MATHEMATICA, 58 (81), N° 1–2, 2016, pp. 101–110

SOME GENERALIZATIONS OF AN INEQUALITY DUE TO A. BEURLING

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Abstract. The purpose of the paper is to provide several generalizations of an inequality due to A. Beurling. These generalizations deal with higher order differentiable functions, as well as with functions of two variables defined on rectangular domains.

MSC 2010. 26D15.
Key words. Beurling's inequality, higher order differentiable function, rectangular domain.

1. INTRODUCTION

C. P. Niculescu [10] proved the following result.

Theorem 1.1 (C. P. Niculescu [10]). *Let E be a Banach space, and let $u : [a,b] \to E$ be a twice differentiable function such that u'' is Bochner integrable. Then*

$$\max\{\|u(a)\|,\ \|u(b)\|\} + \frac{b-a}{4}\int_a^b \|u''(t)\|\mathrm{d}t \geq \sup_{t\in[a,b]} \|u(t)\|. \tag{1.1}$$

In particular, inequality (1.1) ensures that under the assumptions of Theorem 1.1 one has

$$\int_a^b \|u''(t)\|\mathrm{d}t \geq \frac{4}{b-a} \sup_{t\in[a,b]} \|u(t)\| \tag{1.2}$$

whenever u satisfies $u(a) = u(b) = 0$. In the special case of real-valued functions, inequality (1.2) is attributed to A. Beurling (see, for instance, [1, 6, 7] or the monograph [9, p. 305]) and it has attracted the interest of numerous mathematicians. Several improvements and generalizations of (1.2) have been established, especially in the context of higher order differentiable functions (see [4, 11, 12]).

The purpose of the present paper is to present other generalizations of (1.2). More precisely, in section 2 we prove a new generalization of inequality (1.2) for higher order differentiable functions, while in section 3 we establish two generalizations of (1.2) for rectangular domains.

2. A BEURLING-TYPE INEQUALITY FOR HIGHER ORDER DIFFERENTIABLE FUNCTIONS

THEOREM 2.1. *Let $n \geq 2$ be a natural number, and let $u \in C^{n-2}[a,b] \cap C^n(a,b)$ be a function such that*

$$u^{(k)}(a) = u^{(k)}(b) = 0 \quad \text{for all } k = 0, 1, \ldots, n-2. \tag{2.1}$$

Then one has

$$\int_a^b \left| u^{(n)}(t) \right| \mathrm{d}t \geq \frac{2^{\alpha_n}(n-1)!}{(b-a)^{n-1}} \sup_{t \in [a,b]} |u(t)|, \tag{2.2}$$

where

$$\alpha_n := \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

Proof. Let t_0 be a point in $[a,b]$ such that

$$|u(t_0)| = \sup_{t \in [a,b]} |u(t)|.$$

If $t_0 = a$ or $t_0 = b$, then $u(t) = 0$ for all $t \in [a,b]$ due to (2.1), and inequality (2.2) is obvious. Suppose next that $t_0 \in (a,b)$. Without restricting the generality we may assume that $u(t_0) \geq 0$ (otherwise we replace u by $-u$). By the Taylor-Lagrange mean value theorem it results the existence of two points $\xi_1 \in (a, t_0)$ and $\xi_2 \in (t_0, b)$ such that

$$u(t_0) = \sum_{k=0}^{n-2} \frac{u^{(k)}(a)}{k!} (t_0 - a)^k + \frac{u^{(n-1)}(\xi_1)}{(n-1)!} (t_0 - a)^{n-1}$$

and

$$u(t_0) = \sum_{k=0}^{n-2} \frac{u^{(k)}(b)}{k!} (t_0 - b)^k + \frac{u^{(n-1)}(\xi_2)}{(n-1)!} (t_0 - b)^{n-1}.$$

Taking into account (2.1) we deduce that

$$u^{(n-1)}(\xi_1) = \frac{(n-1)!\, u(t_0)}{(t_0 - a)^{n-1}} \tag{2.4}$$

and

$$(-1)^{n-1} u^{(n-1)}(\xi_2) = \frac{(n-1)!\, u(t_0)}{(b - t_0)^{n-1}}\,. \tag{2.5}$$

Depending on n, we have the following two possible cases.

