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A GENERALIZATION OF STEFFENSEN'S INTEGRAL INEQUALITY FOR THE SUGENO INTEGRAL

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Abstract. In this paper we prove the Bellman generalization of Steffensen's integral inequality for the Sugeno integral

$$\left(⨍_0^1 fg\mathrm{d}\mu\right)^p \le ⨍_0^\lambda f^p\mathrm{d}\mu,$$

where f is a nonincreasing and left continuous function defined on $[0,1]$ with $f(0)=1$, $f(1)=0$, g is a nonincreasing function defined on $[0,1]$ with $0 \le g(t) \le 1$, for all $t \in [0,1]$, $\lambda = \left(⨍_0^1 g\mathrm{d}\mu\right)^p$, and $p \ge 1$.

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1. INTRODUCTION

The concept of a Sugeno integral (fuzzy integral) for a measurable function on a normalized monotone measure space was introduced by Sugeno [15], who also discussed some elementary properties of this integral. Further investigations of the integral were also pursued by Batle and Trillas [3], Wierzchon [18], Dubois and Prade [4], Grabisch et al. [6], and others.

These concepts were envisioned by Sugeno in his efforts to compare membership grade functions of fuzzy sets with probabilities. Since no direct comparison is possible, Sugeno conceived of the generalization of classical measures into fuzzy measures as an analogy of the generalization of classical (crisp) sets into fuzzy sets. Using this analogy, he coined for the nonclassical (nonadditive) measures the term fuzzy measures. Fuzzy measures, according to Sugeno, are obtained by replacing the additivity requirement of classical measures with the weaker requirements of increasing monotonicity (with respect to set inclusion) and continuity. The requirement of continuity was later found to be too restrictive and was replaced with the weaker requirement of semicontinuity. The term of fuzzy measure, in the sense of Sugeno, has been accepted by most researchers working in the area of generalized measures [16, 17].

It is well known that integral inequalities are instrumental in studying the qualitative analysis of solutions of differential and integral equations [10].

The study of inequalities for Sugeno integral was initiated by Román-Flores and Chalco-Cano [12]. Since then, the fuzzy integral counterparts of several

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classical inequalities, including the Chebyshev, Jensen, Minkowski and Hölder inequalities, are given by Flores-Franulič and Román-Flores [5], Agahi et al. [2, 1], Mesiar and Ouyang [8], Román-Flores et al. [13], and others.

Hong et al. [7] considered Steffensen's Integral inequality for Sugeno integral. In this paper, we also consider a similar type of Bellman's inequality for fuzzy integrals.

2. PRELIMINARIES

For convenience of the reader, we give a survey of the relevant materials from [17] and [16], without proofs, thus making our exposition self-contained.

Throughout this paper, X is a nonempty set, Σ is a σ-algebra of subsets of X, and all considered subsets belong to Σ.

Definition 2.1. A set function $\mu : \Sigma \to [0, \infty]$ is called a *fuzzy measure* if the following properties are satisfied

(i) $\mu(\varnothing) = 0$;
(ii) $A \subseteq B$ implies $\mu(A) \le \mu(B)$;
(iii) $A_1 \subseteq A_2 \subseteq ...$ implies $\mu\big(\bigcup_{n=1}^{\infty} A_n\big) = \lim_{n\to\infty} \mu(A_n)$;
(iv) $A_1 \supseteq A_2 \supseteq ...$ and $\mu(A_1) < \infty$ imply $\mu\big(\bigcap_{n=1}^{\infty} A_n\big) = \lim_{n\to\infty} \mu(A_n)$.

If $\mu(X) = 1$, then μ is called a normalized fuzzy measure. When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space. If μ is a fuzzy measure on X and $D \in \Sigma$, we denote by $\mathfrak{F}^{\mu}(D)$ the set of all non-negative μ-measurable functions $f : D \to [0, \infty]$.

