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ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY r-PARACOSYMPLECTIC MANIFOLD

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Abstract. The purpose of the present paper is to study semi-invariant submanifolds of a nearly r-paracosymplectic manifold. We also investigate totally r-paracontact umbilical semi-invariant submanifolds of a nearly r-paracosymplectic manifold. Moreover, we construct an example of a nearly r-paracosymplectic metric manifold which is not r-paracosymplectic.

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1. INTRODUCTION

The geometry of semi-invariant submanifolds of Sasakian manifolds has been studied by Bejancu and Papaghuic [3, 4]. Later on, these submanifold have been studied by several authors, for example: Das et al. [7], Ateken [2], Calin et al.[6]. The study of cosymplectic manifold has been conducted by several authors [5, 10, 12]. In 2005, Endo has studied the curvature tensor of a nearly cosymplectic manifold of constant ϕ-sectional curvature [8]. Recently, the geometry of PR-semi-invariant warped product submanifolds in paracosymplectic manifolds has been studied by Srivastava and Sharma [11].

On the other hand, almost r-paracontact Riemannian manifolds and almost product Riemannian manifolds have been studied by Adati [1]. The study of an almost r-paracontact motivates us to study nearly r-paracosymplectic manifolds.

The paper is organized as follows. In Section 2, we give a brief introduction of semi-invariant submanifolds of a nearly r-paracosymplectic manifold. In Section 3, some propositions on a nearly r-paracosymplectic manifold are given. In section 4, totally umbilical and totally geodesic submanifolds are discussed. In the last section, an example of a nearly r-paracosymplectic manifold is given.

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2. PRELIMINARIES

Let $\bar{M}$ be an $2m+r$-dimensional almost r-paracontact manifold [1] with metric tensor g, a tensor field ϕ of type (1, 1), vector fields ξ_p and 1-forms η^p which satisfy

$$\phi^2 = I - \sum_{p=1}^{r} \eta^p \otimes \xi_p, \tag{1}$$

$$\phi\xi_p = 0,\ \eta^p o\phi = 0,\ \eta^p\xi_q = \delta^p_q, \tag{2}$$

where $p, q = 1, \ldots, r$ and δ^p_q denotes the Kronecker delta. Let $\bar{M}$ be also endowed with a Riemannian metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \sum_{p=1}^{r} \eta^p(X)\eta^p(Y) \tag{3}$$

and

$$g(\xi_p, X) = \eta^p(X). \tag{4}$$

Then we say that, in view of the equations (1)–(4), the manifold $\bar{M}$ admits an almost r-paracontact Riemannian structure. The almost r-paracontact manifold $\bar{M}$ is called nearly r-paracosymplectic manifold if ϕ is killing, i.e.

$$(\widetilde{\nabla}_X\phi)(Y) + (\widetilde{\nabla}_Y\phi)(X) = 0, \tag{5}$$

for any vector fields X and Y on $\bar{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection for the metric tensor g on $\bar{M}$ [12]. On such a nearly r-paracosymplectic manifold vector fields ξ_p are killing, i.e.

$$g(\widetilde{\nabla}_X\xi_p, Y) + g(X, \widetilde{\nabla}_Y\xi_p) = 0. \tag{6}$$

Definition 2.1. An n-dimensional Riemannian submanifold M of a nearly r-paracosymplectic manifold $\bar{M}$ is called a semi-invariant submanifold, if ξ_p are tangents to M and there exists, on M, a pair of orthogonal distributions $(D, D^{\perp})$ such that

(i) $TM = \{D\} \oplus \{D^{\perp}\} \oplus \{\xi_p\}$;
(ii) the distribution D is invariant under ϕ, i.e. $\phi D_x = D_x$, for all $x \in M$;
(iii) the distribution $D^{\perp}$ is anti-invariant under ϕ,

i.e. $\phi(D^{\perp}_x) \subset T^{\perp}_x(M)$, for all $x \in M$, where $T_x(M)$ and $T^{\perp}_x(M)$ are the tangent space and the normal space of M at $x \in M$.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical distribution). A semi-invariant submanifold M is said to be invariant (resp. anti-invariant) submanifold, if we have $D^{\perp}_x = \{0\}$ (resp. $D_x = \{0\}$), for each $x \in M$. We also call M proper, if neither D nor $D^{\perp}$ is null.

