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STRONGLY RELATIVE REGULAR MODULES AND EXCELLENT EXTENSIONS OF RINGS

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Abstract. We study the transfer of some strong relative regularity of modules under excellent extensions of rings. In particular, we show that if S is an excellent extension of a ring R, then R is strongly regular if and only if so is S.

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Key words. Strongly regular module, strongly regular ring, excellent extension of rings.

1. INTRODUCTION

Regular rings in the sense of von Neumann [13], and regular modules in the sense of Zelmanowitz [14] have been given a categorical generalization by Dăscălescu, Năstăsescu, Tudorache and Dăuş [5]. Given two objects M and N of a category $\mathcal{A}$, N is called M*-regular* if for every morphism $f : M \to N$ there exists a morphism $g : N \to M$ such that $f = fgf$ [5, Definition 2.1]. If the category $\mathcal{A}$ is abelian, then N is M-regular if and only if for every morphism $f : M \to N$, $\mathrm{Ker}(f)$ is a direct summand of M and $\mathrm{Im}(f)$ is a direct summand of N [5, Proposition 3.1]. The study of relative regular objects in categories was continued by Dăscălescu, Năstăsescu and Tudorache [4], Dăuş [6], Crivei and Kör [2], while regular homomorphisms and modules were further investigated by Kasch and Mader [7], Nicholson and Zhou [9], Lee, Rizvi and Roman [8].

An important subclass of the class of von Neumann regular rings consists of strongly regular rings, introduced by Arens and Kaplansky [1]. Recently, Crivei and Olteanu [3] introduced and studied a categorical generalization of strongly regular rings (see Definition 2.2). Their results can be further used for studying the transfer of some strong relative regularity properties of modules under excellent extensions of rings, which is the objective of the present paper. In our main theorem we prove that if S is an excellent extension of a ring R and M is a strongly R-regular right R-module, then M is a strongly S-regular right S-module. As a consequence, we extend a result of Parmenter and Stewart [10, Proposition 3.5], and show that if S is an excellent extension of a ring R, then R is strongly regular if and only if so is S. For an excellent extension S of a ring R, we also prove that if M is a strongly R-regular right R-module, then $M \otimes_R S$ is a strongly S-regular right S-module. Our properties complement some corresponding results known for von Neumann regular rings [10] and relative regular modules [6].

2. PRELIMINARIES

2.1. Strongly relative regular modules

We recall the definition of strongly regular ring and strongly relative regular object of an abelian category, which will be used in a module category.

Definition 2.1 ([1]). A ring R is called *strongly regular* if for every $a \in R$ there exists an element $b \in R$ such that $a = a^2 b$.

Definition 2.2 ([3]). Let M and N be objects of an abelian category $\mathcal{A}$. Then N is called:

(1) *strongly M-regular* if for every morphism $f : M \to N$, $\mathrm{Ker}(f)$ is a fully invariant direct summand of M and $\mathrm{Im}(f)$ is a fully invariant direct summand of N.
(2) *strongly self-regular* if N is strongly N-regular.

The next proposition shows how the above concepts relate.

Proposition 2.3 ([3]). *Let M be an object of an abelian category $\mathcal{A}$. Then M is strongly self-regular if and only if its endomorphism ring $\mathrm{End}_{\mathcal{A}}(M)$ is strongly regular.*

We also need the following results on the behaviour of strong relative regularity with respect to direct summands and direct sums.

Proposition 2.4 ([3]). *Let M and N be objects of an abelian category $\mathcal{A}$, M' a direct summand of M and N' a direct summand of N. If N is strongly M-regular, then N' is strongly M'-regular.*

Theorem 2.5 ([3]). *Let $\mathcal{A}$ be an abelian category.*

(1) *Let M and $N_1, \dots, N_n$ be objects of $\mathcal{A}$. Then $\bigoplus_{i=1}^n N_i$ is strongly M-regular if and only if N_i is strongly M-regular for every $i \in \{1, \dots, n\}$.*
(2) *Let $M_1, \dots, M_n$ and N be objects of $\mathcal{A}$. Then N is strongly $\bigoplus_{i=1}^n M_i$-regular if and only if N is strongly M_i-regular for every $i \in \{1, \dots, n\}$.*

2.2. Excellent extensions of rings

We recall some types of extensions of rings, which will be used in the sequel.

