MATHEMATICA, 60 (83), N° 1, 2018, pp. 32–47

REFINING LAH-RIBARIĆ INTEGRAL INEQUALITY FOR DIVISIONS OF MEASURABLE SPACE

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper, we establish some refinements of Lah-Ribarić inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given. Some examples related to Hermite-Hadamard inequality for convex functions are provided as well.

MSC 2010. Primary 26D15; Secondary 26D10.

Key words. Jensen's inequality, convex functions, Lebesgue integral, weighted means, Lah-Ribarić inequality, special means.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For the μ-integrable positive μ-a.e. weight w consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, w(t)\, \mathrm{d}\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w \mathrm{d}\mu$ instead of $\int_\Omega w(t)\, \mathrm{d}\mu(t)$ etc.

We say that the family of measurable sets $F_n(\Omega) = \{\Omega_i\}_{i \in \{1,\ldots,n\}}$ is a n-division for Ω if $\Omega = \bigcup_{i=1}^n \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $\mu(\Omega_i) > 0$ for any $i \in \{1, \ldots, n\}$. In this situation, if $f \in L_w(\Omega, \mu)$, then $f \in L_w(\Omega_i, \mu)$ for any $i \in \{1, \ldots, n\}$ and $\int_\Omega f w \mathrm{d}\mu = \sum_{i=1}^n \int_{\Omega_i} f w \mathrm{d}\mu$. Also, $\int_\Omega w \mathrm{d}\mu = \sum_{i=1}^n \int_{\Omega_i} w \mathrm{d}\mu$ with $\int_{\Omega_i} w \mathrm{d}\mu > 0$ for any $i \in \{1, \ldots, n\}$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and consider the functional $\psi(\Phi, w, f, \cdot) : \mathfrak{D}_n(\Omega) \to \mathbb{R}$ defined by

$$\psi(\Phi, f, w, F_n(\Omega)) := \frac{1}{\int_\Omega w \mathrm{d}\mu} \sum_{i=1}^n \Phi\left(\frac{\int_{\Omega_i} f w \mathrm{d}\mu}{\int_{\Omega_i} w \mathrm{d}\mu}\right) \int_{\Omega_i} w \mathrm{d}\mu. \tag{1}$$

The following result has been obtained in [14].

DOI: 10.24193/mathcluj.2018.1.04

THEOREM 1.1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$ with $\int_{\Omega_i} w \mathrm{d}\mu > 0$ for any $i \in \{1, ..., n\}$ we have*

$$\frac{\int_\Omega (\Phi \circ f) w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \geq \psi(\Phi, f, w, F_n(\Omega)) \geq \Phi\left(\frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu}\right), \tag{2}$$

where $n \geq 2$.

For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j \in J} w_j$. If $\Phi : [m, M] \to \mathbb{R}$ is a convex function and $x_j \in [m, M]$, $j \in J$, then Jensen's inequality states that

$$\frac{1}{W_J} \sum_{j \in J} w_j \Phi(x_j) \geq \Phi\left(\frac{1}{W_J} \sum_{j \in J} w_j x_j\right).$$

Assume that, for $n \geq 2$, the family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i \in \{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^n J_i$ and $J_i \cap J_j = \emptyset$, for any i, $j \in \{1, ..., n\}$ with $i \neq j$.

For a given $n \geq 2$, we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\psi(\Phi, f, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ defined by

$$\psi(\Phi, f, w, F_n(J)) := \frac{1}{W_J} \sum_{i=1}^n W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j\right).$$

From the inequality (2) for the discrete measure we have

$$\begin{aligned} \frac{1}{W_J} \sum_{j \in J} w_j \Phi(x_j) &\geq \frac{1}{W_J} \sum_{i=1}^n W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j\right) \\ &\geq \Phi\left(\frac{1}{W_J} \sum_{j \in J} w_j x_j\right), \end{aligned} \tag{3}$$

for any $F_n(J) \in \mathfrak{D}_n(J)$.

The following reverse of Jensen's inequality is known in the literature as *Lah-Ribarić inequality* [20]:

$$\begin{aligned} &\frac{\int_\Omega (\Phi \circ f) w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \\ &\leq \frac{1}{M - m}\left[\left(M - \frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu}\right) \Phi(m) + \left(\frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} - m\right) \Phi(M)\right], \end{aligned} \tag{4}$$

provided $\Phi : [m, M] \to \mathbb{R}$ is a convex function, $f : \Omega \to [m, M]$ is a μ-measurable function and such that f, $\Phi \circ f \in L_w(\Omega, \mu)$.

For other results and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-Divergence measure etc., see [1], [3]-[16], [17]-[19] and [22, 23].

Motivated by the above results we establish in this paper some refinements of Lah-Ribarić inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given. Some examples related to Hermite-Hadamard inequality for convex functions are provided as well.

2. THE RESULTS

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ and for $a, b \in I$ with $a < b$ consider the function $\Delta(\Phi; a, b, \cdot) : \mathbb{R} \to \mathbb{R}$ defined by

$$\Delta(\Phi; a, b, t) = \frac{(b-t)\Phi(a) + (t-a)\Phi(b)}{b-a}.$$

This is the straight line that connects the points $(a, \Phi(a))$ and $(b, \Phi(b))$.