Let $\widetilde{\nabla}$ (rep. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\bar{M}$ (resp. M). The Gauss and Weingarten formulas for M are, respectively, given by

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}_X Y + h(X,Y) \tag{7}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{8}$$

for $X \in TM$, $V \in T^{\perp}M$, where h (resp. A) is the second fundamental form (resp. tensor) of M is $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection.

Moreover, we have

$$g(A_V X, Y) = g(h(X,Y), V) \tag{9}$$

For a vector field tangent to M, we put

$$X = PX + QX + \sum_{p=1}^{r} \eta^p(X)\xi_p, \tag{10}$$

where PX and QX belong to the distribution D and $D^{\perp}$, respectively (see [3]). For a vector field V normal to M, we put

$$\phi V = BV + CV, \tag{11}$$

where BV (resp. CV) belong to the tangential (resp. normal) component of ϕV.

3. SOME RESULTS

In this section, we shall establish some propositions on a semi-invariant submanifold M of a nearly r-paracosymplectic manifold $\bar{M}$.

PROPOSITION 3.1. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$2(\widetilde{\nabla}_X \phi)(Y) = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y], \tag{12}$$

for all $X, Y \in D$.

Proof. Let $\widetilde{\nabla}$ be a Riemannian connection of the enveloping manifold $\bar{M}$. We have

$$(\widetilde{\nabla}_X \phi)(Y) = \widetilde{\nabla}_X(\phi Y) - \phi\widetilde{\nabla}_X Y. \tag{13}$$

Making use of the Gauss formula (7), the above equation (13) takes the form

$$(\widetilde{\nabla}_X \phi)(Y) = \nabla_X(\phi Y) + h(X, \phi Y) - \phi\widetilde{\nabla}_X Y. \tag{14}$$

Interchanging X, Y in the above equation (14), we get

$$(\widetilde{\nabla}_Y \phi)(X) = \nabla_Y(\phi X) + h(Y, \phi X) - \phi\widetilde{\nabla}_Y X. \tag{15}$$

Since the structure tensor ϕ is killing, in view of the equation (5), the above equation (15) takes the form

$$-(\widetilde{\nabla}_X \phi)(Y) = \nabla_Y(\phi X) + h(Y, \phi X) - \phi \widetilde{\nabla}_Y X. \tag{16}$$

Subtracting the equation (16) from (14) yields the required result. □

PROPOSITION 3.2. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$2(\widetilde{\nabla}_X \phi)(Y) = A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_X(\phi Y) - \nabla^{\perp}_Y(\phi X) - \phi[X, Y],$$

for all $X, Y \in D^{\perp}$.

Proof. We have from the equation (13)

$$(\widetilde{\nabla}_X \phi)(Y) = \widetilde{\nabla}_X(\phi Y) - \phi \widetilde{\nabla}_X Y. \tag{17}$$

In view of the equation (8), the above equation takes the form

$$(\widetilde{\nabla}_X \phi)(Y) = -A_{\phi Y} X + \nabla^{\perp}_X(\phi Y) - \phi \widetilde{\nabla}_X Y. \tag{18}$$

Interchanging X and Y in the above equation and using the fact that ϕ is killing, we obtain

$$-(\widetilde{\nabla}_X \phi)(Y) = -A_{\phi X} Y + \nabla^{\perp}_Y(\phi X) - \phi \widetilde{\nabla}_Y X. \tag{19}$$

Subtracting (19) from (18) and using the fact that $\widetilde{\nabla}$ is a Riemannian connection on $\bar{M}$, we get the required result. □

PROPOSITION 3.3. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$2(\widetilde{\nabla}_X \phi)(Y) = \nabla^{\perp}_X(\phi Y) - \nabla^{\perp}_Y(\phi X) - A_{\phi Y} X - h(X, Y) + \phi[X, Y] \tag{20}$$

for all $X \in D, Y \in D^{\perp}$.

