MATHEMATICA, 60 (83), N$^{\circ}$ 1, 2018, pp. 3–11

SOME NEW RESULTS ON THE JOIN GRAPH OF GIVEN GROUPS

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Abstract. Recently, H. Ahmadi et al. defined the join graph associated to a finite group. They studied this graph not only on graph theoretic points, but also on group theoretic cases. They classified finite groups with planar complete join graph and with domination number 1. Moreover, some graph theoretical properties, such as its regularity, clique, chromatic number, bounds for its diameter and girth, was discussed in their research. In this paper, we approach to this graph in the other cases. For instance, the eccentric connectivity, total eccentricity, Wiener index, hyper Wiener index, first, second and third Zagreb index, domination and Hosoya polynomial are computed for the join graph of the class of the groups $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$, where p is a prime number and n is a positive integer.

MSC 2010. 05C25, 20K01, 20B05.

Key words. Join graph, topological index, polynomials associated to a graph.

1. INTRODUCTION

Suppose G is a finite group. H. Ahmadi et al. defined an undirected simple graph $\Delta(G)$ whose vertices are the proper subgroups of G, which are not contained in the Frattini subgroup of G. Moreover, two vertices H and K are joined by an edge if and only if $G = \langle H, K\rangle$. This is called the join graph associated to the group G. They classified finite groups using planar graphs, complete graphs and those with domination number 1. Moreover, some graph theoretical properties, such as its regularity, clique, chromatic number, bounds for its diameter and girth, was discussed in the series of papers [1, 2, 3].

In this paper, we study the join graph of some special groups and, in the process of the verification of the graph theoretical properties of their join graph, we gain some precise knowledge about their subgroups.

In chemical graph theory, the Wiener index is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the molecule. For the connected graph Γ, the Wiener index

The authors thank the referee for his helpful comments and suggestions. B. Tolue is the corresponding author.

DOI: 10.24193/mathcluj.2018.1.01

is defined as
$$W(\Gamma)=\sum_{\{u,v\}\subseteq V(\Gamma)} d(u,v).$$
The Wiener index is named after Harry Wiener, who introduced it in 1947. Based on its success, many other topological indices were introduced. The Wiener index was extended to the hyper Wiener index. The hyper Wiener index is
$$WW(\Gamma)=\frac{1}{2}\sum_{\{u,v\}\subseteq V(\Gamma)} d(u,v)+\frac{1}{2}\sum_{\{u,v\}\subseteq V(\Gamma)} d^2(u,v).$$
The first and second Zagreb index are important topological indices which are defined by Gutman and Trinajestic [6]. The graph invariants $M_1(\Gamma)$, $M_2(\Gamma)$ and $\mathrm{irr}(\Gamma)$, known as the first, second and third Zagrebs, are defined as follows:
$$\begin{aligned}
M_1(\Gamma) &= \sum_{v\in V(\Gamma)} (\deg(v))^2,\\
M_2(\Gamma) &= \sum_{uv\in E(\Gamma)} \deg(u)\deg(v),\\
\mathrm{irr}(\Gamma) &= \sum_{uv\in E(\Gamma)} |\deg(u)-\deg(v)|,
\end{aligned}$$
respectively. The third Zagreb is called the irregularity index too. The subset of vertices T is called a dominating set for the graph Γ, if every vertex outside T is joined to at least one vertex in T. The size of the smallest dominating set is called the domination number and is denoted by $\gamma(\Gamma)$. Alikhani and Peng [4] have introduced the domination polynomial of a graph. The domination polynomial of Γ is
$$D(\Gamma,x)=\sum_{i=\gamma(\Gamma)}^{|V(\Gamma)|} d(\Gamma,i)x^i,$$
where $d(\Gamma,i)$ is the number of dominating sets of size i. The Hosoya polynomial of a graph was introduced in Hosoya's paper [7] in 1988 and received a lot of attention. The Hosoya polynomial of the graph Γ is
$$H(\Gamma,x)=\sum_{\{u,v\}\subseteq V(\Gamma)} x^{d(u,v)}.$$
In the next section, we consider the group $\mathbb{Z}_{p^2}\times\mathbb{Z}_p$ and its subgroups, where p is a prime number. The structure of $\Delta(\mathbb{Z}_{p^2}\times\mathbb{Z}_p)$ is clarified. The number of induced and non-induced cycles with k vertices is presented by a formula for this graph. We observe that the induced subgraph of $\Delta(\mathbb{Z}_{p^2}\times\mathbb{Z}_p)$ on the center vertices is K_p. Furthermore, some graph parameters, such as the eccentric connectivity, total eccentricity, Wiener index, hyper Wiener index, first, second and third Zagreb index, dominating and Hosoya polynomial of

