# VIETORIS TOPOLOGY ON HYPERSPACES ASSOCIATED TO A NONCOMMUTATIVE COMPACT SPACE

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**Abstract.** We study some topological spaces that can be considered as hyperspaces associated to noncommutative spaces. More precisely, for a NC compact space associated to a unital $C^*$-algebra, we consider the set of closed projections of the second dual of the $C^*$-algebra as the hyperspace of closed subsets of the NC space. We endow this hyperspace with an analog of Vietoris topology. In the case that the NC space has a quantum metric space structure in the sense of Rieffel we study the analogs of Hausdorff and infimum distances on the hyperspace. We also formulate some problems about distances between sub-circles of a quantum torus.

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**Key words.** $C^*$-algebra, state space, closed projection, hyperspace, Vietoris topology, Hausdorff distance, infimum distance.

## 1. INTRODUCTION

This note is a contribution to Noncommutative Topology. We introduce and study some topological spaces that can be considered as the hyperspaces associated to noncommutative spaces. More precisely, let $\mathfrak{q}A$ denote the (imaginary) NC compact Hausdorff space associated to a unital $C^*$-algebra $A$. We consider the set of nonzero closed projections of the second dual $A^{**}$ as the hyperspace $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ of nonempty closed subsets of $\mathfrak{q}A$. In the case that $A$ is commutative these closed projections are canonically identified with closed subsets of the Gelfand space of $A$. (To the best of our knowledge the study of closed projections as closed subsets of NC spaces goes back to Akemann [1, 2, 3]. Closed projections have been considered also in some recent papers, see [10, 11] and references therein.) There is a canonical bijection between closed projections in $A^{**}$ and weak*-closed faces of the state space $\mathcal{S}A$ of $A$ (see Section 4). Thus we can identify the hyperspace $\mathfrak{S}_{\mathrm{clc}}\mathcal{S}A$ of such subsets of $\mathcal{S}A$ with $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$. We have a canonical Vietoris topology on $\mathfrak{S}_{\mathrm{clc}}\mathcal{S}A$ induced from the weak*-topology of $\mathcal{S}A$. In the case that $A$ is commutative it is proved in Section 2 that this Vietoris topology coincides with the vietoris topology on the hyperspace of closed subsets of the Gelfand space of $A$. Suppose that $\mathfrak{q}A$ has a quantum metric space structure in the sense of Rieffel [23, 24, 25].

This induces Hausdorff and infimum distances on $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$. Again if $A$ is commutative it is proved that these distances coincide with the usual Hausdorff and infimum distances (see the last paragraph of Section 4).

The notion of *quantum (or NC) metric space* have been considered by many authors, see [25, 15, 14, 12, 16, 17, 29, 27] and references therein. The main subjects studied in most of the mentioned papers are variations of the quantum Gromov-Hausdorff distance and quantum metric spaces defined by Rieffel [25]. The notions introduced by Rieffel [25] are based on order unit spaces. Since our attention here is to NC Topology we are more interested in order unit spaces arising from $C^*$-algebras.

The plan of the paper is as follows. In Section 2 we consider some properties of hyperspaces associated to ordinary topological spaces. Also we consider Hausdorff and infimum distances. In Section 3 we review the notion of quantum metric space. In Section 4 we introduce our main object $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$, the hyperspace of closed subsets of a compact NC space. In Section 5 we study the Vietoris topology on $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$. In Section 6 using the infimum distance we define an analog of Lipschitz seminorm for quantum metric spaces. At last in Section 7 we consider some questions and problems on finite NC spaces and quantum tori.

## 2. HYPERSPACE OF CLOSED SUBSETS OF AN ORDINARY TOPOLOGICAL SPACE

Let $X$ be a compact Hausdorff space. We denote by $\mathfrak{S}_{\mathrm{cl}}X$ the set of all nonempty closed subsets of $X$. For every open $U \subseteq X$, let $U^- := \{K \in \mathfrak{S}_{\mathrm{cl}}X : K \cap U \neq \emptyset\}$ and $U^+ := \{K \in \mathfrak{S}_{\mathrm{cl}}X : K \subseteq U\}$. The smallest topology on $\mathfrak{S}_{\mathrm{cl}}X$ containing all $U^{\pm}$'s is called Vietoris topology. The space $\mathfrak{S}_{\mathrm{cl}}X$ together with the Vietoris topology is called the hyperspace of closed sets in $X$. It is easy to see ([13, Exercise 3.12]) that the hyperspace is compact and Hausdorff. Also the subspace topology of $X$, where $X$ is considered as a subspace of $\mathfrak{S}_{\mathrm{cl}}X$ via the canonical embedding $x \mapsto \{x\}$, coincides with the original topology of $X$. Let $\mathcal{C}X$ denote the $C^*$-algebra of complex valued continuous functions on $X$. We always endow the state space $\mathcal{SC}X$ of $\mathcal{C}X$ with weak* topology. We also identify $\mathcal{SC}X$ with the space of Borel regular probability measures on $X$. Then the map $\delta : x \mapsto \delta_x$ is a homeomorphism from $X$ onto the space of pure states of $\mathcal{C}X$ where $\delta_x$ denote the point mass measure concentrated at $x$. For a nonempty closed subset $K$ of $X$ let $\mathcal{F}_K$ denote the set of those measures $\mu$ in $\mathcal{SC}X$ with $\mathrm{Spt}(\mu) \subseteq K$. Then $\mathcal{F}_K$ is a weak*-closed face of $\mathcal{SC}X$. Also note that $\mathcal{F}_K$ is the weak* closed convex hull of $\{\delta_x : x \in K\}$.