Case I: n is even. Then we have

$$\begin{aligned}
(2.6)\qquad \int_a^b \left|u^{(n)}(t)\right| \mathrm{d}t &\geq \int_{\xi_1}^{\xi_2} \left|u^{(n)}(t)\right| \mathrm{d}t \geq \left|\int_{\xi_1}^{\xi_2} u^{(n)}(t)\mathrm{d}t\right| \\
&= \left|u^{(n-1)}(\xi_1) - u^{(n-1)}(\xi_2)\right| \\
&= (n-1)!\, u(t_0) \left(\frac{1}{(t_0-a)^{n-1}} + \frac{1}{(b-t_0)^{n-1}}\right).
\end{aligned}$$

Since the function $f(x) := 1/x^{n-1}$ is convex on $(0, \infty)$, by Jensen's inequality we have

$$f(t_0 - a) + f(b - t_0) \geq 2f\left(\frac{b-a}{2}\right),$$

i.e.,

$$(2.7)\qquad \frac{1}{(t_0-a)^{n-1}} + \frac{1}{(b-t_0)^{n-1}} \geq \frac{2^n}{(b-a)^{n-1}}.$$

By (2.6) and (2.7) it follows that (2.2) holds.

Case II: n is odd. Since $u^{(n-2)}(a) = u^{(n-2)}(b) = 0$, it follows that there is some $\xi \in (a, b)$ such that $u^{(n-1)}(\xi) = 0$. If $\xi = \xi_1$ or $\xi = \xi_2$, then by (2.4) and (2.5) it follows that $u(t_0) = 0$, whence (2.2) holds. If $\xi \neq \xi_1$ and $\xi \neq \xi_2$ then we have

$$\begin{aligned}
2\int_a^b \left|u^{(n)}(t)\right| \mathrm{d}t &\geq \int_{\min(\xi,\xi_1)}^{\max(\xi,\xi_1)} \left|u^{(n)}(t)\right| \mathrm{d}t + \int_{\min(\xi,\xi_2)}^{\max(\xi,\xi_2)} \left|u^{(n)}(t)\right| \mathrm{d}t \\
&\geq \left|\int_{\xi}^{\xi_1} u^{(n)}(t)\mathrm{d}t\right| + \left|\int_{\xi}^{\xi_2} u^{(n)}(t)\mathrm{d}t\right| \\
&= \left|u^{(n-1)}(\xi_1)\right| + \left|u^{(n-1)}(\xi_2)\right| \\
&\geq u^{(n-1)}(\xi_1) + u^{(n-1)}(\xi_2) \\
&= (n-1)!\, u(t_0) \left(\frac{1}{(t_0-a)^{n-1}} + \frac{1}{(b-t_0)^{n-1}}\right).
\end{aligned}$$

Using (2.7) we deduce that

$$\int_a^b \left|u^{(n)}(t)\right| \mathrm{d}t \geq \frac{2^{n-1}(n-1)!}{(b-a)^{n-1}}\, u(t_0),$$

i.e., (2.2) holds in this case, too. □

In fact, Theorem 2.1 works even in the general framework of Banach spaces.

THEOREM 2.2. *Let $E \neq \{0\}$ be a Banach space, let $n \geq 2$ be a natural number, and let $u : [a, b] \to E$ be a function satisfying the following conditions:*

(i) $u \in C^{n-2}([a, b], E) \cap C^n((a, b), E)$;

(ii) $u^{(k)}(a) = u^{(k)}(b) = 0$ *for all* $k = 0, 1, \ldots, n-2$;

(iii) $u^{(n)}$ *is Bochner integrable.*

Then

$$\int_a^b \left\| u^{(n)}(t) \right\| \mathrm{d}t \geq \frac{2^{\alpha_n}(n-1)!}{(b-a)^{n-1}} \sup_{t\in[a,b]} \|u(t)\|,$$

where α_n *is defined by* (2.3)*.*

Proof. According to the Weierstrass theorem, there exists a point $t_0 \in [a,b]$ such that $\|u(t_0)\| = \sup_{t\in[a,b]} \|u(t)\|$. Let x^* be an arbitrary linear functional in the dual space E^*, whose norm equals 1. By virtue of Theorem 2.1 we have

$$\begin{aligned} |x^*(u(t_0))| &\leq \sup_{t\in[a,b]} |x^*(u(t))| \leq \frac{(b-a)^{n-1}}{2^{\alpha_n}(n-1)!} \int_a^b |x^*(u^{(n)}(t))|\mathrm{d}t \\ &\leq \frac{(b-a)^{n-1}}{2^{\alpha_n}(n-1)!} \int_a^b \|u^{(n)}(t)\|\mathrm{d}t. \end{aligned}$$

Choosing now $x^* \in E^*$ such that $\|x^*\| = 1$ and $|x^*(u(t_0))| = \|u(t_0)\|$ we obtain the conclusion. □

3. MULTIVARIATE BEURLING-TYPE INEQUALITIES FOR RECTANGULAR DOMAINS

The purpose of this section is to establish two multivariate generalizations of Beurling's inequality for functions defined on rectangular domains in $\mathbb{R}^2$. Throughout this section $A := [a,b] \times [c,d]$ denotes such a rectangle.

