{\beta}\lambda z}, \quad c(z,\lambda)=-\frac{\mu(z+\overline{\lambda}\beta)}{\overline{\beta}z+\overline{\lambda}}, \\ \tau(z,\lambda)=\frac{\overline{\beta}z+\overline{\lambda}}{1+\overline{\beta}\lambda z}. \end{cases} \tag{9}$$

Simple computations show that the inequality (8) can be written as

$$\begin{aligned} &\left|f''(z)+\gamma\left(\frac{f'(z)-\alpha}{z}\right)-\frac{b(z,\lambda)+|z|^2\,|\tau(z,\lambda)|^2\,c(z,\lambda)}{1-|z|^2\,|\tau(z,\lambda)|^2}\right| \\ &\qquad\leq\frac{|z|\,|\tau(z,\lambda)|\,|b(z,\lambda)+c(z,\lambda)|}{1-|z|^2\,|\tau(z,\lambda)|^2}. \end{aligned} \tag{10}$$

Now, we have

$$1-|z|^2\,|\tau(z,\lambda)|^2=\frac{1-|\beta|^2\,|z|^4-\left(1-|\beta|^2\right)|\lambda|^2|z|^2+2\left(1-|z|^2\right)\Re\left(\overline{\beta}\lambda z\right)}{\left|1+\overline{\beta}\lambda z\right|^2},$$

$$b(z,\lambda)+c(z,\lambda)=\frac{\mu\left(1-|\lambda|^2\right)\left(1-|\beta|^2\right)z}{\left(1+\overline{\beta}\lambda z\right)\left(\overline{\beta}z+\overline{\lambda}\right)},$$

$$\begin{aligned} b(z,\lambda)+|z|^2\,|\tau(z,\lambda)|^2\,c(z,\lambda)&=\frac{\mu\left[\lambda z+\beta\right]}{1+\overline{\beta}\lambda z}-|z|^2\left|\frac{\overline{\beta}z+\overline{\lambda}}{1+\overline{\beta}\lambda z}\right|^2\frac{\mu(z+\overline{\lambda}\beta)}{\overline{\beta}z+\overline{\lambda}} \\ &=\frac{\mu\left(1-|z|^2\right)\left[\beta\left(1+|z|^2\right)+\beta^2\overline{\lambda}\overline{z}+\lambda z\right]}{\left|1+\overline{\beta}\lambda z\right|^2}. \end{aligned}$$

Set

$$\frac{b(z,\lambda)+|z|^2\,|\tau(z,\lambda)|^2\,c(z,\lambda)}{1-|z|^2\,|\tau(z,\lambda)|^2}=q(z,\lambda),$$

$$\frac{|z|\,|\tau(z,\lambda)|\,|b(z,\lambda)+c(z,\lambda)|}{1-|z|^2\,|\tau(z,\lambda)|^2}=r(z,\lambda).$$

All these relations together with (10) give (6). Equality in (6) occurs when $f(z)=F_{i\theta,\lambda}(z)$, for $z\in E$. Conversely, if equality in (6) occurs for some $z\in E\backslash\{0\}$, then equality must hold in (7). Thus, by the Schwarz lemma, there exists $\theta\in\mathbb{R}$ such that $w_f(z)=z\delta(az,\lambda)$, for all $z\in E$. This implies $f(z)=F_{i\theta,\lambda}(z)$. □

The case $\lambda=0$ leads us to the following result.

Corollary 3.2. *Let $f \in R(0)$. Then*

$$\left| f''(z) + \gamma \left(\frac{f'(z) - \alpha}{z} \right) - \frac{\mu\beta \left(1 - |z|^4\right)}{1 - |\beta|^2 |z|^4} \right| \leq \frac{\left(1 - |\beta|^2\right) |z|^2}{1 - |\beta|^2 |z|^4}.$$

The special case $\gamma = 0$ in the above corollary gives us the known result [2]. For $|\beta| = 1$, the above corollary gives us

$$\left| f''(z) + \gamma \left(\frac{f'(z) - \alpha}{z} \right) - \mu\beta \right| = 0,$$

which further yields

$$f(z) = \alpha z + \mu\beta \frac{z^2}{\gamma + 1}.$$

Geometrically, Proposition 1 means that the functional

$$zf''(z) + \gamma \left(f'(z) - \alpha\right)$$

lies in the closed disk centred at $q(z, \lambda)$ with radius $r(z, \lambda)$. From this fact we have the below corollary.