$\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$, are computed. Finally, we consider the join graph of the group $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$ and some of its properties are verified.

2. THE JOIN GRAPH OF THE GROUP $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$

There are many papers devoted to the subgroups of various finite abelian groups, for instance see the recent paper [8]. Let us start with the following result about the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$.

THEOREM 2.1. *If $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then the number of the subgroups of order p is $p+1$ and the number of subgroups of order p^2 is $p+1$.*

Proof. Let H be a non-trivial subgroup of G. Therefore $|H| = p, p^2$. Clearly the subgroups generated by $(0,b)$ are of order p and all of them are equal, where $0 \neq b \in \mathbb{Z}_p$. On the other hand, the elements of order p in $\mathbb{Z}_{p^2}$ form a subgroup of order p; more precisely, the subgroup generated by an element $(a,0) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, where $a = kp,\ 1 \leq k \leq p-1$. Note that all such subgroups are equal. Moreover, all the elements of the form $(a,b) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ generate a subgroup of order p such that a and b are of order p in $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p$, respectively. The number of elements of order p in both $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p$ is $(p-1)$. Thus there are $(p-1)^2$ elements of order p in $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. But the subgroups generated by them are not distinct. Actually, the number of repetitive subgroups is $(p-1)(p-2)$. Therefore, the number of distinct subgroups of this type is $p-1$. Hence the number of subgroups of order p is $p+1$. The subgroups generated by $(1,b)$ are of order p^2, where $b \in \mathbb{Z}_{p^2},\ 0 \leq b \leq p-1$. Moreover, $\langle (0,1), (p,1) \rangle$ is a non-cyclic group of order p^2. □

Since the Frattini subgroup of G is $\Phi(G) = \{(0,0), (p,0), (2p,0), (3p,0), \ldots, ((p-1)p, 0)\}$, the join graph of the group $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has $2p+1$ vertices and

$$\binom{2p+1}{2} - \binom{p+1}{2}$$

edges. Moreover, all the vertices can be classified in two parts: the upper part vertices $\langle (0,1) \rangle, \langle (p,1) \rangle, \langle (p,2) \rangle, \ldots, \langle (p,p-1) \rangle,\ \langle (0,1),(p,1) \rangle$ and the lower part vertices $\langle (1,0) \rangle, \langle (1,1) \rangle, \langle (1,2) \rangle, \ldots, \langle (1,p-1) \rangle$. The vertices which are located in the upper part are not joined to each other, but they are adjacent to all the vertices of lower part. Therefore, the degree of all the upper part vertices is p and, since the lower part vertices are adjacent, their degree is $2p$. Furthermore, elementary formulas for this graph follow by its structure, for instance $\mathrm{diam}(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = 2$, $\mathrm{girth}(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = 3$, $\omega(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = p+1$, $\chi(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = p+1$ and $\alpha((\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = p+1$.

