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ON THE CONCEPT OF φ-ENTROPY

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Abstract. In this paper, the concept of φ-entropy is defined and some of the its properties are proved. It is a type of generalized entropy with generalized properties. It is invariant under topological conjugacy and satisfies a generalized version of Jaccob's Theorem. Finally, we will extract the Kolmogorov entropy as a special case, by setting φ to be the identity function.

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1. INTRODUCTION

The Rudolf Clausius's study of the Carnot cycle [13] led to the concept of entropy. The thermodynamic definition was developed in 1850s and describes how to measure the entropy of an isolated system in thermodynamic equilibrium. We mention that an extensive thermodynamic variable is useful in characterizing the Carnot cycle.

A statistical equivalent definition of entropy was developed by Ludwig Boltzmann in 1870s. It is interpreted in statistical mechanics as the measure of uncertainty or the degree to which the probability of the system is spread out over different possible microstates. Mathematically, the entropy is the expected value of the logarithm of the probability that a microstate will be occupied, i.e. $S = -k_B \sum_i p_i \log p_i$, where k_B is the Boltzmann constant and the summation is over all possible microstates of the system and p_i is the probability that the system is in the i-th microstate.

A similar formulation was considered by Shannon in order to introduce the concept of entropy in information theory [23]. Using this idea, Kolmogorov [12] introduced the concept of entropy in ergodic theory. The definition of entropy was improved by Sinai in 1959 [27]. The definition was given as

$$h_\mu(T) = \sup_\xi \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\xi),$$

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where the supremum is taken over all finite partitions ξ of X, and $\bigvee_{i=0}^{n-1} T^{-i}\xi$ is the partition generated by events of n successive observations and

$$H(\bigvee_{i=0}^{n-1} T^{-i}\xi) = -\sum_{A\in\bigvee_{i=0}^{n-1} T^{-i}\xi} \mu(A)\log\mu(A).$$

Adler, Konheim, and McAndrew [1] introduced the topological entropy as an invariant of the topological conjugacy and also as an analogue of the measure theoretic entropy. The definition of the topological entropy for continuous maps on compact spaces, using open covers of the phase space, is given by

$$h_{top}(T) = \sup_{\alpha} \lim_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\alpha),$$

where the supremum is taken over all open covers α of X and $N(\alpha)$ is the number of sets in a finite subcover of α with the smallest cardinality.

Later, Dinaburg [7] and Bowen [4] gave a new, but equivalent, definition of the topological entropy that led to the variational principle which connected the topological entropy and the measure theoretic entropy as follows:

$$h_{top}(T) = \sup_{\mu\in M(X,T)} h_{\mu}(T).$$

Since then, the concepts of entropy and information have been studied extensively from two main points of view, namely: generalization and localization.

The generalized forms of Shanon entropy have been extensively discussed, see [2, 3, 8, 9, 10, 11, 19, 20, 22, 24, 25, 26, 28]. They usually have similar properties to the Shannon entropy and they are given as a special case of the last. In [17], the topological entropy of a continuous dynamical system is generalized, in the sense that, it is considered as a linear operator, called entropy operator, with the norm equal to the topological entropy. A local study of the entropy operator leads to the functional entropy [18], which generalizes the Kolmogorov entropy.

On the other hand, the localization of the entropy of a dynamical system has been studied by many authors. Shannon [23], McMillan [14] and Breiman [5] considered local approaches to the entropy, based on the Theorem of Shannon-McMillan-Breiman. Another interesting topological version of the theorem of Shannon-McMillan-Breiman was given by Brin and Katok [6]. A delicate local approach of entropy for smooth dynamics was considered by Ruelle [21] and Pesin [15]. Later on, different versions of the entropy of a dynamical system were defined.

The generalization of the concept of entropy, via a local approach, is considered in this paper. We define an entropy type quantity via a convex function $\varphi : [0,\infty) \to [0,\infty)$. It generalizes the Kolmogorov entropy, in the sense that, it has similar properties to the Kolmogorov entropy and covers the Kolmogorov entropy as a special case. It is also a local entity.