THEOREM 3.1. *Let* $u : A \to \mathbb{R}$ *be a function satisfying the following conditions:*

(i) u *is continuous on* A*;*
(ii) *all partial derivatives* u_x*,* u_{xy}*,* u_{xyx}*, and* u_{xyxy} *exist and are continuous on* A*;*
(iii) $u(\partial A) = u_{xy}(\partial A) = \{0\}$.

Then one has

$$\iint_A |u_{xyxy}(x,y)|\mathrm{d}x\mathrm{d}y \geq \frac{16}{(b-a)(d-c)} \sup_{(x,y)\in A} |u(x,y)|. \tag{3.1}$$

Proof. Let (x_0, y_0) be a point in A such that $|u(x_0,y_0)| = \sup_{(x,y)\in A} |u(x,y)|$. If $(x_0,y_0) \in \partial A$, then $u(x,y) = 0$ for all $(x,y) \in A$ due to (iii), and in this case (3.1) is obvious. Suppose next that (x_0,y_0) is an interior point of A. Without restricting the generality we may assume that $u(x_0,y_0) \geq 0$ (otherwise we replace u by $-u$). By the mean value theorem for functions of two variables it results the existence of a point $(\xi_1,\xi_2) \in (a,x_0) \times (c,y_0)$ such that

$$u(a,c) - u(x_0,c) - u(a,y_0) + u(x_0,y_0) = (x_0-a)(y_0-c)u_{xy}(\xi_1,\xi_2).$$

Taking into account (iii), we get

$$\frac{u(x_0,y_0)}{(x_0-a)(y_0-c)} = u_{xy}(\xi_1,\xi_2). \tag{3.2}$$

Set $A_1 := [a, \xi_1] \times [c, \xi_2]$. Due to (iii) we also have

$$u_{xy}(\xi_1, \xi_2) = \iint_{A_1} u_{xyxy}(x, y)\mathrm{d}x\mathrm{d}y \le \iint_{A_1} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

By (3.2) and (3.3) we deduce that

$$\frac{u(x_0, y_0)}{(x_0 - a)(y_0 - c)} \le \iint_{A_1} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y. \tag{3.4}$$

Analogously, there exist points $(\eta_1, \eta_2) \in (x_0, b) \times (y_0, d)$, $(\xi_1', \xi_2') \in (a, x_0) \times (y_0, d)$, $(\eta_1', \eta_2') \in (x_0, b) \times (c, y_0)$ such that

$$\frac{u(x_0, y_0)}{(b - x_0)(d - y_0)} = u_{xy}(\eta_1, \eta_2) \le \iint_{A_2} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y, \tag{3.5}$$

$$\frac{u(x_0, y_0)}{(x_0 - a)(d - y_0)} = -u_{xy}(\xi_1', \xi_2') \le \iint_{A_3} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y, \tag{3.6}$$

$$\frac{u(x_0, y_0)}{(b - x_0)(y_0 - c)} = -u_{xy}(\eta_1', \eta_2') \le \iint_{A_4} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y, \tag{3.7}$$

where $A_2 := [\eta_1, b] \times [\eta_2, d]$, $A_3 := [a, \xi_1'] \times [\xi_2', d]$, and $A_4 := [\eta_1', b] \times [c, \eta_2']$. By adding the inequalities (3.4), (3.5), (3.6), and (3.7), and taking into account that

$$\sum_{i=1}^{4} \iint_{A_i} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y \le \iint_{A} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y,$$

we obtain

$$\begin{aligned} \iint_{A} |u_{xyxy}(x, y)|\mathrm{d}x\mathrm{d}y \\ \ge u(x_0, y_0)\Bigg(&\frac{1}{(x_0 - a)(y_0 - c)} + \frac{1}{(b - x_0)(d - y_0)} \\ &+ \frac{1}{(x_0 - a)(d - y_0)} + \frac{1}{(b - x_0)(y_0 - c)}\Bigg). \end{aligned} \tag{3.8}$$