EXAMPLE 2.2. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. It has 3 subgroups of order 2, $\langle (0,1) \rangle$, $\langle (2,0) \rangle$ and $\langle (2,1) \rangle$. Moreover, $\langle (1,0) \rangle$, $\langle (1,1) \rangle$ and $\langle (0,1),(2,1) \rangle$ are its subgroups of order 4. Clearly $\Phi(G) = \langle (2,0) \rangle$ and so one can draw $\Delta(G)$ as in the Figure 2.1. Taking into account Figure 2.1, one can deduce that

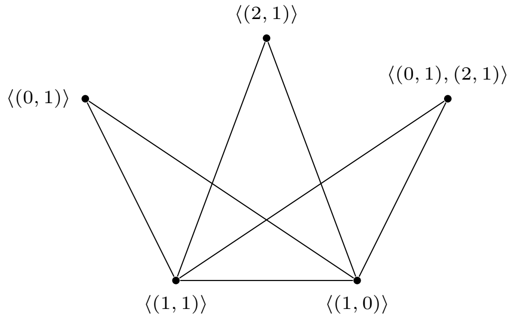

Fig. 2.1 – $\Delta(\mathbb{Z}_4 \times \mathbb{Z}_2)$

$W(\Delta(G)) = 13,\ WW(\Delta(G)) = 16,\ M_1(\Delta(G)) = 44,\ M_2(\Delta(G)) = 64$ and $irr(\Delta(G)) = 12$.

A graph Γ is a split graph if and only if there is a partition $V(\Gamma) = C \uplus I$, where C is a complete and I an independent set (see [5] for more details). Split graphs were first studied by Földes and Hammer and independently introduced by Tyshkevich and Chernyak. It is clear that $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ splits into the upper and lower parts which are independent and complete sets, respectively.

A chordless cycle in a graph, or an induced cycle, is a cycle such that no two vertices of the cycle are connected by an edge that does not itself belong to the cycle.

THEOREM 2.3. *The join graph of the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has $\binom{p}{3} + (p+1)\binom{p}{2}$ induced cycles with $k = 3$ vertices.*

Note that for the join graph of the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, there is no induced cycle with $k > 3$. Since $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is a split graph and split graphs are graphs that are both chordal and complements of chordal graphs, chordal and chordal graphs have no induced cycles with more than 3 vertices.

The adjacency matrix of the join graph of the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has the following structure.

PROPOSITION 2.4. *The adjacency matrix A of the join graph of the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ is an $m \times m$ matrix, where $m = 2p+1$. For the first $p+1$ rows of A, the first $p+1$ arrays are zero and the rest p arrays are 1. From the $(p+2)th$ row on, all the arrays are 1 except the diagonal arrays.*

THEOREM 2.5. *The number of non-induced cycles with k vertices for the join graph $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is obtained by the following formula:*

$$\binom{2p+1}{k} - \binom{p+1}{k} - p\binom{p+1}{k-1} - \sum_{i=2}^{k-2} \binom{p}{k-i}\binom{p+1}{i}.$$

Proof. If $k = 3$, then the number of cycles with 3 vertices of the complete graph K_{2p+1} is $\binom{2p+1}{3}$, but $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is not a complete graph. Therefore, some of these cycles are additional, so we omit them. The $p+1$ vertices on the upper part are not adjacent and they are not the good candidates for a triangle. Moreover, any p vertices of the lower part with two vertices in the upper part are not suitable too. Hence the result is clear for $k = 3$. Now, we can finish the proof by induction on k. □

Recall the eccentric of the vertex u in the graph Γ is $\xi_\Gamma(u) = \mathrm{Max}\{d(u, v) : v \in V(\Gamma)\}$ and the radius of the graph Γ is the smallest value of the eccentricity Γ. The center of the graph is a subset of vertices for which their eccentricity is equal to the radius of the graph.

Proposition 2.6. *The induced subgraph on the center of the join graph $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is K_p.*

Proof. Since the vertices in the lower part of the graph $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ are mutually adjacent, they have the eccentricity one. The induced subgraph of these p vertices is a complete graph. □

The eccentric connectivity of a graph Γ is defined as

$$\xi^c(\Gamma) = \sum_{u \in V(\Gamma)} \deg_\Gamma(u)\xi_\Gamma(u).$$

Moreover, the sum of all the eccentricities of a graph is called the total eccentricity and is denoted by $\zeta(\Gamma)$, and thus $\zeta(\Gamma) = \sum_{u \in V(\Gamma)} \xi_\Gamma(u)$.