2. BACKGROUNDS

In this section, we state some known results that will be used in the article. Let $T : X \to X$ be a continuous map on the compact metric space X. $M(X)$ denotes the set of all probability measures on the Borel sets of X, equipped with the weak* topology. The set of all probability measures on X preserving T is denoted by $M(X, T)$. We also write $E(X, T)$ for the set of all ergodic measures of T. In the following, we give a list of known theorems that will be used in our discussion.

We start with the following theorem that takes the supremum in the definition of the Kolmogorov entropy over a countable family of partitions.

THEOREM 2.1 ([28, Theorem 8.3]). *Let $T : X \to X$ be a continuous map on the compact metric space X. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X such that $diam(\xi_n) \to 0$, as $n \to \infty$. For every $\mu \in M(X, T)$, we have $h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \xi_n)$.*

In the following theorem, we summarize some facts that we need later on. One may find them in [29].

THEOREM 2.2. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X. Then*

(i) *$M(X, T)$ is a compact subset of $M(X)$;*
(ii) *$M(X, T)$ is a convex set;*
(iii) *$E(X, T)$ is the set of all extreme points of $M(X, T)$.*

THEOREM 2.3 (Choquet). *Suppose that Y is a compact convex metrizable subset of a locally convex space E and $x_0 \in Y$. Then there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y, i.e. $\Phi(x_0) = \int_Y \Phi \mathrm{d}\tau$, for every continuous linear functional Φ on E, and $\tau(ext(Y)) = 1$.*

See Phelps [16] for a proof of Choquet's Theorem. Combining Theorems 2.2 and 2.3, we have the following corollary.

COROLLARY 2.4. *Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then, for each $\mu \in M(X, T)$, there is a unique measure τ on the Borel subsets of the compact metrizable space $M(X, T)$ such that $\tau(E(X, T)) = 1$ and*

$$\int_X f(x)\mathrm{d}\mu(x) = \int_{E(X,T)} \left(\int_X f(x)\mathrm{d}m(x) \right) \mathrm{d}\tau(m),$$

for every bounded measurable function $f : X \to \mathbb{R}$.

If μ and τ are as in Corollary 2.4, then we write $\mu = \int_{E(X,T)} m\mathrm{d}\tau(m)$ and we call this the ergodic decomposition of μ.

THEOREM 2.5 (Jacob). *Let $T : X \to X$ be a continuous map on a compact metrizable space. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m\mathrm{d}\tau(m)$ is the ergodic decomposition of μ, then we have:*

(i) *If ξ is a finite Borel partition of X, then*

$$h_\mu(T,\xi) = \int_{E(X,T)} h_m(T,\xi)\mathrm{d}\tau(m).$$

(ii) $h_\mu(T) = \int_{E(X,T)} h_m(T)\mathrm{d}\tau(m)$ *(both sides could be ∞).*

See [29, Theorem 8.4] for a proof of Jacob's Theorem.

3. φ−ENTROPY

In this section, we introduce the concept of φ−entropy and will state some of its properties. We will finally extract the entropy of a system as a special case. In the following, $\varphi : [0,\infty) \to [0,\infty)$ is an injective convex function.

DEFINITION 3.1. Suppose that $T : X \to X$ is a continuous map on a compact metric space X, $x \in X$ and A is a Borel subset of X. Define

$$\omega_T(x,A) := \limsup_{n\to\infty} \frac{1}{n}\mathrm{card}(\{k \in \{0,1,...,n-1\} : \ T^k(x) \in A\}). \tag{1}$$

Now, let $x \in X$ and $\xi = \{A_1, A_2, ..., A_n\}$ be a finite Borel partition of X. Define

$$\Omega_T(x,\xi) := -\sum_{j=1}^{n} \omega_T(x,A_j)\log\omega_T(x,A_j). \tag{2}$$

(We assume that $\log 0 = -\infty$ and $0 \times \infty = 0$.) Finally, let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X such that $\mathrm{diam}(\xi_n) \to 0$ as $n \to \infty$. We may assume that $\xi_n < \xi_{n+1}$, since otherwise we may replace ξ_n by $\eta_n := \bigvee_{k=0}^{n} \xi_k$.