Proof. By virtue of equations (16) and (18), the above proposition follows in a straightforward manner. □

PROPOSITION 3.4. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$2(\widetilde{\nabla}_X \phi)(\xi_p) = \phi[\xi_p, X] - \nabla_{\xi_p}(\phi X) - h(\phi X, \xi_p), \tag{21}$$

where $p = 1, \ldots, r$ and $X \in D$.

Proof. We can write

$$\widetilde{\nabla}_X \phi)(\xi_p) = \widetilde{\nabla}_X(\phi \xi_p) - \phi \widetilde{\nabla}_X \xi_p. \tag{22}$$

By virtue of the equation (2), the above equation takes the form

$$(\widetilde{\nabla}_X \phi)\xi_p = -\phi \widetilde{\nabla}_X \xi_p \tag{23}$$

$$-(\widetilde{\nabla}_X \phi)(\xi_p) = -\left\{\widetilde{\nabla}_{\xi_p}(\phi X) - \phi \widetilde{\nabla}_{\xi_p} X\right\}$$

$$(24) \qquad -(\widetilde{\nabla}_{\xi_p}\phi)(X) = -\left\{\widetilde{\nabla}_{\xi_p}(\phi X) + h(\phi X, \xi_p)\right\} - \phi\widetilde{\nabla}_{\xi_p}X.$$

In view of the equation (5), we can write the above equation in the form

$$(25) \qquad -(\widetilde{\nabla}_X\phi)(\xi_p) = -\left\{\widetilde{\nabla}_{\xi_p}(\phi X) + h(\phi X, \xi_p)\right\} - \phi\widetilde{\nabla}_{\xi_p}X.$$

Summing (23) and (25) yields the required result. □

Proposition 3.5. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$(26) \qquad 2(\widetilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}(\xi_p) + \phi\widetilde{\nabla}_{\xi_p}X - \phi\widetilde{\nabla}_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X),$$

for any $X \in D^{\perp}$.

Proof. From the equation (23), we have also

$$-(\widetilde{\nabla}_{\xi_p}\phi)(X) = -\left\{\widetilde{\nabla}_{\xi_p}(\phi X) - \phi\widetilde{\nabla}_{\xi_p}X\right\}$$

$$= -\left\{-A_{\phi X}\xi_p - \nabla^{\perp}_{\xi_p}(\phi X)\right\} + \phi\widetilde{\nabla}_{\xi_p}X.$$

Since the structure tensor is killing , the above equation becomes

$$(27) \qquad (\widetilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \nabla^{\perp}_{\xi_p}(\phi X) + \phi\widetilde{\nabla}_{\xi_p}X.$$

Adding the equations (23) and (27), we get

$$(28) \qquad 2(\widetilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \phi\widetilde{\nabla}_{\xi_p}X - \phi\widetilde{\nabla}_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X).$$

By virtue of the equation (7) , the above equation (28) takes the form

$$2(\widetilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \phi\nabla_{\xi_p}X - \phi\nabla_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X).$$

□

4. TOTALLY r-PARACONTACT UMBILICAL SUBMANIFOLD OF PARACOSYMPLECTIC MANIFOLD

Definition 4.1. A semi-invariant submanifold M of the nearly r-paracosymplectic manifold $\bar{M}$ is totally r-paracontact umbilical submanifold if there exists a normal vector field H such that

$$(29) \qquad h(X,Y) = g(\phi X, \phi Y)H + \sum_{p=1}^{r}\{\eta^p(X)h(Y, \xi_p + \eta^p(Y)h(X, \xi_p\},$$

for any $X \in TM$.