For any given $f \in \mathfrak{F}^{\mu}(X)$, we write $F_\alpha = \{x : f(x) \ge \alpha\}$, where $\alpha \in [0, \infty]$.

Definition 2.2. Let $A \in \Sigma$, $f \in \mathfrak{F}^{\mu}(X)$. The *fuzzy integral* of f on A, with respect to μ, which is denoted by $⨍_A f \mathrm{d}\mu$, is defined by

$$⨍_A f \mathrm{d}\mu = \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \bigvee_{\alpha\ge 0} [\alpha \wedge \mu(A \cap F_\alpha)].$$

If $A = X$, then

$$⨍_X f \mathrm{d}\mu = \bigvee_{\alpha\ge 0} [\alpha \wedge \mu(F_\alpha)].$$

When $A = X$, the fuzzy integral may also be denoted by $⨍ f \mathrm{d}\mu$.

Sometimes, the fuzzy integral is also called Sugeno's integral in the literature. From now on, we make the convention that the appearance of a symbol $⨍_A f \mathrm{d}\mu$ implies that $A \in \Sigma$ and $f \in \mathfrak{F}^{\mu}(X)$.

The following theorem gives the most elementary properties of the fuzzy integral.

Theorem 2.3 ([16, Theorem 7.2]). (1) *If* $\mu(A) = 0$*, then* $⨍ f \mathrm{d}\mu = 0$*, for any* $f \in \mathfrak{F}^{\mu}(X)$.

(2) *If* $⨍_A f \mathrm{d}\mu = 0$*, then* $\mu(A \cap \{x : f(x) > 0\}) = 0$.

(3) *If* $f \le g$, *then* $⨍_A f\mathrm{d}\mu \le ⨍_A g\mathrm{d}\mu$.
(4) $⨍_A f\mathrm{d}\mu \le \mu(A)$.
(5) $⨍_A k\mathrm{d}\mu = k \wedge \mu(A)$, *where* k *is a nonnegetive constant.*
(6) $\mu(A \cap f \ge \alpha) \le \alpha \Rightarrow ⨍_A f\mathrm{d}\mu \ge \alpha$.
(7) $\mu(A \cap \{f \ge \alpha\}) \le \alpha \Rightarrow ⨍_A f\mathrm{d}\mu \le \alpha$.
(8) $⨍_A f\mathrm{d}\mu < \alpha \Leftrightarrow\ \exists\ \gamma < \alpha;\ \mu(A \cap \{f \ge y\}) < \alpha$.
(9) $⨍_A f\mathrm{d}\mu > \alpha \Leftrightarrow\ \exists\ \gamma > \alpha;\ \mu(A \cap \{f \ge y\}) > \alpha$.

Theorem 2.4 ([14, Lemma 1]). (Jensen type inequality) *Let* $g : [0,1] \to [0,\infty)$ *be a measurable function. Then*

$$\Big(⨍_0^1 g\mathrm{d}\mu\Big)^p \le ⨍_0^1 g^p\mathrm{d}\mu,$$

for all $p \ge 1$.

The following lemma and theorem are due to D. H. Hong.

Lemma 2.5 ([7, Lemma 1]). *Let* f *be a nonincreasing, convex function defined on* $[0,1]$ *with* $f(0) = 1$ *and* $f(1) = 0$, *and let* μ *be the Lebesgue measure on* $\mathbb{R}$. *Then*

$$⨍_{1-\lambda}^1 f\mathrm{d}\mu \le ⨍_0^1 \lambda f\mathrm{d}\mu,$$

for $\lambda \in [0,1]$.

Theorem 2.6 ([7, Theorem 1]). *Let* f *be a nonincreasing, convex function defined on* $[0,1]$ *with* $f(0) = 1$ *and* $f(1) = 0$, g *be a nonincreasing function defined on* $[0,1]$ *with* $0 \le g(t) \le 1$, *for all* $t \in [0,1]$, *and let* μ *be the Lebesgue measure on* $\mathbb{R}$. *If* $\lambda = ⨍_0^1 g\mathrm{d}\mu$, *then*

$$⨍_{1-\lambda}^1 f\mathrm{d}\mu \le ⨍_0^1 fg\mathrm{d}\mu \le ⨍_0^\lambda f\mathrm{d}\mu,$$

for $\lambda \in [0,1]$.