The map $\mathcal{J}_T(\cdot;\mathfrak{U}) : X \to [0,\infty]$ is defined as follows:

$$\mathcal{J}_T(x;\mathfrak{U}) = \lim_{n\to\infty} \limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n). \tag{3}$$

Note that the sequence $a_n(x) = \limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n)$ is increasing with respect to n and so $\lim_{n\to\infty} a_n(x)$ exists as a nonnegative extended real number. We also write $\mathcal{J}_T$ for $\mathcal{J}_T(\cdot;\mathfrak{U})$, when there is no confusion.

DEFINITION 3.2. Suppose that $T : X \to X$ is a continuous map on a compact metric space X, $\mu \in M(X,T)$ and $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of finite Borel partitions of X such that $\mathrm{diam}(\xi_n) \to 0$. Let $\varphi : [0,\infty) \to [0,\infty)$ be an injective convex function. The φ−entropy of T is defined as follows:

$$\Gamma_\varphi(T;\mu;\mathfrak{U}) := \varphi^{-1}\left(\int_X \varphi \circ \mathcal{J}_T(x;\mathfrak{U})\mathrm{d}\mu(x)\right).$$

The following theorem shows that the definition of the $\varphi-$entropy map is independent of the selection of the sequence $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$.

THEOREM 3.3. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X. Let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ and $\mathcal{V} = \{\eta_n\}_{n\in\mathbb{N}}$ be two sequences of finite Borel partitions of X such that diam$(\xi_n) \to 0$ and diam$(\eta_n) \to 0$ and let $\mu \in M(X,T)$. Then*

$$\Gamma_\varphi(T;\mu;\mathfrak{U}) = \Gamma_\varphi(T;\mu;\mathcal{V}).$$

Proof. First let $\mu \in E(X,T)$. For any Borel set $A \subset X$, applying Birkhoff ergodic Theorem we have $\omega_T(x,A) = \mu(A)$ for almost all $x \in X$. Hence, if ξ is a finite Borel partition of X, then $\Omega_T(x,\xi) = H_\mu(\xi)$ for almost all $x \in X$. Thus, for every $n \in \mathbb{N}$,

$$\limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n) = h_\mu(T,\xi_n)$$

for almost all $x \in X$. Hence, for $n \geq 1$ there exists a Borel set Y_n such that $\mu(Y_n) = 1$ and $\limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n) = h_\mu(T,\xi_n)$, for all $x \in Y_n$. Put $Y = \bigcap_{n=1}^\infty Y_n$. Then $\mu(Y) = 1$ and, for $x \in Y$, we have

$$\limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n) = h_\mu(T,\xi_n),$$

for all $n \geq 1$. Thus, using Theorem 2.1, we have

$$\mathcal{J}_T(x;\mathfrak{U}) = \lim_{n\to\infty} \limsup_{m\to\infty} \frac{1}{m}\Omega_T(x, \bigvee_{i=0}^{m-1} T^{-i}\xi_n) = \lim_{n\to\infty} h_\mu(T,\xi_n) = h_\mu(T)$$

for all $x \in Y$. By a similar method, one can find a Borel set Z with $\mu(Z) = 1$ such that

$$\mathcal{J}_T(x;\mathcal{V}) = h_\mu(T),$$

for all $x \in Z$. Put $X_0 := Y \cap Z$. Then $\mu(X_0) = 1$ and

$$\mathcal{J}_T(x;\mathfrak{U}) = \mathcal{J}_T(x;\mathcal{V}) = h_\mu(T),$$

for all $x \in X_0$. Therefore

$$\begin{aligned}\int_X \varphi\circ\mathcal{J}_T(x;\mathfrak{U})\mathrm{d}\mu(x) &= \int_{X_0} \varphi\circ\mathcal{J}_T(x;\mathfrak{U})\mathrm{d}\mu(x)\\ &= \int_{X_0} \varphi\circ\mathcal{J}_T(x;\mathcal{V})\mathrm{d}\mu(x)\\ &= \int_X \varphi\circ\mathcal{J}_T(x;\mathcal{V})\mathrm{d}\mu(x).\end{aligned} \tag{4}$$

The relation (4) holds for all $\mu \in E(X,T)$.

For $\mu \in M(X,T)$ let, in view of Corollary 2.4, $\mu = \int_{E(X,T)} m\mathrm{d}\tau(m)$ be the ergodic decomposition of μ.