The following lemma holds:

Lemma 2.1. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b, c, d \in I$ with $a < c < d < b$. Then*

$$\Phi(t) \leq \Delta(\Phi; c, d, t) \leq \Delta(\Phi; a, b, t) \tag{5}$$

for any $t \in [c, d]$.

Proof. By the convexity of Φ we have for any $t \in [c, d]$ that

$$\begin{aligned}\Delta(\Phi; c, d, t) - \Phi(t) &= \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c} - \Phi(t) \\ &= \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c} - \Phi\left(\frac{(d-t)c + (t-c)d}{d-c}\right) \geq 0.\end{aligned}$$

We observe that for $t \in [a, b]$,

$$y = \frac{(b-t)\Phi(a) + (t-a)\Phi(b)}{b-a}$$

is the equation of the segment joining the points $(a, \Phi(a))$ and $(b, \Phi(b))$ while

$$y = \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c}, \; t \in [c, d]$$

is the equation of the segment joining the points $(c, \Phi(c))$ and $(d, \Phi(d))$.

Since the function Φ is convex on I the segment on the smaller interval $[c, d]$ is under the segment on the larger interval $[a, b]$ containing $[c, d]$.

These prove the desired inequality (5). □

For a division $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,...,n\}} \in \mathfrak{D}_n(\Omega)$ and the measurable essentially bounded function $f : \Omega \to \mathbb{R}$ we denote $M_i := \operatorname{essup}_{x\in\Omega_i} f(x) < \infty$ and $m_i := \operatorname{essinf}_{x\in\Omega_i} f(x) > -\infty$. We also consider

$$M := \operatorname*{essup}_{x\in\Omega} f(x) < \infty \text{ and } m := \operatorname*{essinf}_{x\in\Omega} f(x) > -\infty.$$

Obviously, $M \geq M_i$ and $m \leq m_i$ for any $i \in \{1, ..., n\}$.

We assume in what follows that $M_i > m_i$ for any $i \in \{1, ..., n\}$.

We define the functional

$$\begin{aligned} &\sigma(\Phi, f, w, F_n(\Omega)) \\ &\quad := \frac{1}{\int_\Omega w \mathrm{d}\mu} \sum_{i=1}^n \left(\int_{\Omega_i} w \mathrm{d}\mu \right) \Delta \left(\Phi; m_i, M_i, \frac{\int_{\Omega_i} f w \mathrm{d}\mu}{\int_{\Omega_i} w \mathrm{d}\mu} \right) \\ &\quad = \frac{1}{\int_\Omega w \mathrm{d}\mu} \sum_{i=1}^n \left(\int_{\Omega_i} w \mathrm{d}\mu \right) \times \frac{1}{M_i - m_i} \left[\left(M_i - \frac{\int_{\Omega_i} f w \mathrm{d}\mu}{\int_{\Omega_i} w \mathrm{d}\mu} \right) \Phi(m_i) \right. \\ &\qquad \left. + \left(\frac{\int_{\Omega_i} f w \mathrm{d}\mu}{\int_{\Omega_i} w \mathrm{d}\mu} - m_i \right) \Phi(M_i) \right]. \end{aligned} \tag{6}$$

Observe also that

$$\begin{aligned} &\Delta \left(\Phi; m, M, \frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \right) \\ &= \frac{1}{M - m} \left[\left(M - \frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \right) \Phi(m) + \left(\frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} - m \right) \Phi(M) \right]. \end{aligned}$$

We have the following refinement of Lah-Ribarić inequality:

THEOREM 2.2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,...,n\}} \in \mathfrak{D}_n(\Omega)$ we have*

$$\frac{\int_\Omega w (\Phi \circ f) \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \leq \sigma(\Phi, f, w, F_n(\Omega)) \leq \Delta \left(\Phi; m, M, \frac{\int_\Omega f w \mathrm{d}\mu}{\int_\Omega w \mathrm{d}\mu} \right). \tag{7}$$

Proof. From the second inequality (5) we have for

$$t = \frac{\int_{\Omega_i} f w \mathrm{d}\mu}{\int_{\Omega_i} w \mathrm{d}\mu} \in [m_i, M_i], \ i \in \{1, ..., n\},$$

that

$$\begin{aligned}&\Delta\left(\Phi;m_i,M_i,\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\\&\leq\Delta\left(\Phi;m,M,\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)=\frac{1}{M-m}\left[\left(M-\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\Phi(m)\right.\\&\qquad\left.+\left(\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}-m\right)\Phi(M)\right],\end{aligned}\tag{8}$$

for any $i\in\{1,...,n\}$.

If we multiply by $\int_{\Omega_i}w\mathrm{d}\mu>0$ and sum over i from 1 to n we get

$$\begin{aligned}&\sum_{i=1}^{n}\left(\int_{\Omega_i}w\mathrm{d}\mu\right)\Delta\left(\Phi;m_i,M_i,\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\\&\leq\frac{1}{M-m}\left[\left(M\sum_{i=1}^{n}\int_{\Omega_i}w\mathrm{d}\mu-\sum_{i=1}^{n}\int_{\Omega_i}fw\mathrm{d}\mu\right)\Phi(m)\right.\\&\qquad\left.+\left(\sum_{i=1}^{n}\int_{\Omega_i}fw\mathrm{d}\mu-m\sum_{i=1}^{n}\int_{\Omega_i}w\mathrm{d}\mu\right)\Phi(M)\right]\end{aligned}$$

that is equivalent to the second inequality in (7).