3. MAIN RESULTS

Steffensen [9] proved the following result. *Assume that two integrable functions* $f(t)$ *and* $g(t)$ *are defined on the interval* (a,b), *that* $f(t)$ *never increases, and that* $0 \le g(t) \le 1$ *on* (a,b). *Then*

$$\int_{b-\lambda}^b f(t)\mathrm{d}t \le \int_a^b f(t)g(t)\mathrm{d}t \le \int_a^{a+\lambda} f(t)\mathrm{d}t,$$

where $\lambda = \int_a^b g(t)\mathrm{d}t$.

Some generalizations of Steffensen's inequality may be found in the paper of B. Meidell and T. Hayashi [9]. J. E. Pečarić [11] showed that the Bellman generalization of Steffensen's inequality, with very simple modifications of the conditions, is also true.

THEOREM 3.1 ([11, Theorem 1]). *Let $f : [0,1] \to \mathbb{R}$ be a nonnegative and nonincreasing function and let $g : [0,1] \to \mathbb{R}$ be an integrable function such that $0 \le g(x) \le 1$ ($\forall x \in [0,1]$). If $p \ge 1$, then*

$$\Big(\int_0^1 f(t)g(t)\mathrm{d}t\Big)^p \le \int_0^\lambda f(t)^p \mathrm{d}t,$$

where $\lambda = \big(\int_0^1 g(t)\mathrm{d}t\big)^p$.

D. H. Hong [7] established the Steffensen type inequalities for the Sugeno integral. In this section, we shall obtain a result for the Sugeno integral, which is similar to Bellman's generalization of the Steffensen inequality.

The following example shows that the inequality

$$⨍_{1-\lambda}^1 f^p \mathrm{d}\mu \le \Big(⨍_0^1 fg\mathrm{d}\mu\Big)^p$$

does not hold in general.

EXAMPLE 3.2. Let μ be the usual Lebesgue measure on $\mathbb{R}$. If we take the functions $f(x) = -x+1$, and $g(x) = \frac{9}{10}$, for all $x \in [0,1]$, and $p = 2$, then, after some simple computations, we get that $\lambda = \big(⨍_0^1 (\frac{9}{10})\mathrm{d}\mu\big)^2 = \frac{81}{100}$,

$$\begin{aligned}⨍_{\frac{1}{10}}^1 f\mathrm{d}\mu = ⨍_{\frac{1}{10}}^1 (-x+1)\mathrm{d}\mu &= \bigvee_{\alpha \in [0,\frac{9}{10}]} [\alpha \wedge \mu(1-x \ge \alpha)] \\ &= \bigvee_{\alpha \in [0,\frac{9}{10}]} [\alpha \wedge (\frac{1}{10} - \alpha)] = \frac{1}{20},\end{aligned}$$

$$\begin{aligned}⨍_0^1 fg\mathrm{d}\mu = ⨍_0^1 \frac{9}{10}(-x+1)\mathrm{d}\mu &= \bigvee_{\alpha \in [0,\frac{9}{10}]} [\alpha \wedge \mu(\frac{9}{10}(1-x) \ge \alpha)] \\ &= \bigvee_{\alpha \in [0,\frac{9}{10}]} [\alpha \wedge (-\frac{10}{9}\alpha + 1)] \approx 0.474,\end{aligned}$$

and $⨍_{\frac{19}{100}}^1 (-x+1)^2 \mathrm{d}\mu \approx 0.280$. So

$$\Big(⨍_0^1 fg\mathrm{d}\mu\Big)^p = \Big(⨍_0^1 \frac{9}{10}(-x+1)\mathrm{d}\mu\Big)^2 < ⨍_{\frac{19}{100}}^1 (-x+1)^2\mathrm{d}\mu = ⨍_{1-\lambda}^1 f^p\mathrm{d}\mu.$$