PROPOSITION 2.1. *The map* $\mathcal{F} : K \mapsto \mathcal{F}_K$ *is a homeomorphism from the hyperspace* $\mathfrak{S}_{\mathrm{cl}}X$ *onto a closed subspace of the hyperspace* $\mathfrak{S}_{\mathrm{cl}}\mathcal{SC}X$.

*Proof.* Since both of the hyperspaces are compact Hausdorff spaces and $\mathcal{F}$ is injective it is enough to show that $\mathcal{F}$ is continuous. Let $U, V$ be arbitrary open subsets of $\mathcal{SC}X$. We must show that $\mathcal{F}^{-1}(U^+)$ and $\mathcal{F}^{-1}(V^-)$ are open in $\mathfrak{S}_{\mathrm{cl}}X$. Suppose that $K \in \mathcal{F}^{-1}(U^+)$. Thus $\mathcal{F}_K \subseteq U$. Since $\mathcal{F}_K$ is convex it follows

from [26, Theorem 1.10] that there is a convex open subset $U_0$ of $\mathcal{S}\mathcal{C}X$ with $\mathcal{F}_K \subseteq U_0 \subseteq \overline{U_0} \subseteq U$. We have $K \in (\delta^{-1}(U_0))^+ \subseteq \mathcal{F}^{-1}(U^+)$. Thus $\mathcal{F}^{-1}(U^+)$ is open. Now suppose that $K \in \mathcal{F}^{-1}(V^-)$. Thus there are $\mu \in \mathcal{F}_K \setminus V^c$ and open subset $W$ of $\mathcal{S}\mathcal{C}X$ such that $\mu \in W$ and $W \cap V^C = \emptyset$. It follows that there exist $x_1, \ldots, x_n \in \mathrm{Spt}\mu \subseteq K$, $t_1, \ldots, t_n > 0$ with $\sum_{i=1}^n t_i = 1$, and an open subset $W_0$ of $\mathcal{S}\mathcal{C}X$, such that $\sum_{i=1}^n t_i\delta_{x_i} \in W_0 \subseteq W$. Thus there are open subsets $O_1, \ldots, O_n$ of $X$ with $x_i \in O_i$, and with the property that if $y_i \in O_i$ then $\sum_{i=1}^n t_i\delta_{y_i} \in W_0$. We have $K \in \cap_{i=1}^n O_i^- \subseteq \mathcal{F}^{-1}(V^-)$. Thus $\mathcal{F}^{-1}(V^-)$ is open. This completes the proof. $\square$

Now suppose that $X$ is metrizable and let $d$ be a compatible metric on $X$. The Hausdorff distance $\mathcal{H}_d$ (associated to $d$) on $\mathfrak{S}_{\mathrm{cl}}X$ is defined by

$$\mathcal{H}_d(K, K') = \inf\{r > 0 : K \subseteq \mathrm{Ball}(K', r), K' \subseteq \mathrm{Ball}(K, r)\} \quad (K, K' \in \mathfrak{S}_{\mathrm{cl}}X),$$

where $\mathrm{Ball}(K, r) = \{y \in X : d(x, y) < r, \exists x \in K\}$. It is well known that $\mathcal{H}_d$ is a metric and the topology induced by $\mathcal{H}_d$ coincides with the Vietoris topology ([13, Theorem 3.1]). Also the mapping $x \mapsto \{x\}$ is an isometric embedding of $X$ into $\mathfrak{S}_{\mathrm{cl}}X$. The Lipschitz seminorm $\mathcal{L}_d$ for (self-adjoint) elements of $\mathcal{C}X$ is defined by

$$(1) \qquad \mathcal{L}_d(f) := \sup\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\} \quad (f \in \mathcal{C}X_{\mathrm{sa}}).$$

This seminorm satisfies the Leibniz inequality:

$$\mathcal{L}_d(fg) \leq \mathcal{L}_d(f)\|g\|_\infty + \|f\|_\infty\mathcal{L}_d(g).$$

The Lipschitz algebra of $(X, d)$ is defined by $\mathrm{Lip}_d X := \{f \in \mathcal{C}X_{\mathrm{sa}} : \mathcal{L}_d(f) < \infty\}$. This is a real uniformly-dense subalgebra of $\mathcal{C}X_{\mathrm{sa}}$. (For an extensive account on Lipschitz algebras see [28].) The Monge-Kantorovich distance is defined by

$$(2) \qquad \rho_d(\mu, \nu) := \sup\{|\mu(f) - \nu(f)| : \mathcal{L}_d(f) \leq 1\} \quad (\mu, \nu \in \mathcal{S}\mathcal{C}X).$$

It is well known that the topology of $\rho_d$ coincides with weak* topology and also the restriction of $\rho_d$ to the space of pure states of $\mathcal{C}X$ is equal to $d$ where the pure state space is canonically identified with $X$. The metric version of Proposition 2.1 is as follows.