For $n \geq 1$ put $f_n := \min\{\varphi \circ \mathcal{J}_T(\cdot; \mathfrak{U}), n\}$ and $g_n := \min\{\varphi \circ \mathcal{J}_T(\cdot; \mathcal{V}), n\}$. Then:

(1) f_n and g_n are bounded, for all $n \geq 1$.
(2) The sequences $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ are increasing.
(3) $f_n \nearrow \varphi \circ \mathcal{J}_T(\cdot; \mathfrak{U})$ and $g_n \nearrow \varphi \circ \mathcal{J}_T(\cdot; \mathcal{V})$ on X.
(4) The sequences $\{\phi_n\}_{n\geq 1}$ and $\{\psi_n\}_{n\geq 1}$ given by $\phi_n(m) := \int_X f_n \mathrm{d}m(x)$ and $\psi_n(m) := \int_X g_n \mathrm{d}m(x)$, respectively, are increasing.

Now, using Corollary 2.4, the Monotone Convergence Theorem and the equality in (4), we have

$$\begin{aligned}
\Gamma_\varphi(T; \mu; \mathfrak{U}) &= \varphi^{-1}\left(\int_X \varphi \circ \mathcal{J}_T(x; \mathfrak{U}) \mathrm{d}\mu(x)\right) \\
&= \varphi^{-1}\left(\int_X \lim_{n\to\infty} f_n(x) \mathrm{d}\mu(x)\right) \\
&= \varphi^{-1}\left(\lim_{n\to\infty} \int_X f_n(x) \mathrm{d}\mu(x)\right) \\
&= \varphi^{-1}\left(\lim_{n\to\infty} \int_{E(X,T)} \left(\int_X f_n(x) dm(x)\right) \mathrm{d}\tau(m)\right) \\
&= \varphi^{-1}\left(\int_{E(X,T)} \left(\int_X \varphi \circ \mathcal{J}_T(x; \mathfrak{U}) \mathrm{d}m(x)\right) d\tau(m)\right) \\
&= \varphi^{-1}\left(\int_{E(X,T)} \left(\int_X \varphi \circ \mathcal{J}_T(x; \mathcal{V}) dm(x)\right) \mathrm{d}\tau(m)\right) \\
&= \varphi^{-1}\left(\int_{E(X,T)} \left(\int_X \lim_{n\to\infty} g_n(x) dm(x)\right) \mathrm{d}\tau(m)\right) \\
&= \varphi^{-1}\left(\lim_{n\to\infty} \int_{E(X,T)} \left(\int_X g_n(x) \mathrm{d}m(x)\right) \mathrm{d}\tau(m)\right) \\
&= \varphi^{-1}\left(\lim_{n\to\infty} \int_X g_n(x) \mathrm{d}\mu(x)\right) \\
&= \varphi^{-1}\left(\int_X \varphi \circ \mathcal{J}_T(x; \mathcal{V}) \mathrm{d}\mu(x)\right) \\
&= \Gamma_\varphi(T; \mu; \mathcal{V}).
\end{aligned}$$

The proof is complete. □

REMARK 3.4. By Theorem 3.6, we conclude that the definition of $\varphi-$entropy is independent of the selection of the sequence of finite Borel partitions. Therefore, given any sequence of finite Borel partitions $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ with $\mathrm{diam}(\xi_n) \to 0$ we have the (unique) $\varphi-$entropy $\Gamma_\varphi(T; \mu)$. So we can write $\Gamma_\varphi(T; \mu)$ for $\Gamma_\varphi(T; \mu; \mathfrak{U})$ with no confusion.

THEOREM 3.5. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X, $\mu \in M(X,T)$, and $\varphi : [0,\infty) \to [0,\infty)$ is an injective convex function. Let $\mu = \int_{E(X,T)} m \mathrm{d}\tau(m)$ be the ergodic decomposition of μ. Then:*

(i) *If φ is increasing, then*

$$\Gamma_\varphi(T;\mu) \geq \int_{E(X,T)} \Gamma_\varphi(T;m)\mathrm{d}\tau(m).$$

(ii) *If φ is decreasing, then*

$$\Gamma_\varphi(T;\mu) \leq \int_{E(X,T)} \Gamma_\varphi(T;m)\mathrm{d}\tau(m).$$