It is obvious that the Hessian matrix of the function $f(x, y) := \frac{1}{xy}$ is positive definite on $(0, \infty) \times (0, \infty)$, whence f is convex on $(0, \infty) \times (0, \infty)$. By Jensen's inequality we have

$$\begin{aligned} &f(x_0 - a, y_0 - c) + f(b - x_0, d - y_0) + f(x_0 - a, d - y_0) + f(b - x_0, y_0 - c) \\ &\quad \ge 4f\left(\frac{b - a}{2}, \frac{d - c}{2}\right), \end{aligned}$$

i.e.,

$$\frac{1}{(x_0-a)(y_0-c)}+\frac{1}{(b-x_0)(d-y_0)}+\frac{1}{(x_0-a)(d-y_0)}+\frac{1}{(b-x_0)(y_0-c)}$$
$$\geq \frac{16}{(b-a)(d-c)}\,.$$

This inequality and (3.8) prove (3.1). □

Next we present our second Beurling-type inequality for rectangular domains, which is also the main result of our paper.

THEOREM 3.2. *Let $u : A \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) *u is continuous on A;*
(ii) *all partial derivatives u_x, u_{xx}, u_{xxy}, and u_{xxyy} exist and are continuous on A;*
(iii) *$u(a,y) = u(b,y) = 0$ for all $y \in [c,d]$;*
(iv) *$u_{xx}(x,c) = u_{xx}(x,d) = 0$ for all $x \in [a,b]$.*

Then the following inequality holds

$$\iint_A |u_{xxyy}(x,y)|\mathrm{d}x\mathrm{d}y \geq \frac{16}{(b-a)(d-c)} \sup_{(x,y)\in A} |u(x,y)|. \tag{3.9}$$

In the proof of Theorem 3.2 we need the following two auxiliary results.

LEMMA 3.3. *Let $g : [a,b] \to \mathbb{R}$ be a function which is twice continuously differentiable on $[a,b]$, and let $f(x) := |g(x)|$ for all $x \in [a,b]$. Then the following assertions are true:*

1° *f is right differentiable on $[a,b)$ and*

$$f'_+(x) = \big(\operatorname{sgn} g(x)\big) g'(x) + \big(1 - |\operatorname{sgn} g(x)|\big)\,|g'(x)| \tag{3.10}$$

for all $x \in [a,b)$. In particular, $|f'_+(x)| = |g'(x)|$ for all $x \in [a,b)$.

2° *f'_+ is right differentiable on $[a,b)$ and*

$$\begin{aligned} f''_+(x) &:= \big(f'_+\big)'_+(x) \\ &= \big(\operatorname{sgn} g(x)\big) g''(x) + \big(1 - |\operatorname{sgn} g(x)|\big)\big(\operatorname{sgn} g'(x)\big)\, g''(x) \\ &\qquad + \big(1 - |\operatorname{sgn} g(x)|\big)\big(1 - |\operatorname{sgn} g'(x)|\big)\,|g''(x)| \end{aligned} \tag{3.11}$$

for all $x \in [a,b)$. In particular, $|f''_+(x)| = |g''(x)|$ for all $x \in [a,b)$.

Proof. 1° Let $x_0 \in [a,b)$ arbitrarily chosen. If $g(x_0) \neq 0$, then there exists $r > 0$ such that $[x_0, x_0+r) \subseteq [a,b)$ and $\operatorname{sgn} g(x) = \operatorname{sgn} g(x_0)$ for all $x \in [x_0, x_0+r)$. Then

$$f(x) = \big(\operatorname{sgn} g(x)\big)\, g(x) = \big(\operatorname{sgn} g(x_0)\big)\, g(x), \quad \text{for all } x \in [x_0, x_0+r).$$

Therefore, f is right differentiable at x_0 and $f'_+(x_0) = \big(\operatorname{sgn} g(x_0)\big)\, g'(x_0)$, proving the validity of (3.10).

If $g(x_0) = 0$, then for all $x \in [a, b)$ with $x > x_0$ we have

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{|g(x)|}{x - x_0} = \left| \frac{g(x) - g(x_0)}{x - x_0} \right| .$$

Therefore, there exists the limit

$$\lim_{x \searrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \left| \lim_{x \searrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \right| = |g'(x_0)|.$$

In other words, f is right differentiable at x_0 and $f'_+(x_0) = |g'(x_0)|$. This shows that (3.10) holds in this case, too.