PROPOSITION 2.2. *$K \mapsto \mathcal{F}_K$ is an isometric embedding from $(\mathfrak{S}_{\mathrm{cl}}X, \mathcal{H}_d)$ into $(\mathfrak{S}_{\mathrm{cl}}\mathcal{S}\mathcal{C}X, \mathcal{H}_{\rho_d})$.*

*Proof.* Let $h$ denote the Hausdorff distance of $\mathcal{F}_K$ and $\mathcal{F}_{K'}$. Suppose that $\mathcal{H}_d(K, K') < r$. Then for every $x \in K$ there is $y \in K'$ such that $d(x, y) < r$. Let $t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ and let $x_1, \ldots, x_n \in K$. Then it is easily verified that $\rho_d(\sum_{i=1}^n t_i\delta_{x_i}, \sum_{i=1}^n t_i\delta_{y_i}) < r$ where $y_i \in K'$ is such that $d(x_i, y_i) < r$. This shows that $\mathcal{F}_K \subseteq \mathrm{Ball}(\mathcal{F}_{K'}, r + \epsilon)$ for every $\epsilon > 0$. Similarly, we have $\mathcal{F}_{K'} \subseteq \mathrm{Ball}(\mathcal{F}_K, r + \epsilon)$. Thus $h \leq r$, and hence $h \leq \mathcal{H}_d(K, K')$. Now suppose that $h < s$. Let $x \in K$. Then there are $z_1, \ldots, z_n \in K'$ and

$t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^{n} t_i = 1$ and $\rho_d(\delta_x, \sum_{i=1}^{n} t_i \delta_{z_i}) < s$. Let $f$ be the function on $X$ defined by $y \mapsto d(x, y)$. Then $\mathcal{L}_d(f) = 1$ (if $X$ at least has two points). We have

$$\sum_{i=1}^{n} t_i d(x, z_i) = |\delta_x(f) - (\sum_{i=1}^{n} t_i \delta_{z_i})(f)| < s.$$

Thus $d(x, z_{i_0}) < s$ for some $i_0$. This shows that $K \subseteq \text{Ball}(K', s)$. Similarly we have $K' \subseteq \text{Ball}(K, s)$. Thus $\mathcal{H}_d(K, K') \leq s$, and hence $\mathcal{H}_d(K, K') \leq h$. The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

For two subsets $K, K'$ of $X$ their infimum distance is defined by

$$\mathcal{I}_d(K, K') := \inf\{d(x, y) : x \in K, y \in K'\}.$$

In the case that $K$ or $K'$ is empty we let $\mathcal{I}_d(K, K') = \infty$. Note also that in general $\mathcal{I}_d$ is not a metric on $\mathfrak{S}_{\text{cl}}X$.

PROPOSITION 2.3. *Let* $K, K' \in \mathfrak{S}_{\text{cl}}X$. *Then* $\mathcal{I}_d(K, K') = \mathcal{I}_{\rho_d}(\mathcal{F}_K, \mathcal{F}_{K'})$.

*Proof.* Let $I$ denote the infimum distance of $\mathcal{F}_K$ and $\mathcal{F}_{K'}$. For $x \in K, y \in K'$ we have $\delta_x \in \mathcal{F}_K, \delta_y \in \mathcal{F}_{K'}$ and $d(x, y) = \rho_d(\delta_x, \delta_y)$. Thus $I \leq \mathcal{I}_d(K, K')$. Let $I < r$. There are $\mu := \sum_{i=1}^{n} t_i \delta_{x_i} \in \mathcal{F}_K$ and $\nu := \sum_{j=1}^{m} s_j \delta_{y_j} \in \mathcal{F}_{K'}$ such that $\rho_d(\mu, \nu) < r$. Let the function $f$ on $X$ be defined by $x \mapsto \mathcal{I}_d(\{x\}, \{y_1, \ldots, y_n\})$. Then $\mathcal{L}_d(f) = 1$, and we have

$$\sum_{i=1}^{n} t_i f(x_i) = |\mu(f) - \nu(f)| < r.$$

Thus $f(x_{i_0}) < r$ for some $i_0$, and hence there is $j_0$ such that $d(x_{i_0}, y_{j_0}) < r$. This shows that $\mathcal{I}_d(K, K') < r$. Since $r > I$ is arbitrary we conclude that $\mathcal{I}_d(K, K') \leq I$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

It is well known that

(3) $\qquad \mathcal{L}_d(f) = \sup\{\dfrac{|\mu(f) - \nu(f)|}{\rho_d(\mu, \nu)} : \mu, \nu \in \mathcal{SC}X, \mu \neq \nu\} \quad (f \in \mathcal{C}X_{\text{sa}}).$

Also the following formula follows from (1) and Proposition 2.3.