Corollary 3.3. *Let $\gamma : z(t), 0 \leq t \leq 1$ be a C^1-curve in E with $z(0) = 0$ and $z(1) = z_0$. Then we have*

$$V(z_0, \lambda) \subset \overline{\mathbb{D}} \left(Q(\lambda, \gamma), W(\lambda, \gamma)\right) = \{w \in C : |w - Q(\lambda, \gamma)| \leq W(\lambda, \gamma)\},$$

where

$$Q(\lambda, \gamma) = \alpha + \frac{1}{z_0^\gamma} \int_0^1 z^\gamma(t) q(z(t), \lambda) z'(t) \mathrm{d}t,$$

$$W(\lambda, \gamma) = \int_0^1 r(z(t), \lambda) \frac{z(t)}{z_0} |z'(t)| \mathrm{d}t.$$

Proof. Since f is in $R_{\lambda,\mu}(\alpha, \beta, \gamma)$,

$$\frac{1}{z_0^\gamma} \int_0^1 \left[z^\gamma(t)(f'(z(t) - \alpha)\right]' z'(t) \mathrm{d}t = f'(z(1)) - \alpha = f'(z_0) - \alpha.$$

Now, from Proposition 2, it follows that

$$\left|f'(z_0) - Q(\lambda, \gamma)\right| = \left| f'(z_0) - \alpha - \frac{1}{z_0^\gamma} \int_0^1 z^\gamma(t) q(z(t), \lambda) z'(t) \mathrm{d}t \right| \mathrm{d}t$$

$$= \left| \int_0^1 \left[f''(z(t) + \gamma \left(\frac{f'(z(t)) - \alpha}{z(t)} \right) - q(z(t), \lambda) \right] \left(\frac{z(t)}{z_0} \right)^\gamma (z'(t))^2 \mathrm{d}t \right|$$

$$\leq \int_0^1 r(z(t), \lambda) \left| \left(\frac{z(t)}{z_0} \right)^\gamma z'(t) \right| |z'(t)| \mathrm{d}t = W(\lambda, \gamma).$$

This implies the required result. □

Proposition 3.4. *Let $\theta \in (-\pi, \pi]$ and $z_0 \in E\backslash\{0\}$. Then $H'_{e^{i\theta},\lambda}(z_0) \in \partial V(z_0, \lambda)$. Moreover, for some $\theta \in (-\pi, \pi]$ and $f \in R_{\lambda,\mu}(\alpha, \beta, \gamma)$,*

$$f'(z_0) = H'_{e^{i\theta},\lambda}(z_0) \implies f(z) = H_{e^{i\theta},\lambda}(z).$$

Proof. We have for $z \in E$

$$\begin{aligned} H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) &= \frac{\mu\left[\delta(az, \lambda)z + \beta\right]}{1 + \overline{\beta}\delta(az, \lambda)z} \\ &= \frac{\mu\left[(az + \lambda)z + \beta(1 + a\overline{\lambda}z)\right]}{1 + \left(a\overline{\lambda} + \overline{\beta}\lambda\right)z + a\overline{\beta}z^2} \end{aligned}$$

Thus, from (9), it follows that

$$H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) - b(z, \lambda) = \frac{\mu\left(1 - |\lambda|^2\right)\left(1 - |\beta|^2\right)az^2}{\left[1 + \left(a\overline{\lambda} + \overline{\beta}\lambda\right)z + a\overline{\beta}z^2\right]\left[1 + \overline{\beta}\lambda z\right]}$$

$$H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) + c(z, \lambda) = \frac{-\mu\left(1 - |\lambda|^2\right)\left(1 - |\beta|^2\right)z}{\left[1 + \left(a\overline{\lambda} + \overline{\beta}\lambda\right)z + a\overline{\beta}z^2\right]\left[\overline{\beta}z + \overline{\lambda}\right]},$$