For μ-almost every $x\in\Omega_i$ we have $f(x)\in[m_i,M_i]$ and then by the first inequality in (5) we have

$$\Phi(f(x))\leq\Delta(\Phi;m_i,M_i,f(x))$$

namely,

$$\Phi(f(x))\leq\frac{1}{M_i-m_i}\left[(M_i-f(x))\Phi(m_i)+(f(x)-m_i)\Phi(M_i)\right]\tag{9}$$

μ-almost every $x\in\Omega_i$ and for any $i\in\{1,...,n\}$.

If we multiply by $w\geq0$ μ-almost everywhere and integrate on Ω_i we get

$$\begin{aligned}&\int_{\Omega_i}w(\Phi\circ f)\mathrm{d}\mu\\&\leq\frac{1}{M_i-m_i}\times\left[\left(M_i\int_{\Omega_i}w\mathrm{d}\mu-\int_{\Omega_i}fw\mathrm{d}\mu\right)\Phi(m_i)\right.\\&\qquad\left.+\left(\int_{\Omega_i}fw\mathrm{d}\mu-m_i\int_{\Omega_i}w\mathrm{d}\mu\right)\Phi(M_i)\right]\\&=\frac{\int_{\Omega_i}w\mathrm{d}\mu}{M_i-m_i}\left[\left(M_i-\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\Phi(m_i)\right.\\&\qquad\left.+\left(\frac{\int_{\Omega_i}fw\,\mathrm{d}\,\mu}{\int_{\Omega_i}w\mathrm{d}\mu}-m_i\right)\Phi(M_i)\right]\end{aligned}\tag{10}$$

for any $i \in \{1, ..., n\}$.

Now, if we sum the inequality (10) over i from 1 to n we get the first inequality in (7). □

The following lemma holds:

LEMMA 2.3. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b, c, d \in I$ with $a < c < d < b$. Then*

$$0 \leq [\Delta(\Phi; c, d, t) - \Phi(t)](d - c) \leq [\Delta(\Phi; a, b, t) - \Phi(t)](b - a) \tag{11}$$

for any $t \in [c, d]$.

Proof. We observe that for any $t \in (c, d)$ we also have

$$\begin{aligned}
\Delta(\Phi; c, d, t) - \Phi(t) &= \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c} - \Phi(t) \\
&= \frac{(d-t)\Phi(c) + (t-c)\Phi(d) - (d-c)\Phi(t)}{d-c} \\
&= \frac{(d-t)\Phi(c) + (t-c)\Phi(d) - (d-t+t-c)\Phi(t)}{d-c} \\
&= \frac{(t-c)(\Phi(d) - \Phi(t)) - (d-t)(\Phi(t) - \Phi(c))}{d-c} \\
&= \frac{(t-c)(d-t)}{d-c}\left(\frac{\Phi(d) - \Phi(t)}{d-t} - \frac{\Phi(t) - \Phi(c)}{t-c}\right)
\end{aligned}$$

giving that

$$\begin{aligned}
&[\Delta(\Phi; c, d, t) - \Phi(t)](d-c) \\
&\quad = (t-c)(d-t)\left(\frac{\Phi(d) - \Phi(t)}{d-t} - \frac{\Phi(t) - \Phi(c)}{t-c}\right).
\end{aligned} \tag{12}$$

Similarly we have

$$\begin{aligned}
&[\Delta(\Phi; a, b, t) - \Phi(t)](b-a) \\
&\quad = (t-a)(b-t)\left(\frac{\Phi(b) - \Phi(t)}{b-t} - \frac{\Phi(t) - \Phi(a)}{t-a}\right),
\end{aligned} \tag{13}$$

for any $t \in I$.

It is know that, since $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then for any $\alpha \in I$ the function $\psi : I \setminus \{\alpha\} \to \mathbb{R}$,

$$\psi(s) := \frac{\Phi(s) - \Phi(\alpha)}{s - \alpha}$$

is monotonic nondecreasing on $I \setminus \{\alpha\}$.

Then for $t \in (c, d)$ we have

$$\frac{\Phi(d) - \Phi(t)}{d-t} \leq \frac{\Phi(b) - \Phi(t)}{b-t}$$

and

$$\frac{\Phi(t)-\Phi(c)}{t-c}=\frac{\Phi(c)-\Phi(t)}{c-t}\geq\frac{\Phi(a)-\Phi(t)}{a-t}=\frac{\Phi(t)-\Phi(a)}{t-a}$$

giving that

$$\frac{\Phi(d)-\Phi(t)}{d-t}-\frac{\Phi(t)-\Phi(c)}{t-c}\leq\frac{\Phi(b)-\Phi(t)}{b-t}-\frac{\Phi(t)-\Phi(a)}{t-a} \tag{14}$$

for any $t\in(c,d)$.

We also have

$$0\leq(t-c)(d-t)\leq(t-a)(b-t) \tag{15}$$

for any $t\in(c,d)$.

Therefore, by (14) and (15) we get

$$\begin{aligned}&(t-c)(d-t)\left(\frac{\Phi(d)-\Phi(t)}{d-t}-\frac{\Phi(t)-\Phi(c)}{t-c}\right)\\&\leq(t-a)(b-t)\left(\frac{\Phi(b)-\Phi(t)}{b-t}-\frac{\Phi(t)-\Phi(a)}{t-a}\right)\end{aligned} \tag{16}$$

for any $t\in(c,d)$.