Proof. Let $\mathcal{J}_T$ be defined as in (3). For $n \geq 1$, let $f_n := \min\{\varphi \circ \mathcal{J}_T, n\}$. Then $\{f_n\}_{n\geq 1}$ is an increasing sequence of bounded measurable maps such that $f_n \nearrow \varphi \circ \mathcal{J}_T$. Applying Corollary 2.4, the Monotone Convergence Theorem and the Jensen inequality, we get

$$\begin{aligned}
\varphi\left(\Gamma_\varphi(T;\mu)\right) &= \int_X \varphi \circ \mathcal{J}_T(x)\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty} \int_X f_n(x)\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty} \int_{E(X,T)} \left(\int_X f_n(x)\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \int_{E(X,T)} \left(\int_X \varphi \circ \mathcal{J}_T(x)\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \int_{E(X,T)} \varphi\left(\Gamma_\varphi(T;m)\right)\mathrm{d}\tau(m) \\
&\geq \varphi\left(\int_{E(X,T)} \Gamma_\varphi(T;m)\mathrm{d}\tau(m)\right).
\end{aligned} \tag{5}$$

Finally, (i) and (ii) follow from (5). □

THEOREM 3.6. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X, $\mu_1, ..., \mu_n \in M(X,T)$ and $\varphi : [0,\infty) \to [0,\infty)$ is an injective convex function. Let $\lambda_1, ..., \lambda_n \in [0,1]$ be such that $\sum_{i=1}^n \lambda_i = 1$. Then:*

(i) *If φ is increasing, then*

$$\Gamma_\varphi(T;\sum_{i=1}^n \lambda_i\mu_i) \geq \lambda_i\Gamma_\varphi(T;\mu_i).$$

(ii) *If φ is decreasing, then*

$$\Gamma_\varphi(T;\sum_{i=1}^n \lambda_i\mu_i) \leq \lambda_i\Gamma_\varphi(T;\mu_i).$$

Proof. Since φ is convex, then

$$\begin{aligned}\varphi\left(\Gamma_\varphi(T;\sum_{i=1}^n\lambda_i\mu_i)\right)&=\int_X\varphi\circ\mathcal{J}_T\mathrm{d}(\sum_{i=1}^n\lambda_i\mu_i)\\&=\sum_{i=1}^n\lambda_i\int_X\varphi\circ\mathcal{J}_T\mathrm{d}\mu_i\\&=\sum_{i=1}^n\lambda_i\varphi\left(\Gamma_\varphi(T;\mu_i)\right)\\&\geq\varphi\left(\sum_{i=1}^n\lambda_i\Gamma_\varphi(T;\mu_i)\right).\end{aligned}\tag{6}$$

Finally, (i) and (ii) follow from (6). □

We recall that two continuous maps $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ are said to be *topologically conjugate*, if there is a homeomorphism $h : X_1 \to X_2$ such that $h \circ T_1 = T_2 \circ h$.

The following theorem shows the invariance of $\varphi-$entropy under the topological conjugacy.

THEOREM 3.7. *Suppose that $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ are topologically conjugate continuous maps via the homeomorphism $h : X_1 \to X_2$. Then*

$$\Gamma_\varphi(T_1;\mu)=\Gamma_\varphi(T_2;\mu h^{-1}),$$

for all $\mu \in M(X_1, T_1)$.

Proof. For $x \in X$ and for the Borel set $A \subseteq X_1$, we have $\omega_{T_1}(x, A) = \omega_{T_2}(h(x), h(A))$. Therefore, $\Omega_{T_1}(x,\xi) = \Omega_{T_2}(h(x), h(\xi))$, for any finite Borel partition ξ. Now, for any sequence $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ of finite Borel partitions of X with $\mathrm{diam}(\xi_n) \to 0$, by the definitions of $\mathcal{J}_{T_1}(\cdot;\mathfrak{U})$ and $\mathcal{J}_{T_2}(\cdot;h(\mathfrak{U}))$, we have $\mathcal{J}_{T_1}(\cdot;\mathfrak{U}) = \mathcal{J}_{T_2}(\cdot;h(\mathfrak{U}))\circ h$. Note that $h(\mathfrak{U}) = \{h(\xi_n)\}_{n\in\mathbb{N}}$ and $\mathrm{diam}(h(\xi_n)) \to 0$. Let $\mu \in M(X_1, T_1)$. Then