and hence we have

$$\begin{aligned} H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) - q(z, \lambda) &= H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) \\ &\quad - \frac{b(z, \lambda) + |z|^2\left|\tau(z, \lambda)\right|^2 c(z, \lambda)}{1 - |z|^2\left|\tau(z, \lambda)\right|^2} \end{aligned}$$

$$= \frac{1}{1 - |z|^2\left|\tau(z, \lambda)\right|^2}\left[\begin{array}{c} H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) - b(z, \lambda) \\ -|z|^2\left|\tau(z, \lambda)\right|^2\left(H''_{a,\lambda}(z) + \gamma\left(\frac{H'_{a,\lambda}(z) - \alpha}{z}\right) + c(z, \lambda)\right) \end{array}\right]$$

$$= \frac{\mu\left(1 - |\lambda|^2\right)\left(1 - |\beta|^2\right)z^2}{1 - |\beta|^2|z|^4 - \left(1 - |\beta|^2\right)|\lambda|^2|z|^2 + 2\left(1 - |z|^2\right)\Re\left(\overline{\beta}\lambda z\right)}\frac{\overline{J(a, z)}}{J(a, z)},$$

where

$$J(a, z) = 1 + \left(a\overline{\lambda} + \overline{\beta}\lambda\right)z + a\overline{\beta}z^2$$

Putting $a = e^{i\theta}$, we obtain

$$H''_{e^{i\theta},\lambda}(z) + \gamma\left(\frac{H'_{e^{i\theta},\lambda}(z) - \alpha}{z}\right) - q(z, \lambda) = r(z, \lambda)\frac{\left|J(e^{i\theta}, z)\right|^2}{|z|^2}\frac{e^{i\theta}z^2}{\left(J(e^{i\theta}, z)\right)^2}.$$

From this we note that

$$H''_{e^{i\theta},\lambda}(z)+\gamma\left(\frac{H'_{e^{i\theta},\lambda}(z)-\alpha}{z}\right)-q(z,\lambda)=r(z,\lambda)\frac{G'(z)}{|G'(z)|}. \tag{11}$$

Since the function s is starlike in E, for any $z_0\in E\backslash\{0\}$, the linear segment joining 0 and $s(z_0)$ lies entirely in $s(E)$. Let Γ_0 be the curve defined by

$$\Gamma_0 : z(t)=s^{-1}\left(ts(z_0)\right),\ t\in[0,1].$$

This relation, together with (11), leads to

$$G'(z(t))z'(t)=3t^2G(z_0),\quad t\in[0,1]. \tag{12}$$

This relation, together with (11), leads to

$$\begin{aligned}
&H'_{e^{i\theta},\lambda}(z_0)-Q(\lambda,\gamma_0)\\
=&\int_0^1\left(H''_{e^{i\theta},\lambda}(z)+\gamma\left(\frac{H'_{e^{i\theta},\lambda}(z)-\alpha}{z}\right)-q(z(t),\lambda)\right)\left(\frac{z(t)}{z_0}\right)^{\gamma}(z'(t))^2\mathrm{d}t\\
=&\int_0^1 r(z(t),\lambda)\frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|}\left(\frac{z(t)}{z_0}\right)^{\gamma}z'(t)|z'(t)|\mathrm{d}t\\
=&\frac{z_0}{\gamma+1}\frac{G(z_0)}{|G(z_0)|}\int_0^1 r(z(t),\lambda)|z'(t)|\mathrm{d}t\\
=&\frac{G(z_0)}{|G(z_0)|}W(\lambda,\gamma_0).
\end{aligned}$$

That is

$$H'_{e^{i\theta},\lambda}(z_0)-Q(\lambda,\gamma_0)=\frac{G(z_0)}{|G(z_0)|}W(\lambda,\gamma_0). \tag{13}$$

This implies that $H'_{e^{i\theta},\lambda}(z_0)\in\partial\overline{\mathbb{D}}\left(Q(\lambda,\gamma_0),W(\lambda,\gamma_0)\right)$. Hence, from Corollary 1, we have $H'_{e^{i\theta},\lambda}(z_0)\in\partial V(\lambda,z_0)$.