Definition 2.6 ([11]). Let R be a subring of a ring S. Then S is called a:

(1) *finite normalizing extension* of R if R and S have the same identity 1, and there exists a finite set $\{a_1, \dots, a_n\} \subseteq S$ such that $S = \sum_{i=1}^n a_i R$, $a_i R = R a_i$ for every $i \in \{1, \dots, n\}$. In this case $\{a_1, \dots, a_n\}$ is called the *set of R-normalizing generators* of S.
(2) *free normalizing extension* of R if it is a finite normalizing extension with a set $\{a_1, \dots, a_n\}$ of R-normalizing generators of S such that $a_1 = 1$ and S is free with basis $\{a_1, \dots, a_n\}$ both as a left and right R-module.

(3) *excellent extension of* R if it is a free normalizing extension and S is right R-projective (i.e., if N_S is a submodule of M_S such that N_R is a direct sumand of M_R, then N_S is a direct sumand of M_S).

3. RESULTS

We begin with the following useful remark on fully invariant submodules of a module.

REMARK 3.1. Let K be a submodule of a right R-module M. Then K is a fully invariant submodule of M if and only if for every homomorphism $h: M \to M$, $\mathrm{Im}(hk) = h(K) \subseteq K = \mathrm{Im}(k)$ if and only if for every homomorphism $h: M \to M$, $hk = k\alpha$ for some homomorphism $\alpha : K \to K$.

LEMMA 3.2. *Let $\varphi : R \to S$ be a ring homomorphism. Let M be a right S-module and K a direct summand of M such that K is a fully invariant R-submodule of the right R-module M. Then K is a fully invariant submodule of the right S-module M.*

Proof. Consider the following covariant functors between module categories:

- extension of scalars $\varphi^* : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ given on objects by $\varphi^*(M) = M \otimes_R S$;
- restriction of scalars $\varphi_* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ given on objects by $\varphi_*(N) = N$.

Then (φ^*, φ_*) is an adjoint pair of functors [12, Chapter IX, p.105]. Let $k : K \to M$ be the inclusion S-homomorphism and let $h : M \to M$ be an S-homomorphism. Then $\varphi_*(k) : \varphi_*(K) \to \varphi_*(M)$ is the inclusion R-homomorphism and $\varphi_*(h) : \varphi_*(M) \to \varphi_*(M)$. Since $\varphi_*(K)$ is a fully invariant R-submodule of M, by Remark 3.1 there exists some R-homomorphism $\alpha : \varphi_*(K) \to \varphi_*(K)$ such that $\varphi_*(h)\varphi_*(k) = \varphi_*(k)\alpha$. Since K is a direct summand of the right S-module M, there exists an S-homomorphism $p : M \to K$ such that $pk = 1_K$. Then $\varphi_*(phk) = \varphi_*(p)\varphi_*(h)\varphi_*(k) = \varphi_*(p)\varphi_*(k)\alpha = \varphi_*(pk)\alpha = \alpha$, which implies that $\varphi_*(hk) = \varphi_*(h)\varphi_*(k) = \varphi_*(k)\alpha = \varphi_*(k)\varphi_*(phk) = \varphi_*(kphk)$. Since the functor φ_* is faithful, it follows that $hk = kphk$, which shows that K is a fully invariant S-submodule of M, again by Remark 3.1. □

Now we give the main result of the paper.