2° Let $x_0 \in [a, b)$. If $g(x_0) \neq 0$, then there exists $r > 0$ such that $[x_0, x_0 + r) \subseteq [a, b)$ and $\operatorname{sgn} g(x) = \operatorname{sgn} g(x_0) = \pm 1$ for all $x \in [x_0, x_0 + r)$. By 1° it follows that

$$f'_+(x) = \big(\operatorname{sgn} g(x)\big)\, g'(x) = \big(\operatorname{sgn} g(x_0)\big)\, g'(x) \quad \text{for all } x \in [x_0, x_0 + r).$$

Therefore, f'_+ is right differentiable at x_0 and $f''_+(x_0) = \big(\operatorname{sgn} g(x_0)\big)\, g''(x_0)$, proving the validity of (3.11).

Now suppose that $g(x_0) = 0$. By 1° it follows that $f'_+(x_0) = |g'(x_0)|$. Assume first that $g'(x_0) \neq 0$. Then there exists $r > 0$ such that $[x_0, x_0 + r) \subseteq [a, b)$ and $\operatorname{sgn} g'(x) = \operatorname{sgn} g'(x_0) = \pm 1$ for all $x \in [x_0, x_0 + r)$. Therefore, g is strictly monotone on $[x_0, x_0 + r)$, whence $\operatorname{sgn} g(x) = \operatorname{sgn} g'(x_0)$ for all $x \in (x_0, x_0 + r)$. By 1° it results that

$$f'_+(x) = \big(\operatorname{sgn} g'(x_0)\big)\, g'(x) \quad \text{for all } x \in [x_0, x_0 + r).$$

Consequently, f'_+ is right differentiable at x_0 and $f''_+(x_0) = \big(\operatorname{sgn} g'(x_0)\big)\, g''(x_0)$, i.e., (3.11) holds in this case, too.

Finally, assume that $g(x_0) = g'(x_0) = 0$. Depending on $g''(x_0)$, we distinguish the following two possible cases.

Case I: $g''(x_0) \neq 0$.

Then the continuity of g'' ensures the existence of an $r > 0$ such that $[x_0, x_0 + r) \subseteq [a, b)$ and $\operatorname{sgn} g''(x) = \operatorname{sgn} g''(x_0) = \pm 1$ for all $x \in [x_0, x_0 + r)$. It results that g' is strictly monotone on $[x_0, x_0 + r)$, whence $\operatorname{sgn} g'(x) = \operatorname{sgn} g''(x_0)$ for all $x \in (x_0, x_0 + r)$, because $g'(x_0) = 0$. Thus g is strictly monotone on $[x_0, x_0+r)$ and $\operatorname{sgn} g(x) = \operatorname{sgn} g''(x_0)$ for all $x \in (x_0, x_0+r)$. By 1° it follows that $f'_+(x) = \big(\operatorname{sgn} g''(x_0)\big)\, g'(x)$ for all $x \in [x_0, x_0+r)$. Consequently, f'_+ is right differentiable at x_0 and $f''_+(x_0) = \big(\operatorname{sgn} g''(x_0)\big)\, g''(x_0) = |g''(x_0)|$, proving the validity of (3.11).

Case II: $g''(x_0) = 0$.

Taking into consideration 1°, for every $x \in (x_0, b)$ we have

$$\left| \frac{f'_+(x) - f'_+(x_0)}{x - x_0} \right| = \frac{|f'_+(x)|}{x - x_0} = \frac{|g'(x)|}{x - x_0} = \left| \frac{g'(x) - g'(x_0)}{x - x_0} \right| .$$

Since $\lim_{x \searrow x_0} \frac{g'(x) - g'(x_0)}{x - x_0} = g''(x_0) = 0$, we deduce that f'_+ is right differentiable at x_0 and $f''_+(x_0) = 0 = |g''(x_0)|$. This completes the proof. $\square$

LEMMA 3.4. *Let $f : [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$, right differentiable on (a,b), and satisfies $f(a) = f(b) = 0$. Then*

$$\sup_{x\in[a,b]} |f(x)| \le \frac{b-a}{4} \bigvee_{a+0}^{b-0} (f'_+),$$

where $\displaystyle\bigvee_{a+0}^{b-0}(f'_+) = \lim_{\alpha\searrow a,\, \beta\nearrow b} \bigvee_{\alpha}^{\beta}(f'_+)$.