Now, we prove the uniqueness part. Suppose that $f'(z_0)=H'_{e^{i\theta},\lambda}(z_0)$ for some $\theta\in(-\pi,\pi]$ and $f\in R_{\lambda,\mu}(\alpha,\beta,\gamma)$. Let

$$g(t)=\frac{\overline{G(z_0)}}{|G(z_0)|}\left(f''(z(t)+\gamma\left(\frac{f'(z(t))-\alpha}{z(t)}\right)-q(z(t),\lambda)\right)\left(\frac{z(t)}{z_0}\right)^{\gamma}(z'(t))^2,$$

where $\gamma_0(t)=z(t),\quad t\in[0,1]$. Then the function g is continuous and satisfies $|g(t)|\le r(z(t),\lambda)|z'(t)|$. Further, from (13), we have

$$\begin{aligned}
\int_0^1\Re g(t)\mathrm{d}t \;=\;& \int_0^1\Re\Bigg[\frac{\overline{G(z_0)}}{|G(z_0)|}\Bigg(f''(z(t)+\gamma\left(\frac{f'(z(t))-\alpha}{z(t)}\right)\\
&-q(z(t),\lambda)\Bigg)\left(\frac{z(t)}{z_0}\right)^{\gamma}(z'(t))^2\Bigg]\mathrm{d}t
\end{aligned}$$

$$= \Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}\left(f'(z_0) - Q(z(t),\gamma_0)\right)\right]$$
$$= \Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}\left(H'_{e^{i\theta},\lambda}(z_0) - Q(z(t),\gamma_0)\right)\right]$$
$$= \int_0^1 \Re r(z(t),\lambda)|z'(t)|\mathrm{d}t.$$

Thus $g(t) = r(z(t),\lambda)|z'(t)|$, for all $t \in [0,1]$. From (11) and (12), this implies that $f''(z(t) + \gamma\left(\frac{f'(z)-\alpha}{z}\right) = H''_{e^{i\theta},\lambda}(z) + \gamma\left(\frac{H'_{e^{i\theta},\lambda}(z)-\alpha}{z}\right)$ on γ_0. The identity theorem for analytic functions yields us $f(z) = H_{e^{i\theta},\lambda}(z)$, $z \in E$. □

4. MAIN THEOREM

THEOREM 4.1. *Let $|\lambda| < 1$, $z_0 \in E\backslash\{0\}$. Then boundary $\partial V(\lambda, z_0)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \mapsto H'_{e^{i\theta},\lambda}(z_0) = \alpha + \frac{\mu z_0}{\overline{\beta}(\gamma+1)} + \frac{\mu}{\beta e^{i\theta} z_0^{\gamma}} \log \frac{\left(1-\frac{z_0}{\zeta_2}\right)^{\frac{\zeta_2^{\gamma}\left(|\beta|^2-1\right)\left(\overline{\lambda}e^{i\theta}\zeta_2+1\right)}{\overline{\beta}(\zeta_1-\zeta_2)}}}{\left(1-\frac{z_0}{\zeta_1}\right)^{\frac{\zeta_1^{\gamma}\left(|\beta|^2-1^2\right)\left(\overline{\lambda}e^{i\theta}\zeta_1+1\right)}{\overline{\beta}(\zeta_1-\zeta_2)}}},$$

where ζ_1, ζ_2 are the zeros of the equation

$$1 + \left(e^{i\theta}\overline{\lambda} + \overline{\beta}\lambda\right)x + \overline{\beta}e^{i\theta}x^2 = 0.$$

If $f'(z_0) = H'_{e^{i\theta},\lambda}(z_0)$ for some f in $R_{\lambda,\mu}(\alpha,\beta,\gamma)$ and $\theta \in (-\pi,\pi]$, then $f(z_0) = F_{e^{i\theta},\lambda}(z_0)$.