If $p = 1$, then $\lambda = ⨍_0^1 (\frac{9}{10})\mathrm{d}\mu = \frac{9}{10}$ and

$$⨍_{\frac{1}{10}}^1 (-x+1)\mathrm{d}\mu < ⨍_0^1 \frac{9}{10}(-x+1)\mathrm{d}\mu.$$

In what follows, we show that some kind of generalization of the second part of Steffensen's inequality is valid for the Sugeno integral.

THEOREM 3.3. *Let f be a nonincreasing, left continuous function defined on $[0,1]$ with $f(0)=1$ and $f(1)=0$, g be a nonincreasing function defined on $[0,1]$ with $0 \le g(t) \le 1$, for all $t \in [0,1]$, and let μ be the Lebesgue measure on $\mathbb{R}$. If $p \ge 1$, then*

$$\Big(⨍_0^1 fg\mathrm{d}\mu\Big)^p \le ⨍_0^\lambda f^p\mathrm{d}\mu,$$

where $\lambda = \big(⨍_0^1 g\mathrm{d}\mu\big)^p$.

Proof. First, we show that $\lambda \wedge ⨍_0^1 f^p\mathrm{d}\mu = ⨍_0^\lambda f^p\mathrm{d}\mu$. Since f is nonincreasing and left continuous, $h := f^p$ is nonincreasing and left continuous, too. Therefore

$$\{x : h(x) \ge \alpha\} = [0, h^{-1}(\alpha)],$$

where $h^{-1}(y) = \sup\{x : h(x) \ge y\}$. We have that

$$\begin{aligned}
⨍_0^\lambda h\mathrm{d}\mu &= \bigvee_{\alpha\ge 0}\big(\alpha \wedge \mu([0,\lambda] \cap \{x : h(x) \ge \alpha\})\big)\\
&= \bigvee_{\alpha\ge 0}\big(\alpha \wedge \mu([0,\lambda] \cap [0, h^{-1}(\alpha)])\big)\\
&= \bigvee_{\alpha\ge 0}\big(\alpha \wedge \mu[0,\lambda] \wedge \mu[0, h^{-1}(\alpha)]\big)\\
&= \lambda \wedge \bigvee_{\alpha\ge 0}\big(\alpha \wedge \mu[0, h^{-1}(\alpha)]\big)\\
&= \lambda \wedge ⨍_0^1 h\mathrm{d}\mu.
\end{aligned}$$

If $\lambda = 0$ or $\lambda = 1$, then the result is trivial. Next, assume that $0 < \lambda < 1$. Let $⨍_0^1 f^p\mathrm{d}\mu \le \lambda < 1$. Then, by Theorem 2.3 (3) and the Jensen type inequality,

$$\Big(⨍_0^1 gf\mathrm{d}\mu\Big)^p \le ⨍_0^1 g^p f^p\mathrm{d}\mu \le ⨍_0^1 f^p\mathrm{d}\mu.$$

Since $⨍_0^\lambda f^p\mathrm{d}\mu = \lambda \wedge ⨍_0^1 f^p\mathrm{d}\mu = ⨍_0^1 f^p\mathrm{d}\mu$, we obtain

$$\Big(⨍_0^1 fg\mathrm{d}\mu\Big)^p \le ⨍_0^\lambda f^p\mathrm{d}\mu.$$

If $0 < \lambda < ⨍_0^1 f^p\mathrm{d}\mu$, then $⨍_0^\lambda f^p\mathrm{d}\mu = \lambda \wedge ⨍_0^1 f^p\mathrm{d}\mu = \lambda$. So $\big(⨍_0^1 gf\mathrm{d}\mu\big)^p \le \big(⨍_0^1 g\mathrm{d}\mu\big)^p = \lambda$ and the proof is complete. $\square$

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