If $t=c$ then (11) becomes

$$0\leq\Delta(\Phi;a,b,c)-\Phi(c)$$

namely

$$0\leq\frac{(b-c)\Phi(a)+(c-a)\Phi(b)}{b-a}-\Phi(c)$$

that is also obvious by the convexity of Φ.

The case $t=d$ is similar and the details are omitted. □

The following result also holds:

THEOREM 2.4. *Let $\Phi:[m,M]\to\mathbb{R}$ be a convex function, $f:\Omega\to[m,M]$ a μ-measurable function such that f, $\Phi\circ f\in L_w(\Omega,\mu)$. Then for any $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}\in\mathfrak{D}_n(\Omega)$ we have*

$$\begin{aligned}0\leq\;&\frac{1}{(M-m)\int_\Omega w\mathrm{d}\mu}\left[\sum_{i=1}^{n}\left(\int_{\Omega_i}(M_i-f)\,w\mathrm{d}\mu\right)\Phi(m_i)\right.\\&+\sum_{i=1}^{n}\left(\int_{\Omega_i}(f-m_i)\,w\mathrm{d}\mu\right)\Phi(M_i)\\&\left.-\sum_{i=1}^{n}(M_i-m_i)\,\Phi\left(\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\int_{\Omega_i}w\mathrm{d}\mu\right]\\&\leq\Delta\left(\Phi;m,M,\frac{\int_\Omega fw\mathrm{d}\mu}{\int_\Omega w\mathrm{d}\mu}\right)-\psi(\Phi,f,w,F_n(\Omega)),\end{aligned} \tag{17}$$

where $\psi(\Phi,f,w,F_n(\Omega))$ is defined by (1).

Proof. From the inequality (11) we have for

$$t=\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\in[m_i,M_i]\,,\ i\in\{1,...,n\}\,,$$

that

$$\begin{aligned}
0&\le\left[\Delta\left(\Phi;m_i,M_i,\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)-\Phi\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)\right](M_i-m_i)\\
&\le\left[\Delta\left(\Phi;m,M,\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)-\Phi\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)\right](M-m)
\end{aligned}\tag{18}$$

for any $i\in\{1,...,n\}$.

This inequality is equivalent to

$$\begin{aligned}
0&\le\left(M_i-\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)\Phi(m_i)+\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}-m_i\right)\Phi(M_i)\\
&-\Phi\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)(M_i-m_i)\\
&\le\left(M-\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)\Phi(m)+\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}-m\right)\Phi(M)\\
&-(M-m)\,\Phi\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)
\end{aligned}\tag{19}$$

for any $i\in\{1,...,n\}$.

If we multiply this inequality by $\int_{\Omega_i} w\mathrm{d}\mu>0$ we get

$$\begin{aligned}
0&\le\left(\int_{\Omega_i}(M_i-f)\,w\mathrm{d}\mu\right)\Phi(m_i)\\
&\quad+\left(\int_{\Omega_i}(f-m_i)\,w\mathrm{d}\mu\right)\Phi(M_i)\\
&-(M_i-m_i)\,\Phi\left(\frac{\int_{\Omega_i} fw\,\mathrm{d}\,\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)\int_{\Omega_i} w\mathrm{d}\mu\\
&\le\left(M\int_{\Omega_i} w\mathrm{d}\mu-\int_{\Omega_i} fw\mathrm{d}\mu\right)\Phi(m)\\
&+\left(\int_{\Omega_i} fw\mathrm{d}\mu-m\int_{\Omega_i} w\mathrm{d}\mu\right)\Phi(M)\\
&-(M-m)\int_{\Omega_i} w\mathrm{d}\mu\Phi\left(\frac{\int_{\Omega_i} fw\mathrm{d}\mu}{\int_{\Omega_i} w\mathrm{d}\mu}\right)
\end{aligned}\tag{20}$$

for any $i\in\{1,...,n\}$.

Now, if we sum the inequality (20) over i from 1 to n we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n}\left(\int_{\Omega_i}(M_i-f)\,w\mathrm{d}\mu\right)\Phi(m_i) \\
&\quad+\sum_{i=1}^{n}\left(\int_{\Omega_i}(f-m_i)\,w\mathrm{d}\mu\right)\Phi(M_i) \\
&\quad-\sum_{i=1}^{n}(M_i-m_i)\,\Phi\left(\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right)\int_{\Omega_i}w\mathrm{d}\mu \\
&\leq \left(M\sum_{i=1}^{n}\int_{\Omega_i}w\mathrm{d}\mu-\sum_{i=1}^{n}\int_{\Omega_i}fw\mathrm{d}\mu\right)\Phi(m) \\
&\quad+\left(\sum_{i=1}^{n}\int_{\Omega_i}fw\mathrm{d}\mu-m\sum_{i=1}^{n}\int_{\Omega_i}w\mathrm{d}\mu\right)\Phi(M) \\
&\quad-(M-m)\sum_{i=1}^{n}\int_{\Omega_i}w\mathrm{d}\mu\Phi\left(\frac{\int_{\Omega_i}fw\mathrm{d}\mu}{\int_{\Omega_i}w\mathrm{d}\mu}\right) \\
&=\left(M\int_{\Omega}w\mathrm{d}\mu-\int_{\Omega}fw\mathrm{d}\mu\right)\Phi(m) \\
&\quad+\left(\int_{\Omega}fw\mathrm{d}\mu-m\int_{\Omega}w\mathrm{d}\mu\right)\Phi(M) \\
&\quad-(M-m)\,\psi(\Phi,f,w,F_n(\Omega))\int_{\Omega}w\mathrm{d}\mu,
\end{aligned}
\tag{21}
$$

which is equivalent to the desired result (17). □

The following result also holds.