Proof. We will show that the curve $(-\pi,\pi] \ni \theta \to F'_{e^{i\theta},\lambda}(z_0)$ is simple. Let us assume that $H'_{e^{i\theta_1},\lambda}(z_0) = H'_{e^{i\theta_2},\lambda}(z_0)$ for some θ_1,$\theta_2 \in (-\pi,\pi]$ with $\theta_1 \neq \theta_2$. Then the use of Proposition 3 yield us that $H_{e^{i\theta_1},\lambda}(z_0) = H_{e^{i\theta_2},\lambda}(z_0)$, which further gives the following relation

$$\tau\left(\frac{w_{H'_{e^{i\theta_1},\lambda}}(z)}{z},\lambda\right) = \tau\left(\frac{w_{H'_{e^{i\theta_2},\lambda}}(z)}{z},\lambda\right).$$

This implies that

$$\frac{\left(\overline{\lambda}^2+\overline{\beta}\right)e^{i\theta_1}z + \overline{\lambda} + \overline{\beta}\lambda}{\left(\overline{\lambda}+\overline{\beta}\lambda\right)e^{i\theta_1}z + 1 + \overline{\beta}\lambda^2} = \frac{\left(\overline{\lambda}^2+\overline{\beta}\right)e^{i\theta_2}z + \overline{\lambda} + \overline{\beta}\lambda}{\left(\overline{\lambda}+\overline{\beta}\lambda\right)e^{i\theta_2}z + 1 + \overline{\beta}\lambda^2}.$$

After some simplifications, we obtain $ze^{i\theta_1} = ze^{i\theta_2}$, which leads us to a contradiction. Therefore the curve is simple.

As $V(\lambda, z_0)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V(\lambda, z_0)$ is a simple closed curve. From Proposition 3, the curve $\partial V(\lambda, z_0)$ contains the curve $(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta},\lambda}(z_0)$. Since a simple closed curve cannot contain any simple closed curve other than itself, $\partial V(\lambda, z_0)$ is given by $(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta},\lambda}(z_0)$.

Now, we calculate

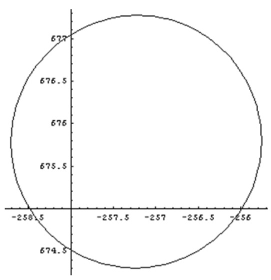
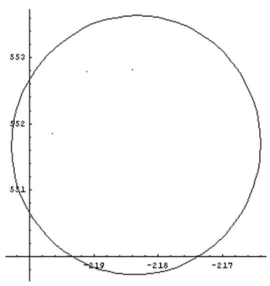
$$\begin{aligned} H'_{e^{i\theta},\lambda}(z_0) &= \alpha + \frac{\mu}{z_0^{\gamma}} \int_0^{z_0} \zeta^{\gamma} \frac{\left[\delta(e^{i\theta}\zeta, \lambda)\zeta + \beta\right]}{1 + \overline{\beta}\delta(e^{i\theta}\zeta, \lambda)\zeta} d\zeta \\ &= \alpha + \frac{\mu}{z_0^{\gamma}} \int_0^{z_0} \zeta^{\gamma} \frac{\left[e^{i\theta}\zeta^2 + \left[\lambda + \beta\overline{\lambda}e^{i\theta}\right]\zeta + \beta\right]}{1 + \left(e^{i\theta}\overline{\lambda} + \overline{\beta}\lambda\right)\zeta + \overline{\beta}e^{i\theta}\zeta^2} d\zeta \\ &= \alpha + \frac{\mu z_0}{\overline{\beta}(\gamma + 1)} + \frac{\mu}{\beta e^{i\theta} z_0^{\gamma}} \log \frac{\left(1 - \frac{z_0}{\zeta_2}\right)^{\frac{\zeta_2^{\gamma}\left(|\beta|^2 - 1\right)\left(\overline{\lambda}e^{i\theta}\zeta_2 + 1\right)}{\overline{\beta}(\zeta_1 - \zeta_2)}}}{\left(1 - \frac{z_0}{\zeta_1}\right)^{\frac{\zeta_1^{\gamma}\left(|\beta|^2 - 1^2\right)\left(\overline{\lambda}e^{i\theta}\zeta_1 + 1\right)}{\overline{\beta}(\zeta_1 - \zeta_2)}}}. \end{aligned}$$