(4)
$$\mathcal{L}_d(f) = \sup_{\lambda < \lambda' \in \mathbb{R}} \frac{\lambda' - \lambda}{\mathcal{I}_d(f^{-1}\lambda', f^{-1}\lambda)} = \sup_{\lambda < \lambda' \in \mathbb{R}} \frac{\lambda' - \lambda}{\mathcal{I}_{\rho_d}(\mathcal{F}_{f^{-1}\lambda'}, \mathcal{F}_{f^{-1}\lambda})} \quad (f \in \mathcal{C}X_{\text{sa}}).$$

### 3. COMPACT QUANTUM METRIC SPACES

For the theory of order unit spaces we refer the reader to [5]. We denote the state space of an order unit space $B$ by $\mathcal{S}B$. This space is always considered with the weak* topology. Let $A$ be a unital $C^*$-algebra with the self-adjoint part $A_{\text{sa}}$ and state space $\mathcal{S}A$. Suppose that $B$ is any real linear subspace of $A_{\text{sa}}$ that contains $1_A$. Then $B$ together with the usual partial ordering

between self-adjoint elements and $1_A$ as order unit becomes an order unit space. Moreover if $B$ is dense in $A_{\mathrm{sa}}$ (with the norm topology) then the mapping $\mu \mapsto \mu|_B$ defines an affine homeomorphism from $\mathcal{S}A$ with the weak* topology onto $\mathcal{S}B$.

It is clear that the state space of an order unit space with weak* topology is a compact convex subset of a locally convex Hausdorff space. The converse of this fact is also well known (see the details after Corollary II.2.3 of [5]). Indeed, let $E$ be a compact convex subset of a locally convex Hausdorff space; if $\mathcal{A}E$ denotes the order unit space of all continuous affine real valued functions on $E$ with the constant function $1_E$ as order unit, then $E$ and $\mathcal{S}\mathcal{A}E$ are affinely homeomorphic via the map $x \mapsto (f \mapsto f(x))$ $(x \in E)$. Thus there is no difference that we formulate our results in terms of order unit spaces or else using compact convex sets.

Let $B$ be an order unit space and $L$ be a seminorm on $B$. By analogy with Formula (2) we define a pseudo-metric on $\mathcal{S}B$ as follows.

$$(5) \qquad \rho_L(\mu,\nu) := \sup\{|\mu(b) - \nu(b)| : L(b) \leq 1\} \quad (\mu, \nu \in \mathcal{S}B).$$

Note that in general $\rho_L$ does not separate the points and may take value $+\infty$.

DEFINITION 3.1 ([23, 24, 25]). Let $B$ be an order unit space with order unit $e$ and let $L$ be a seminorm on $B$ satisfying the following two conditions:

   i) $L(b) = 0$ if and only if $b = \lambda e$ for some $\lambda \in \mathbb{R}$.
   ii) The topology induced by $\rho_L$, given by Formula (5), coincides with the weak*-topology on $\mathcal{S}B$.

Then the pair $(B, L)$ is called a compact quantum metric space. Also if a unital $C^*$-algebra $A$ is given such that $B$ as an order unit space is a subspace of $A_{\mathrm{sa}}$ containing $1_A$, and $B$ is dense in $A_{\mathrm{sa}}$ w.r.t. the $C^*$-norm, then $(A, B, L)$ is called a $C^*$-algebraic compact quantum metric space. In the case that $(B, L)$ is understood we say that $\mathfrak{q}A$ is a $C^*$-algebraic quantum metric space.

Let $(X, d)$ be an ordinary compact metric space. Then $(\mathcal{C}X, \mathrm{Lip}_d X, \mathcal{L}_d)$ is a $C^*$-algebraic compact quantum metric space. Also by (2) and (5) we have $\rho_{\mathcal{L}_d} = \rho_d$. Thus the structure of $(X, d)$ is completely recovered by $(\mathcal{C}X, \mathrm{Lip}_d X, \mathcal{L}_d)$. We remark that there are examples of $C^*$-algebraic compact quantum metric spaces $(A, B, L)$ with $A = \mathcal{C}X$ for a compact space $X$ such that $L$ does not arise from any ordinary metric $d$ on $X$ i.e. $L \neq \mathcal{L}_d$, see [23, Example 7.1 and Theorem 8.1]. For other examples of nonclassical quantum metric spaces we refer the reader to the list of papers in Introduction.