THEOREM 2.5. *Let $\Phi:[m,M]\to\mathbb{R}$ be a convex function, $f:\Omega\to[m,M]$ a μ-measurable function such that f, $\Phi\circ f\in L_w(\Omega,\mu)$. Then for any $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}\in\mathfrak{D}_n(\Omega)$ we have*

$$
\begin{aligned}
0 &\leq \frac{1}{(M-m)\int_{\Omega}w\mathrm{d}\mu}\left[\sum_{i=1}^{n}\Phi(m_i)\left(\int_{\Omega_i}(M_i-f)\,w\mathrm{d}\mu\right)\right. \\
&\quad\left.+\sum_{i=1}^{n}\Phi(M_i)\int_{\Omega_i}(f-m_i)\,w\mathrm{d}\mu-\sum_{i=1}^{n}(M_i-m_i)\int_{\Omega_i}w(\Phi\circ f)\,\mathrm{d}\mu\right] \\
&\leq \Delta\left(\Phi;m,M,\frac{\int_{\Omega}fw\mathrm{d}\mu}{\int_{\Omega}w\mathrm{d}\mu}\right)-\int_{\Omega}w(\Phi\circ f)\,\mathrm{d}\mu.
\end{aligned}
\tag{22}
$$

Proof. For μ-almost every $x\in\Omega_i$ we have $f(x)\in[m_i,M_i]$, $i\in\{1,\dots,n\}$ and then by the inequality (11) we get

$$
\begin{aligned}
0 &\leq [\Delta(\Phi;m_i,M_i,f(x))-\Phi(f(x))](M_i-m_i) \\
&\leq [\Delta(\Phi;m,M,f(x))-\Phi(f(x))](M-m)
\end{aligned}
\tag{23}
$$

for μ-almost every $x \in \Omega_i$.

This is equivalent to

$$\begin{aligned} 0 &\leq (M_i - f(x))\,\Phi(m_i) + (f(x) - m_i)\,\Phi(M_i) - \Phi(f(x))\,(M_i - m_i) \\ &\leq (M - f(x))\,\Phi(m) + (f(x) - m)\,\Phi(M) - \Phi(f(x))\,(M - m) \end{aligned}$$

for μ-almost every $x \in \Omega_i$ and every $i \in \{1, ..., n\}$.

If we multiply by $w \geq 0$ μ-almost everywhere and integrate on Ω_i we get

$$\begin{aligned} &0 \leq \Phi(m_i)\left(\int_{\Omega_i}(M_i - f)\,w\mathrm{d}\mu\right) + \Phi(M_i)\int_{\Omega_i}(f - m_i)\,w\mathrm{d}\mu \\ &-(M_i - m_i)\int_{\Omega_i} w\,(\Phi \circ f)\,\mathrm{d}\mu \\ &\leq \left(M\int_{\Omega_i} w\mathrm{d}\mu - \int_{\Omega_i} fw\mathrm{d}\mu\right)\Phi(m) + \left(\int_{\Omega_i} fw\mathrm{d}\mu - m\int_{\Omega_i} w\mathrm{d}\mu\right)\Phi(M) \\ &-(M - m)\int_{\Omega_i} w\,(\Phi \circ f)\,\mathrm{d}\,\mu \end{aligned}$$

for every $i \in \{1, ..., n\}$.

If we sum over i from 1 to n we get

$$\begin{aligned} &0 \leq \sum_{i=1}^{n}\Phi(m_i)\left(\int_{\Omega_i}(M_i - f)\,w\mathrm{d}\mu\right) + \sum_{i=1}^{n}\Phi(M_i)\int_{\Omega_i}(f - m_i)\,w\mathrm{d}\mu \\ &-\sum_{i=1}^{n}(M_i - m_i)\int_{\Omega_i} w\,(\Phi \circ f)\,\mathrm{d}\mu \\ &\leq \left(M\sum_{i=1}^{n}\int_{\Omega_i} w\mathrm{d}\mu - \sum_{i=1}^{n}\int_{\Omega_i} fw\mathrm{d}\mu\right)\Phi(m) \\ &+\left(\sum_{i=1}^{n}\int_{\Omega_i} fw\mathrm{d}\mu - m\sum_{i=1}^{n}\int_{\Omega_i} w\mathrm{d}\mu\right)\Phi(M) \\ &-(M - m)\sum_{i=1}^{n}\int_{\Omega_i} w\,(\Phi \circ f)\,\mathrm{d}\mu, \end{aligned}$$

which is equivalent to (22). □

3. DISCRETE INEQUALITIES

Assume that, for $n \geq 2$, we have a family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i\in\{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^{n} J_i$ and $J_i \cap J_j = \emptyset$ for any i, $j \in \{1, ..., n\}$ with $i \neq j$.