THEOREM 3.3. *Let S be an extension of a ring R and let M be a right S-module.*

(1) *Suppose that S is a free normalizing extension of R. Then M is a strongly R-regular right R-module if and only if M is a strongly S-regular right R-module.*
(2) *Suppose that S is R-projective. If M is a strongly S-regular right R-module, then M is a strongly S-regular right S-module.*
(3) *Suppose that S is an excellent extension of R. If M is a strongly R-regular right R-module, then M is a strongly S-regular right S-module.*

Proof. (1) Let $\{a_1, \dots, a_n\}$ be the set of R-normalizing generators of S. Suppose that M is a strongly R-regular right R-module. For every $i \in \{1, \dots, n\}$, $a_iR \cong R$ as right R-modules, hence M is a strongly a_iR-regular right R-module. By Theorem 2.5, it follows that M is strongly $\bigoplus_{i=1}^n a_iR$-regular, that is, M is a strongly S-regular right R-module.

Conversely, suppose that M is a strongly S-regular right R-module. Since R is a direct summand of S, it follows that M is a strongly R-regular right R-module by Proposition 2.4.

(2) Let $f : S \to M$ be an S-homomorphism. Then f is also an R-homomorphism. Since M is a strongly S-regular right R-module, $\mathrm{Ker}(f)_R$ is a fully invariant direct summand of S_R and $\mathrm{Im}(f)_R$ is a fully invariant direct summand of M_R. Since S is right R-projective, it follows that $\mathrm{Ker}(f)_S$ is a direct summand of S_S and $\mathrm{Im}(f)_S$ is a direct summand of M_S. Moreover, $\mathrm{Ker}(f)$ is a fully invariant submodule of S_S and $\mathrm{Im}(f)$ is a fully invariant submodule of M_S by Lemma 3.2. Hence M is a strongly S-regular right S-module.

(3) This follows by (1) and (2). □

Parmenter and Stewart showed that if S is an excellent extension of a ring R, then R is von Neumann regular if and only if S is von Neumann regular [10, Corollary 3.3]. Also, they proved that if S is a finite normalizing extension of a ring R and S is strongly regular, then R is strongly regular [10, Proposition 3.5]. Our next corollary completes the above results and shows that strong regularity transfers in both ways in case of excellent extensions.

COROLLARY 3.4. *Let S be an excellent extension of a ring R. Then R is a strongly regular ring if and only if S is a strongly regular ring.*

Proof. Suppose that R is a strongly regular ring. Then R is a strongly self-regular right R-module by Proposition 2.3. For every $i \in \{1, \dots, n\}$, $a_iR \cong R$ as right R-modules, hence a_iR is a strongly R-regular right R-module. By Proposition 2.3 it follows that $S = \bigoplus_{i=1}^n a_iR$ is a strongly R-regular right R-module. Then S is a strongly S-regular right S-module by Theorem 3.3. Hence S is a strongly regular ring by Proposition 2.3.

Conversely, suppose that S is a strongly regular ring. Let $a \in R \subseteq S$. Then there exists $b \in S$ such that $a = a^2b$. Since S is a free extension of R with basis $\{a_1 = 1, a_2, \dots, a_n\}$, there exist unique $b_1, \dots, b_n \in R$ such that $b = b_1 + a_2b_2 + \dots + a_nb_n$. It follows that $a = a^2b_1$, which shows that R is a strongly regular ring. □

EXAMPLE 3.5. Corollary 3.4 does not hold if one only assumes that S is a finite (or even free) normalizing extension of R. Indeed, following [6, Remark 2.6], let $R = K$ be a field and $S = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$. Then S is a free normalizing extension of R which is not an excellent extension, R is clearly strongly regular, but S is not strongly regular.

COROLLARY 3.6. *Let S be an excellent extension of a ring R. If M is a strongly R-regular right R-module, then $M \otimes_R S$ is a strongly S-regular right S-module.*

Proof. Let $\{a_1, \ldots, a_n\}$ be the set of R-normalizing generators of S. Since M is a strongly R-regular right R-module, M^n is also a strongly R-regular right R-module by Theorem 2.5. But we have

$$M \otimes_R S \cong M \otimes_R \left(\bigoplus_{i=1}^{n} Ra_i \right) \cong M \otimes_R R^n \cong M^n.$$

Hence $M \otimes_R S$ is a strongly R-regular right R-module. Finally, $M \otimes_R S$ is a strongly S-regular right S-module by Theorem 3.3. □

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