Proposition 2.7. *Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then:*

(i) $\xi^c(\Delta(G)) = 2p(2p + 1)$.

(ii) $\zeta(\Delta(G)) = 3p + 2$.

Proof. (i) By the structure of the graph which was explained after Theorem 2.1, we deduce that the upper and lower part vertices have eccentricity $2, 1$ and degree $p, 2p$, respectively. The rest of the assertion is clear. □

Theorem 2.8. *Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then:*

(i) *The Wiener index of the join graph of the group $\Delta(G)$ is obtained by*

$$2\binom{p+1}{2} + \left[\binom{2p+1}{2} - \binom{p+1}{2}\right] = \binom{p+1}{2} + \binom{2p+1}{2}.$$

(ii) *The hyper Wiener index is* $WW(\Delta(G)) = 2\binom{p+1}{2} + \binom{2p+1}{2}$.

Proof. (i) By the structure of the graph, there are $p+1$ vertices such that the shortest path between every pair of them is 2. We can chose them in $\binom{p+1}{2}$

ways. Thus for computing the Wiener index, 2 will be added $\binom{p+1}{2}$ times. Consider the other p vertices that join all the vertices. Then $\binom{2p+1}{2}$ times 1 is added to the Wiener index. Since $p+1$ vertices have been counted before, the subtraction of $\binom{p+1}{2}$ yields the result.
(ii) By the definition of hyper Wiener index, it is enough to compute $\sum_{\{u,v\}\subseteq V(\Delta(G))} d^2(u,v)$, so by the first part, the result follows. □

THEOREM 2.9. *Let* $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.

(i) *The first Zagreb index is* $M_1(\Delta(G)) = 5p^3 + p^2$.

(ii) *The second Zagreb index is* $M_2(\Delta(G)) = 2p^3(p+1) + 4p^2\binom{p}{2}$.

(iii) *The third Zagreb index (irregularity) is* $\mathrm{irr}(\Delta(G)) = p^2(p+1)$.

Proof. (i) Again, by the structure of the graph, the degree of the vertices is clarified, so the assertion follows by the definition of the first Zagreb.
(ii) Let $e = uv$ be an edge of $\Delta(G)$. It is enough to consider two cases. Firstly, suppose the vertices u and v are in the lower part. Therefore, their degrees is $2p$ and the number of such edges is $\binom{p}{2}$. Secondly, assume u is located in the upper part and v in the lower part. Thus, the number of such edges is $p(p+1)$ and, by considering the degree of u and v, the result is clear.
(iii) By the definition of third Zagreb, it is enough to compute $|\deg(u) - \deg(v)|$ for all edges such that one of their ends is located in the lower part and another in the upper part. □

THEOREM 2.10. *The domination polynomial of the graph* $\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ *is*

$$\begin{aligned} &D(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p), x) \\ &= \left[\binom{2p+1}{1} - \binom{p+1}{1}\right]x + \left[\binom{2p+1}{2} - \binom{p+1}{2}\right]x^2 \\ &+ \cdots + \left[\binom{2p+1}{p} - \binom{p+1}{p}\right]x^p \\ &+ \left[\binom{2p+1}{p+1} - \binom{p+1}{p+1}\right]x^{p+1} + \cdots + x^{2p+1}. \end{aligned}$$

Proof. By definition of the domination polynomial, it is enough to compute the number of dominating sets of size i, $1 \le i \le 2p+1$. Since the adjacent vertices are clarified by the discussion after Theorem 2.1, the assertion is clear. □

By computation the Hosoya polynomial is obtained as follows:

$$H(\Delta(\mathbb{Z}_{p^2} \times \mathbb{Z}_p) = \binom{p+1}{2}x^2 + \left[\binom{2p+1}{2} - \binom{p+1}{2}\right]x.$$

Clearly its roots are zero and $\dfrac{\binom{p+1}{2}-\binom{2p+1}{2}}{\binom{p+1}{2}}$.