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, $\{x_j\}_{j\in J} \subset I$ and put $m := \min_{j\in J}\{x_j\}$ and $M := \max_{j\in J}\{x_j\}$. Also let $m_{J_i} := \min_{j\in Ji}\{x_j\}$ and $M_{J_i} = \max_{j\in Ji}\{x_j\}$ and assume that $m_{J_i} < M_{J_i}$ for $i \in \{1, ..., n\}$. For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j\in J} w_j$.

Consider the discrete version of the functional (6)

$$\begin{aligned}
&\sigma\left(\Phi, x, w, F_n(J)\right) \\
&\quad := \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\Delta\left(\Phi; m_{J_i}, M_{J_i}, \frac{\sum_{j\in J_i} w_j x_j}{W_{J_i}}\right) \\
&\quad = \frac{1}{W_J}\sum_{i=1}^{n}\frac{W_{J_i}}{M_{J_i}-m_{J_i}} \\
&\quad\times\left[\left(M_{J_i}-\frac{\sum_{j\in J_i} w_j x_j}{W_{J_i}}\right)\Phi(m_i)+\left(\frac{\sum_{j\in J_i} w_j x_j}{W_{J_i}}-m_{J_i}\right)\Phi\left(M_{J_i}\right)\right]. \qquad (24)
\end{aligned}$$

If we write the inequality (7) for the *discrete measure* we get

$$\frac{1}{W_J}\sum_{j\in J} w_j\Phi(x_j) \le \sigma\left(\Phi, x, w, F_n(J)\right) \le \Delta\left(\Phi; m, M, \frac{1}{W_J}\sum_{j\in J} w_j x_j\right). \qquad (25)$$

From (17) we have

$$\begin{aligned}
0 &\le \frac{1}{(M-m)W_J}\left[\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(M_{J_i}-x_j\right)w_j\right)\Phi\left(m_{J_i}\right)\right. \\
&\quad+\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(x_j-m_{J_i}\right)w_j\right)\Phi\left(M_{J_i}\right) \\
&\quad\left.-\sum_{i=1}^{n}\left(M_{J_i}-m_{J_i}\right)\Phi\left(\frac{\sum_{j\in J_i} x_j w_j}{W_{J_i}}\right)W_{J_i}\right] \qquad (26) \\
&\le \Delta\left(\Phi; m, M, \frac{1}{W_J}\sum_{j\in J} w_j x_j\right)-\psi\left(\Phi, x, w, F_n(J)\right),
\end{aligned}$$

where

$$\psi\left(\Phi, f, w, F_n(J)\right) := \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right).$$

From (22) we also have

$$
\begin{aligned}
0 &\le \frac{1}{(M-m)W_J}\left[\sum_{i=1}^{n}\Phi\left(m_{J_i}\right)\left(\sum_{j\in J_i}\left(M_{J_i}-x_j\right)w_j\right)\right.\\
&\left.+\sum_{i=1}^{n}\Phi\left(M_{J_i}\right)\sum_{j\in J_i}\left(x_j-m_{J_i}\right)w_j-\sum_{i=1}^{n}\left(M_{J_i}-m_{J_i}\right)\sum_{j\in J_i}w_j\Phi\left(x_j\right)\right]\\
&\le \Delta\left(\Phi;m,M,\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)-\frac{1}{W_J}\sum_{j\in J}w_j\Phi\left(x_j\right).
\end{aligned}
\tag{27}
$$

If we write the above inequalities for the positive numbers $x_i>0, i\in\{1,...,n\}$ and the convex power function $\Phi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$ we have

$$
\begin{aligned}
\frac{1}{W_J}\sum_{j\in J}w_jx_j^p &\le \frac{1}{W_J}\sum_{i=1}^{n}\frac{W_{J_i}}{M_{J_i}-m_{J_i}}\times\left[\left(M_{J_i}-\frac{\sum_{j\in J_i}w_jx_j}{W_{J_i}}\right)m_i^p\right.\\
&\qquad\left.+\left(\frac{\sum_{j\in J_i}w_jx_j}{W_{J_i}}-m_{J_i}\right)M_{J_i}^p\right]\\
&\le \frac{1}{M-m}\left[\left(M-\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)m^p\right.\\
&\qquad\left.+\left(\frac{1}{W_J}\sum_{j\in J}w_jx_j-m\right)M^p\right],
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
0 &\le \frac{1}{(M-m)W_J}\left[\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(M_{J_i}-x_j\right)w_j\right)m_{J_i}^p\right.\\
&\qquad+\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(x_j-m_{J_i}\right)w_j\right)M_{J_i}^p\\
&\qquad\left.-\sum_{i=1}^{n}\left(M_{J_i}-m_{J_i}\right)\left(\sum_{j\in J_i}x_jw_j\right)^p W_{J_i}^{1-p}\right]\\
&\le \frac{1}{M-m}\left[\left(M-\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)m^p+\left(\frac{1}{W_J}\sum_{j\in J}w_jx_j-m\right)M^p\right]\\
&\qquad-\frac{1}{W_J}\sum_{i=1}^{n}W_{J_i}^{1-p}\left(\sum_{j\in J_i}w_jx_j\right)^p
\end{aligned}
\tag{29}
$$

and

$$0 \le \frac{1}{(M-m) W_J} \left[\sum_{i=1}^{n} m_{J_i}^p \left(\sum_{j \in J_i} (M_{J_i} - x_j) w_j \right) + \sum_{i=1}^{n} M_{J_i}^p \sum_{j \in J_i} (x_j - m_{J_i}) w_j - \sum_{i=1}^{n} (M_{J_i} - m_{J_i}) \sum_{j \in J_i} w_j x_j^p \right] \le \frac{1}{M-m} \left[\left(M - \frac{1}{W_J} \sum_{j \in J} w_j x_j \right) m^p + \left(\frac{1}{W_J} \sum_{j \in J} w_j x_j - m \right) M^p \right] - \frac{1}{W_J} \sum_{j \in J} w_j x_j^p. \tag{30}$$

4. SOME INEQUALITIES RELATED TO HH-INEQUALITY

It is clear that all inequalities from Section 2 can be written for univariate functions $f : [a,b] \subset \mathbb{R} \to [m,M]$ and the functional defined in (6).