## 4. HYPERSPACE OF CLOSED SETS IN AN NC SPACE

Let $A$ be a unital $C^*$-algebra with the state space $\mathcal{S}A$. Let $A''$ denote the second commutant of $A$ in the universal representation of $A$. By the Sherman Theorem the second dual $A^{**}$ is canonically isomorphic to the von Neumann algebra $A''$ where $A^{**}$ is considered as a $C^*$-algebra with the Arens

product. A projection $p \in A^{**}$ is called closed [1] if there is a decreasing net of positive elements of $A$ that converges to $p$ in the weak* topology. A projection $q$ is called open if $1 - q$ is closed. For every projection $p \in A^{**}$ we let $\mathcal{F}_p := \{\mu \in \mathcal{S}A : \langle \mu, p \rangle = 1\}$. In the case that $A = \mathcal{C}X$ for a compact Hausdorff space $X$, there is a bijection $K \mapsto p_K$ between closed subsets $K$ of $X$ and closed projections of $A^{**}$ such that $\mathcal{F}_K = \mathcal{F}_{p_K}$ with the notations of Section 2.

PROPOSITION 4.1. *The assignment $p \mapsto \mathcal{F}_p$ is a bijection between closed projections of $A^{**}$ and weak\*-closed faces of $\mathcal{S}A$.*

*Proof.* It follows by [21, Theorems 3.6.11 and 3.10.7] or [4, Theorem 2.5]. $\square$

Let $E$ be a compact convex subset of a locally convex Hausdorff space. We denote by $\mathfrak{S}_{\mathrm{clc}}E$ the set of nonempty closed convex subsets of $E$, and by $\mathfrak{S}_{\mathrm{clf}}E$ the set of all closed faces of $E$. Thus we have the chain $\mathfrak{S}_{\mathrm{clf}}E \subset \mathfrak{S}_{\mathrm{clc}}E \subset \mathfrak{S}_{\mathrm{cl}}E$ of Hyperspaces. Throughout the paper these hyperspaces are endowed with Vietoris topology.

Let $B$ be an order unit space. In [25] (imaginary) closed subsets of the quantum space $\mathfrak{q}B$ are identified with elements of $\mathfrak{S}_{\mathrm{clc}}\mathcal{S}B$. As we saw above it is more natural to consider the closed subsets as elements of $\mathfrak{S}_{\mathrm{clf}}\mathcal{S}B$. So by analogy with the notations of Section 2 we would use the symbol $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}B$ instead of $\mathfrak{S}_{\mathrm{clf}}\mathcal{S}B$. Analogously, for a unital $C^*$-algebra $A$ we let $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A := \mathfrak{S}_{\mathrm{clf}}\mathcal{S}A$. In the case that $A = \mathcal{C}X$ for a compact Hausdorff space $X$ it follows from Proposition 2.1 that $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ is homeomorphic to $\mathfrak{S}_{\mathrm{cl}}X$. Suppose that $X$ has a compatible metric $d$ and consider the $C^*$-algebraic quantum metric space $(\mathcal{C}X, \mathrm{Lip}_d X, \mathcal{L}_d)$. Let $\rho := \rho_{\mathcal{L}_d} = \rho_d$. It follows from Proposition 2.2 that the metric spaces $(\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A, \mathcal{H}_\rho)$ and $(\mathfrak{S}_{\mathrm{cl}}X, \mathcal{H}_d)$ are isometrically isomorphic. Also it follows from Proposition 2.3 that the distance functions $\mathcal{I}_\rho$ on $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ and $\mathcal{I}_d$ on $\mathfrak{S}_{\mathrm{cl}}X$ coincide when the two spaces are considered canonically identical.

## 5. VIETORIS TOPOLOGY

Throughout this section $E$ denotes a compact convex subset of a locally convex Hausdorff space. The following result stated as Theorem 5.2 is very well known, at least in the case that $E$ is metrizable; but we did not find in literatures any proof for the general case; however its proof is easy and based on the following lemma. (Let $\Lambda, \Lambda'$ be directed sets and $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $X$. Let $f : \Lambda' \to \Lambda$ be an order preserving function such that $\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda' : f(\lambda') \geq \lambda$. Then the net $(x_{f(\lambda')})_{\lambda' \in \Lambda'}$ is called a subnet of $(x_\lambda)_{\lambda \in \Lambda}$.)

LEMMA 5.1. *Let $X$ be a compact Hausdorff space and $(K_\lambda)_\lambda$ a net in $\mathfrak{S}_{\mathrm{cl}}X$ converging to $K$.*

   (i) *If $(x_\lambda)_\lambda$ is a net in $X$ such that $x_\lambda \to x$ and $x_\lambda \in K_\lambda$, then $x \in K$.*
   (ii) *If $x \in K$, then there are a subnet $(K_{\lambda'})_{\lambda'}$ of $(K_\lambda)_\lambda$ and a net $(x_{\lambda'})_{\lambda'}$ such that $x_{\lambda'} \in K_{\lambda'}$ and $x_{\lambda'} \to x$.*

*Proof.* Straightforward. ☐

THEOREM 5.2. *The hyperspace* $\mathfrak{S}_{\mathrm{clc}}E$ *is a compact Hausdorff space.*