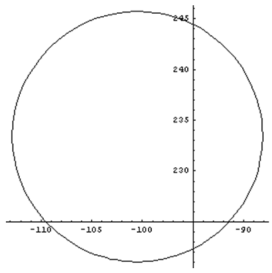
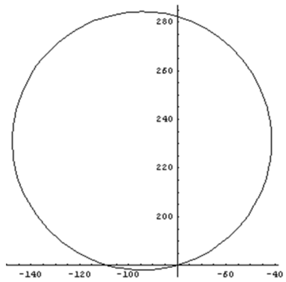
□

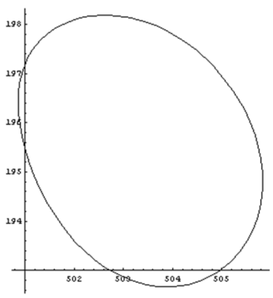
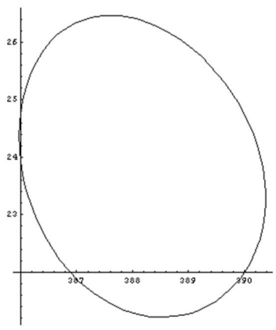
For $\gamma = 0$, we obtain the variability regions shown by Ponnusamy et al. [2].

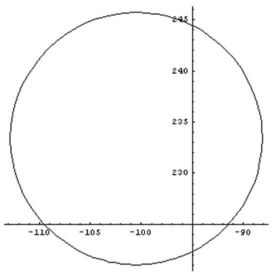
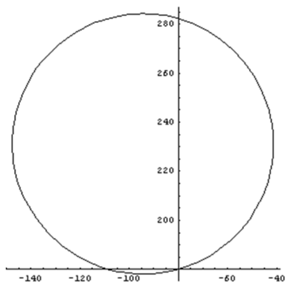
5. GEOMETRIC VIEW OF THEOREM 1

In the below figures, the geometric view of Theorem 1 is given by assigning different values to the involved parameters. All the values except those of γ are taken from the article [2], for comparison purposes. It can also be seen that, when $\gamma = 0$, we obtain the geometric view of [2, Theorem 2.4].

Values of parameters [2]	Theorem 1 $\gamma = 0.9 + 0.5i$	Theorem 2.4 of [2] $\gamma = 0$
$z_0 = 0.00882581 - 0.514124i$ $\alpha = -230.939 + 799.526i$ $\lambda = 0.427174 + 0.0755107i$ $\mu = 509.317$ $\beta = 0.94485 + 0.0416585i$		

Values of parameters [2]	Theorem 1 $\gamma = 10+1.0i$	Theorem 2.4 of [2] $\gamma = 0$
$z_0 = -0.605185 + 0.789592i$ $\alpha = -100.796+233.556i$ $\lambda = 0.0523661+0.167249i$ $\mu = 164.079$ $\beta = 0.00810121 - 0.00819085$		

Values of parameters [2]	Theorem 1 $\gamma = -0.5-0.9i$	Theorem 2.4 of [2] $\gamma = 0$
$z_0 = -0.439619 - 0.843107i$ $\alpha = 306.095+212.047i$ $\lambda = -0.847689 - 0.07592i$ $\mu = 206.29$ $\beta = 0.67079+0.843107i$		

Values of parameters [2]	Theorem 1 $\gamma = 10+1.0i$	Theorem 2.4 of [2] $\gamma = 0$
$z_0 = -0.605185 + 0.789592i$ $\alpha = -100.796+233.556i$ $\lambda = 0.0523661+0.167249i$ $\mu = 164.079$ $\beta = 0.00810121 - 0.00819085$		

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Majmaah Univeristy
College of Science Al-Zulfi
Department of Mathematics
Al-Zulfi, Saudi Arabia
E-mail: w.ulhaq@mu.edu.sa

University of Islamabad
Department of Mathematics
Bahria Golf City, Islamabad, Pakistan
E-mail: wasim474@hotmail.com