We are, however, interested here in the particular case that is related to the celebrated *Hermite-Hadamard inequality*

$$\Phi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b \Phi(t)\, \mathrm{d}t \le \frac{\Phi(a) + \Phi(b)}{2},$$

where $\Phi : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$.

Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function and $f : [a,b] \to [m,M]$ an integrable function. Consider the division of the interval $[a,b]$ given by

$$d_n : a = x_0 < x_{1<} ... < x_{n-1} < x_n = b, \ n \ge 2.$$

If we take $\Omega = [a,b]$ and $\Omega_1 = [a, x_1]$, $\Omega_i = (x_i, x_{i+1}]$ for $i \in \{1, ..., n-1\}$ then $\Omega = \bigcup_{i=1}^n \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \ne j$.

By making use of (6) for this division and $f : [a,b] \subset \mathbb{R} \to [a,b]$, $f(x) = x$, we can consider the functional

$$\sigma(\Phi, d_n) := \frac{1}{b-a} \sum_{i=1}^{n} (x_{i+1} - x_i) \frac{\Phi(x_i) + \Phi(x_{i+1})}{2}. \tag{31}$$

If we use the inequality (7) we have

$$\frac{1}{b-a} \int_a^b \Phi(t)\, \mathrm{d}t \le \frac{1}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) \frac{\Phi(x_i) + \Phi(x_{i-1})}{2} \le \frac{\Phi(a) + \Phi(b)}{2}. \tag{32}$$

This inequality was obtained by the author in 1994 in [2], see also [15, p. 22].

From (17) we have

$$\begin{aligned}0 &\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i - x_{i-1})^2\left[\frac{\Phi(x_i)+\Phi(x_{i-1})}{2} - \Phi\left(\frac{x_{i-1}+x_i}{2}\right)\right] \\ &\leq \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\sum_{i=1}^{n}\Phi\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}),\end{aligned} \tag{33}$$

while from (22) we have

$$\begin{aligned}0 &\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i - x_{i-1})^2 \\ &\quad\times\left[\frac{\Phi(x_i)+\Phi(x_{i-1})}{2} - \frac{1}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i}\Phi(x)\,\mathrm{d}x\right] \\ &\leq \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(t)\,\mathrm{d}t.\end{aligned} \tag{34}$$

If we take in (33) and (34) $\Phi(t) = \frac{1}{t}$, $t \in [a,b] \subset (0,\infty)$ then we get the inequalities

$$\frac{1}{2(b-a)^2}\sum_{i=1}^{n}\frac{(x_i - x_{i-1})^4}{x_i x_{i-1}(x_i + x_{i-1})} \leq \frac{a+b}{2ab} - \frac{2}{b-a}\sum_{i=1}^{n}\frac{x_i - x_{i-1}}{x_i + x_{i-1}}, \tag{35}$$

and

$$\begin{aligned}0 &\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i - x_{i-1})^2\left[\frac{L(x_i,x_{i-1}) - H(x_i,x_{i-1})}{L(x_i,x_{i-1})\,H(x_i,x_{i-1})}\right] \\ &\leq \frac{L(a,b) - H(a,b)}{L(a,b)\,H(a,b)},\end{aligned} \tag{36}$$

where

$$H(\alpha,\beta) := \frac{2\alpha\beta}{\alpha+\beta}$$

is the *harmonic mean* while

$$L(\alpha,\beta) := \frac{\alpha-\beta}{\ln\alpha - \ln\beta}, \alpha \neq \beta,$$

is the *logarithmic mean*.

If we take in (33) and (34) $\Phi(t) = -\ln t$, $t \in [a,b] \subset (0,\infty)$ then we get the inequalities

$$1 \leq \prod_{i=1}^{n}\left(\frac{A(x_{i-1},x_i)}{G(x_{i-1},x_i)}\right)^{\frac{(x_i - x_{i-1})^2}{(b-a)^2}} \leq \frac{\prod_{i=1}^{n}\left(A(x_{i-1},x_i)\right)^{\frac{(x_i - x_{i-1})}{b-a}}}{G(a,b)}, \tag{37}$$

and

$$1 \leq \prod_{i=1}^{n}\left(\frac{I(x_{i-1},x_i)}{G(x_{i-1},x_i)}\right)^{\frac{(x_i - x_{i-1})^2}{(b-a)^2}} \leq \frac{I(a,b)}{G(a,b)}, \tag{38}$$

where

$$G(\alpha,\beta) := \sqrt{\alpha\beta}$$

is the *geometric mean* while

$$I(\alpha,\beta) := \frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right)^{\frac{1}{\beta-\alpha}}, \alpha \neq \beta,$$

is the *identric mean*.