Definition 4.2. If $H = 0$, M is a totally r-paracontact geodesic submanifold of $\bar{M}$.

We can easily verify that the following lemma.

LEMMA 4.3. *Let M be a semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. Then*

$$(\widetilde{\nabla}_X \phi)(\phi X) = \sum_{p=1}^{r} g(X, \xi_p) \widetilde{\nabla}_X \xi_p, \tag{30}$$

for any $X \in TM$.

THEOREM 4.4. *Let M be a proper semi-invariant submanifold of a nearly r-paracosymplectic manifold $\bar{M}$. If M is totally r-paracontact umbilical, then it is also totally r-paracontact geodesic.*

Proof. For any $X \in D$, we have, from equation (30),

$$g((\widetilde{\nabla}_X \phi)\phi X, H) = 0. \tag{31}$$

Making use of the equations (1), (3), (7) and (8), we obtain

$$\begin{aligned} g((\widetilde{\nabla}_X \phi)\phi X, H) &= g(\widetilde{\nabla}_X \phi X, \phi H) + g(\widetilde{\nabla}_X X, H) \\ &= -g((\phi X, \widetilde{\nabla}_X \phi H) - g(X, \widetilde{\nabla}_X H) \\ &= -g((\phi X, A_{\phi H} H) + g(X, A_H X). \end{aligned} \tag{32}$$

Since

$$g((X, A_H X) = g(h(X, X), H), \tag{33}$$

making use of the equation (29), we get

$$g(\phi X, A_{\phi H} X) = g(h(X, \phi X), \phi H) = g(X, \phi X) g(H, \phi H) = 0. \tag{34}$$

Thus, from (31)–(34) follows

$$g(X, X) g(H, H) = 0. \tag{35}$$

Since M is a proper semi-invariant submanifold, from (35) it follows that $H = 0$. Hence M is totally r-paracontact geodesic. □

5. EXAMPLE

In this section, we construct an example of a nearly r-paracosymplectic metric manifold which is not r-paracosymplectic.

Let V^{2m+r} be a real vector space with the basis

$$\{e_0, e_1, e_2, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{2m+r-1}\}.$$

Let L denote the Lie algebra constructed on V [10].

$$[e_j, e_i] = a_i e_{i+j} + a_{i+m+j} e_{i+m+j}, \tag{36}$$

where $i = 1, 2 \ldots, m; j = 0, m+1, \ldots, m+r+1$.

$$[e_j, e_{i+m+r-1}] = a_{i+m+j} e_i - a_i e_{i+m+j}, \tag{37}$$

where $i = 1, 2, \ldots, m$, $j = 0, m+1, \ldots, m+r+1$ and $[e_j, e_i] = 0$ in other cases.

Let L be a solvable Lie algebra isomorphic to the semi-direct maximal abelian ideal and the subalgebra generated by e_0. Let GL be a connected real Lie group whose Lie algebra is L. The $2m+r$-dimension identity matrix gives a left invariant Riemannian metric for GL and we take its Levi-Civita connection.

Define a set

$$\eta^p(e_k) = \delta_{pk},\ p = 1, 2, \ldots, r, k = 0, 1, 2, \ldots, 2m, 2m+1, \ldots, 2m+r-1.$$

We take $\xi_p = e_p$, $p = 1, 2, \ldots, r$. Define

$$\phi(e_p) = 0, \phi(e_{p+i}) = e_{i+n+p}, \ldots, \phi(e_{i+n+p}) = -e_{i+p};$$

for $i = 1, 2, \ldots, n$, $p = 1, 2, \ldots, r$. Then $\{GL, \phi, \xi_p, \eta^p, g\}$ is a nearly r-paracosymplectic metric manifold.

Since $\nabla\phi \neq 0$, $\{GL, \phi, \xi_p, \eta^p, g\}$ is not r-paracosymplectic.

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