Using Kuratowski's Theorem, the planarity of some join graph of specific groups was discussed in [3] and the structure of them are completely clear. For instance, we observe the graph $\Delta(\mathbb{Z}_{p_1p_2p_3})$ in Figure 2.2, where $|a| = p_1$, $|b| = p_2$, $|c| = p_3$ and p_1, p_2, p_3 are distinct prime numbers (see [3, Proposition 2.7]). Obviously, $M_1(\Delta(\mathbb{Z}_{p_1p_2p_3})) = 30$, $M_2(\Delta(\mathbb{Z}_{p_1p_2p_3})) = 36$ and $\mathrm{irr}(\Delta(\mathbb{Z}_{p_1p_2p_3})) = 6$.

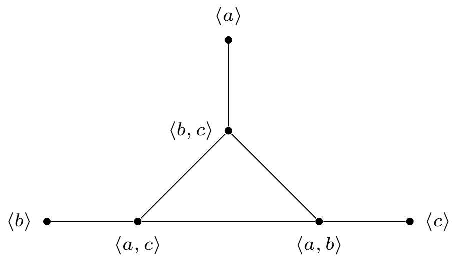

Fig. 2.2 – $\Delta(\mathbb{Z}_{p_1p_2p_3})$

Moreover, the structure of the join graph of the groups $\mathbb{Z}_{p_1^2p_2^n}$, $\mathbb{Z}_{p_1^2p_2p_3}$ and $\mathbb{Z}_{p_1^2p_2^2p_3}$ are verified, so their graph parameters can be obtained by computations [3, Figures (4), (6), (7)].

3. THE JOIN GRAPH OF THE GROUP $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$

In the previous section, Example 2.2 clarifies the structure of the join graph of the group $\mathbb{Z}_4 \times \mathbb{Z}_2$. In the following results, we present some graph parameters of $\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$, where $n \geq 3$.

THEOREM 3.1. *Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_2$, where $n \geq 3$. Then:*

(i) *G has $3n$ subgroups.*
(ii) *G has 3 maximal subgroups such that 2 of them are cyclic.*
(iii) *$\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$ has $2n+1$ vertices and the number of edges is $4n-1$ (see Figure 3.3).*
(iv) *$\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$ has $2n-1$ vertices of degree and eccentric 2. Moreover, there are 2 vertices of degree $2n$ and eccentric 1.*
(v) *The eccentric connectivity of the graph $\xi^c(\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)) = 12n - 4$.*
(vi) *The total eccentricity $\zeta(\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)) = 4n$.*
(vii) *The adjacency matrix is a $(2n+1) \times (2n+1)$ matrix with a zero block submatrix $(2n-1) \times (2n-1)$ such that all the other arrays are 1 except the diagonal arrays which are 0.*

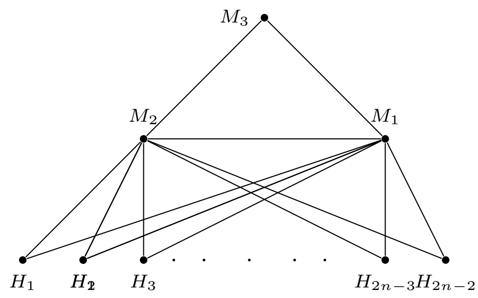

Fig. 3.3 – $\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$, $n \geq 3$

Proof. One can compute all the subgroups of G. The Frattini subgroup is $\Phi(G) = \langle(2,0)\rangle$ while maximal subgroups are $M_1 = \langle(1,0)\rangle$, $M_2 = \langle(1,1)\rangle$ and $M_3 = \langle(2,0),(2,1)\rangle$. The other subgroups which are not inside $\Phi(G)$ are denoted by H_i, $1 \leq i \leq 2n-2$ in the Figure 3.3. The proof of the items are clear by computations, considering the Figure 3.3. □

It is not hard to conclude that there are $2n-1$ cycles of length 3 in $\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$ and the center of the graph is $\{M_1, M_2\}$.