$$\begin{aligned}\Gamma_\varphi(T_1;\mu)&=\varphi^{-1}\left(\int_{X_1}\varphi\circ\mathcal{J}_{T_1}(x;\mathfrak{U})\mathrm{d}\mu(x)\right)\\&=\varphi^{-1}\left(\int_{X_1}\varphi\circ\mathcal{J}_{T_2}(h(x);h(\mathfrak{U}))\mathrm{d}\mu(x)\right)\\&=\varphi^{-1}\left(\int_{X_2}\varphi\circ\mathcal{J}_{T_2}(x;h(\mathfrak{U}))\mathrm{d}\mu h^{-1}(x)\right)\\&=\Gamma_\varphi(T_2;\mu h^{-1}).\end{aligned}$$

□

4. p-ENTROPY AS A SPECIAL CASE

In this section, we consider the φ-entropy for a special case $\varphi(t) = \varphi_p(t) = t^p$, for $p \geq 1$. In this case, we write $\Gamma_p(T;\mu)$, for $\Gamma_{\varphi_p}(T;\mu)$ and call it the p-entropy of T with respect to μ. Indeed, we have

$$\Gamma_p(T;\mu) = \left(\int_X \mathcal{J}_T^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

The special case $p = 1$ is of great importance.

THEOREM 4.1. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X and let $\mu \in M(X,T)$. Then*

$$\Gamma_1(T;\mu) = h_\mu(T),$$

where $h_\mu(T)$ is the Kolmogorov entropy of T.

Proof. Let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X such that $\mathrm{diam}(\xi_n) \to 0$ and $\mathcal{J}_T = \mathcal{J}_T(\cdot;\mathfrak{U})$. Let $m \in E(X,T)$. As in the proof of Theorem 3.3, we have $\mathcal{J}_T(x) = h_m(T)$, for almost all $x \in X$. Therefore

$$\Gamma_1(T;m) = \int_X \mathcal{J}_T(x)\mathrm{d}m(x) = h_m(T).$$

Now, let $\mu \in M(X,T)$. As in Theorem 3.3, let, in view of Corollary 2.4, $\mu = \int_{E(X,T)} m d\tau(m)$ be the ergodic decomposition of μ. For $n \in \mathbb{N}$, let $g_n := \min\{\mathcal{J}_T, n\}$. Then:

(1) g_n is bounded, for all $n \geq 1$;
(2) the sequence $\{g_n\}_{n\geq 1}$ is increasing;
(3) $g_n \to \mathcal{J}_T$ on X;
(4) the sequence $\{\psi_n\}_{n\geq 1}$ given by $\psi_n(m) := \int_X g_n(x)dm(x)$ is increasing.

Now, using Corollary 2.4, Theorem 2.5 (ii) and the Monotone Convergence Theorem, we have

$$\begin{aligned}
\Gamma_1(T;\mu) &= \int_X \mathcal{J}_T(x)\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty} \int_X g_n(x)\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty} \int_{E(X,T)} \left(\int_X g_n(x)\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \int_{E(X,T)} \left(\lim_{n\to\infty} \int_X g_n(x)\mathrm{d}m(x)\right)\mathrm{d}\tau(m)
\end{aligned}$$

$$\begin{aligned} &= \int_{E(X,T)} \left(\int_X \mathcal{J}_T(x) \mathrm{d}m(x) \right) \mathrm{d}\tau(m) \\ &= \int_{E(X,T)} h_m(T) \mathrm{d}\tau(m) \\ &= h_\mu(T). \end{aligned}$$

□

5. CONCLUDING REMARKS

In this paper, a generalized form of entropy of a dynamical system is presented. This quantity depends on a convex map $\varphi : [0,\infty) \to [0,\infty)$ and is called the φ-entropy. Theorem 3.5 is, somehow, a generalized version of Jacob's Theorem. Theorem 3.7 states that the φ-entropy is invariant under the topological conjugacy. By Theorem 4.1, the φ-entropy is equal to the Kolmogorov entropy, for $\varphi = id_X$. In this case, a local approach to the Kolmogorov is presented, in the sense that, integrating the map $\mathcal{J}_T : X \to [0,\infty]$, the result is the Kolmogorov entropy.

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