*Proof.* It is enough to show that $\mathfrak{S}_{\mathrm{clc}}E$ is a closed subset of $\mathfrak{S}_{\mathrm{cl}}E$. Suppose that $(K_\lambda)_\lambda$ is a net in $\mathfrak{S}_{\mathrm{clc}}E$ converging to $K \in \mathfrak{S}_{\mathrm{cl}}E$. We must show that $K$ is convex. Suppose that $x, y \in K$ and $0 \le t \le 1$. By Lemma 5.1(ii) there exist a subnet $(K_{\lambda'})_{\lambda'}$ of $(K_\lambda)_\lambda$ and nets $(x_{\lambda'})_{\lambda'}, (y_{\lambda'})_{\lambda'}$ such that $x_{\lambda'}, y_{\lambda'} \in K_{\lambda'}$ and $x_{\lambda'} \to x, y_{\lambda'} \to y$. Thus $(tx_{\lambda'} + (1 - t)y_{\lambda'})_{\lambda'}$ is a net in $K_{\lambda'}$ converging to $tx + (1 - t)y$. Now it follows from Lemma 5.1(i) that $tx + (1 - t)y \in K$. The proof is complete. ☐

For metrizable $E$ we have the following strong result of Nadler-Quinn-Stavrakos:

THEOREM 5.3. *Suppose that $E$ is metrizable and the real dimension of the smallest real hyperplane containing $E$ is $\ge 2$. Then $\mathfrak{S}_{\mathrm{clc}}E$ is homeomorphic to Hilbert cube.*

*Proof.* This is a restatement of [18, Theorem 2.2]. (Note that in the proof of [18, Theorem 2.2] it is enough that $K$ be metrizable.) ☐

For some results similar to Theorem 5.3 in the case that $E$ is not metrizable see [9]. A direct consequence of Theorem 5.3 is the following.

COROLLARY 5.4. *Let $(B, L)$ be a compact quantum metric space such that the (real vector space) dimension of $B$ is $\ge 2$. Then $\mathfrak{S}_{\mathrm{clc}}\mathcal{S}B$ is homeomorphic to Hilbert cube. In particular, if $\mathfrak{q}A$ is a $C^*$-algebraic quantum metric space such that $A \ne 0, \mathbb{C}$ then $\mathfrak{S}_{\mathrm{clc}}\mathcal{S}A$ is homeomorphic to Hilbert cube.*

Let $\partial_{\mathrm{e}}E$ denote the subspace of extreme points of $E$.

THEOREM 5.5. *If the hyperspace $\mathfrak{S}_{\mathrm{clf}}E$ is compact then $\partial_{\mathrm{e}}E$ is compact.*

*Proof.* Suppose that $\mathfrak{S}_{\mathrm{clf}}E$ is compact. We must show that $\partial_{\mathrm{e}}E$ is a closed subset of $E$. Let $(e_\lambda)_\lambda$ be a net in $\partial_{\mathrm{e}}E$ converging to $x \in E$. Since $e_\lambda$ is an extreme point $\{e_\lambda\}$ is a closed face of $E$. Thus there is a subnet $(e_{\lambda'})_{\lambda'}$ such that $\{e_{\lambda'}\} \to K$ in $\mathfrak{S}_{\mathrm{clf}}E$. Now it follows from Lemma 5.1 that $K = \{x\}$ which means that $x \in \partial_{\mathrm{e}}E$. Thus $\partial_{\mathrm{e}}E$ is a closed subset of $E$. ☐

In general the converse of Theorem 5.5 is not satisfied even if $E$ is finite dimensional, see [22]. We say that $E$ is stable [20] if for every $0 \le t \le 1$ the map $(x, y) \mapsto tx + (1 - t)y$ from $E \times E$ into $E$ is open. Among examples of stable compact convex sets are Bauer simplices [19, Theorem 1]. For a complete account on Bauer simplices see [5]. It is well known that a unital $C^*$-algebra is commutative if and only if its state space with weak*-topology is a Bauer simplex [6, Remark in page 296],[8]. To our knowledge the following result has not been mentioned before in the literatures.

THEOREM 5.6. *If $E$ is stable then $\mathfrak{S}_{\mathrm{clf}}E$ is compact.*

*Proof.* Suppose that $E$ is stable. We must show that $\mathfrak{S}_{\mathrm{clf}}E$ is a closed subset of $\mathfrak{S}_{\mathrm{clc}}E$. Let $(K_\lambda)_\lambda$ be a net in $\mathfrak{S}_{\mathrm{clf}}E$ converging to $K \in \mathfrak{S}_{\mathrm{clc}}E$. We show that $K$ is a face of $E$. Suppose that for some $0 \le t \le 1$ and $x, y \in E$ we have $z := tx + (1-t)y \in K$. By Lemma 5.1(ii) there exist a subnet $(K_{\lambda'})_{\lambda'}$ of $(K_\lambda)_\lambda$ and a net $(z_{\lambda'})_{\lambda'}$ such that $z_{\lambda'} \in K_{\lambda'}$ and $z_{\lambda'} \to z$. Since the map $\phi : (x', y') \mapsto tx' + (1-t)y'$ is open and continuous, for every open $U$ in $E$ containing $z$ there are opens $V, W$ respectively containing $x, y$ such that $\phi(V \times W)$ is an open in $U$. This property enables us to find a subnet $(z_{\lambda''})_{\lambda''}$ of $(z_{\lambda'})_{\lambda'}$ and nets $(x_{\lambda''})_{\lambda''}, (y_{\lambda''})_{\lambda''}$ such that $x_{\lambda''} \to x, y_{\lambda''} \to y$ and $z_{\lambda''} = tx_{\lambda''} + (1-t)y_{\lambda''}$. Since $K_{\lambda''}$ is a face we have $x_{\lambda''}, y_{\lambda''} \in K_{\lambda''}$. Now it follows from Lemma 5.1(i) that $x, y \in K$. The proof is complete.          $\square$