Now, consider the *p-logarithmic mean* defined by

$$L_p(\alpha,\beta) := \left(\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\alpha-\beta)}\right)^{1/p},$$

where $p \in \mathbb{R}\setminus\{-1,0\}$.

From (33) and (34) we have for $p \in (-\infty,0)\cup(1,\infty)\setminus\{-1\}$

$$\begin{aligned} 0 &\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})^2\left[A\left(x_i^p,x_{i-1}^p\right)-A^p(x_{i-1},x_i)\right] \\ &\leq A(a^p,b^p)-\frac{1}{b-a}\sum_{i=1}^{n}A^p(x_{i-1},x_i)(x_i-x_{i-1}) \end{aligned} \tag{39}$$

and

$$\begin{aligned} 0 \leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})^2\left[A\left(x_i^p,x_{i-1}^p\right)-L_p^p(x_{i-1},x_i)\right] \\ \leq A(a^p,b^p)-L_p^p(a,b). \end{aligned} \tag{40}$$

REFERENCES

[1] S. Abramovich, S. Ivelić and J. Pečarić, *Generalizations of Jensen-Steffensen and related integral inequalities for superquadratic functions*, CEJOR Cent. Eur. J. Oper. Res., **8** (2010), 937–949.

[2] S.S. Dragomir, *Some remarks on Hadamard's inequalities for convex functions,* Extracta Math., **9** (1994), 88–94.

[3] S.S. Dragomir, *An improvement of Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **34** (**82**) (1990), 291–296.

[4] S.S. Dragomir, *Some refinements of Ky Fan's inequality*, J. Math. Anal. Appl., **163** (1992), 317–321.

[5] S.S. Dragomir, *Some refinements of Jensen's inequality,* J. Math. Anal. Appl., **168** (1992), 518–522.

[6] S.S. Dragomir, *A further improvement of Jensen's inequality,* Tamkang J. Math., **25** (1994), 29–36.

[7] S.S. Dragomir, *A new improvement of Jensen's inequality,* Indian J. Pure Appl. Math., **26** (1995), 959–968.

[8] S.S. Dragomir, *A refinement of Jensen's inequality with applications for f-divergence measures*, Taiwanese J. Math., **14** (2010), 153–164.

[9] S.S. Dragomir, *A new refinement of Jensen's inequality in linear spaces with applications*, Mathematical and Computer Modelling, **52** (2010), 1497–1505.

[10] S.S. Dragomir, *Some reverses of the Jensen inequality with applications*, Bull. Aust. Math. Soc.,**87** (2013), 177–194.

[11] S.S. Dragomir and N.M. Ionescu, *Some converse of Jensen's inequality and applications*, J. Numer. Anal. Approx. Theory, **23** (1994), 71–78.

[12] S.S. Dragomir and C.J. Goh, *A counterpart of Jensen's discrete inequality for differentialble convex mappings and applications in information theory,* Mathematical and Computer Modelling, **24** (1996), 1–11.

[13] S.S. Dragomir and M.A. Khan, *Refinement of the Jensen integral inequality*, Research Group in Mathematical Inequalities and Applications, Research Report Collection, **18**, Article 94 (2015), 1–10.

[14] S.S. Dragomir, *Refining Jensen's integral inequality for divisions of measurable space*, Research Group in Mathematical Inequalities and Applications, Research Report Collection, **18**, Article 97 (2015), 1–14.

[15] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Research Group in Mathematical Inequalities and Applications, Monographs, Victoria University, 2000.

[16] S.S. Dragomir, J. Pečarić and L.E. Persson, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar., **70** (1996), 129–143.

[17] L. Hortháth, *Inequalities corresponding to the classical Jensen's inequality*, Journal of Mathematical Inequalities, **3** (2009), 189–200.

[18] L. Hortháth, *A refinement of the integral form of Jensen's inequality*, J. Inequal. Appl., **2012**, Article 178 (2012), 1–19.

[19] S. Khalid and J. Pečarić, *On the refinement of the integral Jensen-Steffensen inequality*, J. Inequal. Appl., **2013**, Article 20 (2013), 1–18.

[20] P. Lah and M. Ribarič, *Converse of Jensen's inequality for convex functions*, Publikacije Elektrotehničkog fakulteta, Serija Matematika i fizika, **412/460** (1973), 201–205.

[21] J. Pečarić and S.S. Dragomir, *A refinements of Jensen inequality and applications,* Stud. Univ. Babes-Bolyai Math., **24** (1989), 15–19.

[22] F. Qi and M.L. Yang, *Comparisons of two integral inequalities with Hermite-Hadamard-Jensen's integral inequality*, Research Group in Mathematical Inequalities and Applications, Research report collection, **8** (2005), 1–6.

[23] Z.Y. Song, *Discussion on the integral type Jensen inequality for P-convex functions* (in Chinese), Pure Appl. Math. (Xi'an), **28** (2012), 36–40.

Received July 28, 2017
Accepted August 23, 2017

Victoria University
Mathematics, College of Engineering & Science
PO Box 14428
Melbourne City, MC 8001, Australia

University of the Witwatersrand
School of Computer Science & Applied Mathematics
Private Bag 3, Johannesburg 2050, South Africa
E-mail: `sever.dragomir@vu.edu.au`