THEOREM 3.2. *If $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_2$, where $n \geq 3$, then:*

(i) *The Wiener index of $W(\Delta(G)) = (8n-5)$.*
(ii) *The hyper Wiener index of $WW(\Delta(G)) = 10n-7$.*
(iii) *The first Zagreb index is $M_1(\Delta(G)) = 8n^2 + 8n - 4$.*
(iv) *The second Zagreb index is $M_2(\Delta(G)) = 20n^2 - 8n$.*
(v) *The third Zagreb index (irregularity) is $\mathrm{irr}(\Delta(G)) = 8n^2 - 12n + 4$.*

Proof. (i) By Figure 3.3, all the maximal subgroups are adjacent. All the non-maximal subgroups H_i do not join the non-cyclic maximal subgroup M_3, as they are inside it, and so the distance between them is 2. The distance of cyclic maximal subgroups M_j and non-maximal subgroups H_i is 1, $j = 1, 2$, $1 \leq i \leq 2n-2$. Now, by the definition of the Wiener index $W(\Delta) = 3(1) + (2n-2)2 + 2(2n-2)$, the second part follows similarly.
(iii) The degree of the vertices is clear by Figure 3.3, $\deg(M_3) = 2$, $\deg(M_1) = \deg(M_2) = 2n$ and $\deg(H_i) = 2$, $1 \leq i \leq 2n-2$. Thus, the first Zagreb index is computed.
(iv) The number of edges whose two end vertices are non-maximal and cyclic maximal subgroups is $4n-4$. The degrees of these two ends are 2 and $2n$. Moreover, there are 3 edges that join the maximal subgroups. Thus, $M_2(\Delta(G)) = (4n-4)(2n)2 + 2(2n)2 + (2n)^2$ and the assertion follows. By a similar argument, the fifth part follows. □

THEOREM 3.3. *The domination polynomial of the graph $\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$ is $D(\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2), x) = (x+1)^{(2n-1)}(x+2)x$.*

Proof. By the structure of the graph $\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)$, which is completely clear by Figure 3.3, we can partition all the vertices in two sets $A = \{M_1, M_2\}$ (vertices which join all the other vertices) and $B = \{M_3, H_1, H_2, \ldots, H_{2n-2}\}$ (vertices which just join the vertices of A). Now, it is enough to count the dominating sets of different sizes. Clearly, there are 2 dominating sets of size 1. In order to construct the dominating sets of size 2, at least one of the vertices M_1 or M_2 is inside it, so there are $1 + 2(2n - 1)$ dominating sets of size 2. Similarly, there exist $\binom{2}{2}\binom{2n-1}{i-2} + 2\binom{1}{1}\binom{2n-1}{i-1}$ dominating sets of size i, where $2 \leq i \leq 2n + 1$. Hence the assertion follows. □

PROPOSITION 3.4. *The Hosoya polynomial of the join graph of the group* $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$ *is* $H(\Delta(\mathbb{Z}_{2^n} \times \mathbb{Z}_2), x) = \binom{2n-1}{2}x^2 + (4n - 1)x.$

Proof. By Figure 3.3, M_1, M_2 and M_3 join, so $3x$ appears in the Hosoya polynomial. Also, M_1, M_2 are adjacent to $2n - 2$ vertices (non-maximal subgroups) and so $2(2n-2)x$ is another term of the polynomial. The distance from M_3 to $H_1, H_2, \ldots, H_{2n-2}$ is 2 and we deduce that $\binom{2n-1}{2}x^2$. Hence, the assertion is clear. □

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Received October 7, 2017
Accepted November 21, 2017

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