A direct consequence of Theorems 5.5 and 5.6 is the following result.

COROLLARY 5.7. *Let $A$ be a unital $C^*$-algebra. If $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ is compact then the space of pure states of $A$ is weak\*-compact. If $\mathcal{S}A$ is stable then $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ is compact.*

## 6. INFIMUM DISTANCE AND AN ANALOG OF LIPSCHITZ SEMINORM

Let $A$ be a unital $C^*$-algebra and let $\rho$ be a compatible metric on $\mathcal{S}A$. Let $L_1 : A_{\mathrm{sa}} \to [0, \infty]$ be a seminorm given by the analog of Formula (3):

$$L_1(a) = \sup\left\{ \frac{|\mu(a) - \nu(a)|}{\rho(\mu, \nu)} : \mu, \nu \in \mathcal{S}A, \mu \ne \nu \right\}.$$

Let $H$ denote the Hilbert space of the universal representation of $A$, and $\mathcal{B}H$ be the algebra of bounded operators on $H$. Then by definition we have $A \subset A'' \subseteq \mathcal{B}H$. For $a \in A_{\mathrm{sa}}$ let $E_a$ denotes the spectral measure of $a$ defined on the Borel subsets of $\mathbb{R}$ where $a$ is considered as an element of $\mathcal{B}H$. It is well known that for every closed subset $S$ of $\mathbb{R}$ the projection $E_a(S)$ is a closed projection in $A''$. (The converse is also true [2, Theorem A1],[3], that is if $a \in A''_{\mathrm{sa}}$ and $E_a(S)$ is a closed projection for every closed subset $S \subseteq \mathbb{R}$ then $a \in A$.) For $a \in A_{\mathrm{sa}}$ and $\lambda \in \mathbb{R}$ let $\mathcal{F}_{a,\lambda}$ denote the weak\*-closed face of $\mathcal{S}A$ corresponding to the closed projection $E_a(\{\lambda\})$ as in Proposition 4.1. (In the case that $E_a(\{\lambda\}) = 0$ we let $\mathcal{F}_{a,\lambda} = \emptyset$.) In Section 2 we restated the definition of Lipschitz seminorm for an ordinary metric space as Formula (4). Now analogously we define a function $L_2 : A_{\mathrm{sa}} \to [0, \infty]$ by

$$L_2(a) = \sup_{\lambda < \lambda' \in \mathbb{R}} \frac{\lambda' - \lambda}{\mathcal{I}_\rho(\mathcal{F}_{a,\lambda'}, \mathcal{F}_{a,\lambda})}.$$

We have $L_2 \le L_1$ but in general $L_2$ is not a seminorm.

QUESTION 6.1. *Under which conditions is $L_2$ a seminorm (with Leibniz property) on any commutative subalgebra of $A$?*

Suppose $\rho$ is induced by a $C^*$-algebraic quantum metric structure $(A, B, L)$ i.e. $\rho = \rho_L$. Then we have $L_2(a) \le L_1(a) \le L(a)$ for $a \in B$. As we saw

in Section 2 in the classical case $(A, B, L) = (\mathcal{C}X, \mathrm{Lip}_d X, \mathcal{L}_d)$ we have $L_2 = L_1 = L$. By Theorem 4.1 of [23] we know that if $L$ is lower semicontinuous (which means $\{a \in B : L(a) \leq 1\}$ is closed in $B$ w.r.t. the $C^*$-norm) then $L_1(a) = L(a)$ for every $a \in B$.

### 7. SOME QUESTIONS AND PROBLEMS

We saw that for a unital commutative $C^*$-algebra $A$, $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ is compact.

PROBLEM 7.1. *Characterize those unital $C^*$-algebras $A$ such that $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ is compact.*

QUESTION 7.2. *For which $C^*$-algebras $A$, is $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}A$ (path or locally path) connected? (See [7] in the classical case.)*

Let $\mathcal{M}_n$ denote the $C^*$-algebra of $n \times n$ matrixes. In NC Geometry $\mathfrak{q}\mathcal{M}_n$ is usually considered as the finite NC space with $n$ points. Since $\mathcal{M}_n = \mathcal{M}^{**}$ any projection in $\mathcal{M}_n$ is closed and open ([1, Proposition II.18]), and hence $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}\mathcal{M}_n$ as a set is canonically identified with $\cup_{i=1}^n \mathrm{Gr}(i, n)$ where $\mathrm{Gr}(i, n)$ denote the Grassmannian manifold of $i$-dimensional subspaces of $\mathbb{C}^n$.

QUESTION 7.3. *Is the subspace of $\mathfrak{S}_{\mathrm{cl}}\mathfrak{q}\mathcal{M}_n$ containing projections of rank $i$ homeomorphic to $\mathrm{Gr}(i, n)$?*

Let $X$ be a compact Hausdorff space and $C$ be a $C^*$-subalgebra of $\mathcal{C}X$ containing $1_X$. Let $Z$ denote the pure state space of $C$ with weak*-topology. We have a canonical continuous surjective map $\Gamma : X \to Z$ defined by $\Gamma(x)(c) = c(x)$ $(c \in C)$. It is easily checked that the topology of $Z$ is the quotient topology under $\Gamma$. Also $\Gamma$ induces the family $\{K_z\}_{z \in Z}$ of nonempty disjoint closed subsets of $X$ parameterized by $Z$ where $K_z := \Gamma^{-1}(z)$. A generalization of this notion is as follows.

DEFINITION 7.4. Let $A$ be a unital $C^*$-algebra and $C$ be a $C^*$-subalgebra of $A$ containing the unit. Let $Z$ denote the pure state space of $C$. For every $z \in Z$ let $\mathcal{F}_z := \{\mu \in \mathcal{S}A : \mu(c) = z(c), c \in C\}$. Then $\mathcal{F}_z$ is a weak*-closed face of $\mathcal{S}A$. We say that $\{\mathcal{F}_z\}_{z \in Z}$ is the family of closed subsets of $\mathfrak{q}A$ parameterized by $\mathfrak{q}C$.

Let $0 \leq \theta < 1$. The quantum torus $\mathbb{T}_\theta^2 := \mathfrak{q}\mathcal{C}\mathbb{T}_\theta^2$ is the NC space associated to the universal $C^*$-algebra $\mathcal{C}\mathbb{T}_\theta^2$ generated by two unitary elements $u, v$ satisfying $uv = e^{2\pi i\theta}vu$. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. We identify the $C^*$-subalgebra generated by $v$ with $\mathcal{C}\mathbb{T}$ via the *-isomorphism given by the assignment $v \mapsto \mathrm{id}_\mathbb{T}$ where $\mathrm{id}_\mathbb{T} \in \mathcal{C}\mathbb{T}$ is the identity function. For every $z \in \mathbb{T}$ let $\mathbb{T}_{\theta,z} := \{\mu \in \mathcal{S}\mathcal{C}\mathbb{T}_\theta^2 : \mu(f) = f(z), f \in \mathcal{C}\mathbb{T}\}$. Then we call $\{\mathbb{T}_{\theta,z}\}_{z \in \mathbb{T}}$ the family of $v$-sub-circles in $\mathbb{T}_\theta^2$. The name is justified as follows. It is clear that $\mathcal{C}\mathbb{T}_0^2$ can be identified with $\mathcal{C}\mathbb{T}^2$ via the *-isomorphism given by the assignments $u \mapsto \mathrm{id}_1, v \mapsto \mathrm{id}_2$ where $\mathrm{id}_1, \mathrm{id}_2 \in \mathcal{C}\mathbb{T}^2$ are respectively the projection functions on the first and second components of $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. Then $\mathbb{T}_{0,z}$ is identified

with the set of Borel probability measures $\mu$ on $\mathbb{T}^2$ such that the support of $\mu$ is contained in the sub-circle $\{(w, z) \in \mathbb{T}^2 : w \in \mathbb{T}\}$.

It is not hard to see that the map $z \mapsto \{(w, z) \in \mathbb{T}^2 : w \in \mathbb{T}\}$ from $\mathbb{T}$ into $\mathfrak{S}_{\mathrm{cl}}\mathbb{T}^2$ is continuous with Vietoris topology. So it is natural to ask the following questions.

QUESTION 7.5. *Is the map $z \mapsto \mathbb{T}_{\theta,z}$ ($\theta \neq 0$) from $\mathbb{T}$ into $\mathfrak{S}_{\mathrm{cl}}\mathbb{T}_\theta^2$ continuous? Is the family of $v$-sub-circles in $\mathbb{T}_\theta^2$ compact or (path) connected?*

If we have a (Riemannian) metric on $\mathbb{T}^2$ we can ask about the Hausdorff and infimum distances of sub-circles. Analogously we have the following problem.

PROBLEM 7.6. *Consider $\mathbb{T}_\theta^2$ as a $C^*$-algebraic quantum metric space described in [24, 25] and find the Hausdorff and infimum distances between two arbitrary sub-circles $\mathbb{T}_{\theta,z}$ and $\mathbb{T}_{\theta,z'}$.*

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