Proof. Let $x_0 \in [a,b]$ be a point such that $|f(x_0)| = \sup_{x\in[a,b]} |f(x)|$. If $x_0 = a$ or $x_0 = b$, then f vanishes on $[a,b]$ and there is nothing to prove. Suppose that $x_0 \in (a,b)$. Without losing the generality we may assume that $f(x_0) \ge 0$ (otherwise we replace f by $-f$). According to the mean-value theorem for right differentiable functions (see, for instance, [2, Theorem 3.2.9]) we have

$$\inf_{x\in(a,x_0)} f'_+(x) \le \frac{f(x_0) - f(a)}{x_0 - a} = \frac{f(x_0)}{x_0 - a} \le \sup_{x\in(a,x_0)} f'_+(x)$$

and, analogously,

$$-\sup_{x\in(x_0,b)} f'_+(x) \le \frac{f(x_0)}{b - x_0} \le -\inf_{x\in(x_0,b)} f'_+(x).$$

Now let $\varepsilon > 0$ be arbitrarily chosen. By the above inequalities it follows that there exist two points $x_1 \in (a,x_0)$ and $x_2 \in (x_0,b)$ such that

$$\frac{f(x_0)}{x_0 - a} - \varepsilon \le f'_+(x_1) \quad \text{and} \quad \frac{f(x_0)}{b - x_0} - \varepsilon \le -f'_+(x_2),$$

whence

$$f(x_0)\left(\frac{1}{x_0 - a} + \frac{1}{b - x_0}\right) \le 2\varepsilon + f'_+(x_1) - f'_+(x_2) \le 2\varepsilon + \bigvee_{a+0}^{b-0}(f'_+).$$

Letting $\varepsilon \searrow 0$ and taking into account (2.7), we get

$$\bigvee_{a+0}^{b-0}(f'_+) \ge f(x_0)\left(\frac{1}{x_0 - a} + \frac{1}{b - x_0}\right) \ge \frac{4}{b-a} f(x_0).$$

$\square$

Proof of Theorem 3.2. Let (x_0, y_0) be a point in A such that

$$|u(x_0, y_0)| = \sup_{(x,y)\in A} |u(x,y)|,$$

and let $u_0 : [a,b] \to \mathbb{R}$ be the function defined by $u_0(x) := u(x, y_0)$. Then $u_0 \in C^2[a,b]$ and $u_0(a) = u_0(b) = 0$, due to (ii) and (iii). By Theorem 2.1 we have

$$|u(x_0,y_0)| = \sup_{x\in[a,b]} |u_0(x)| \le \frac{b-a}{4}\int_a^b |u_0''(x)|\mathrm{d}x,$$

i.e.,

$$|u(x_0,y_0)| \le \frac{b-a}{4}\int_a^b |u_{xx}(x,y_0)|\mathrm{d}x.$$

Further, let $F:[c,d]\to\mathbb{R}$ be the function defined by

$$F(y) := \int_a^b |u_{xx}(x,y)|\mathrm{d}x.$$

Since u_{xx} is continuous on A, it follows that F is continuous on $[c,d]$. Let $y_1\in[c,d]$ be a point such that $F(y_1)=\sup_{y\in[c,d]}F(y)$. Then we have

$$|u(x_0,y_0)| \le \frac{b-a}{4}F(y_0) \le \frac{b-a}{4}F(y_1). \tag{3.12}$$

Next, let $f:A\to\mathbb{R}$ be the function defined by $f(x,y):=|u_{xx}(x,y)|$. Since $u_{xx}(x,\cdot)$ is differentiable on $[c,d]$ for every $x\in[a,b]$, by Lemma 3.3 it follows that $f(x,\cdot)$ is right differentiable on $[c,d)$ for all $x\in[a,b]$ and

$$\left|\left(f_+'\right)_y(x,y)\right| = |u_{xxy}(x,y)| \quad \text{for all } (x,y)\in[a,b]\times[c,d).$$

Because u_{xxy} is continuous on A, a standard argument based on the dominated convergence theorem shows that F is right differentiable on $[c,d)$ and

$$F_+'(y) = \int_a^b \left(f_+'\right)_y(x,y)\mathrm{d}x \quad \text{for all } y\in[c,d).$$

By Lemma 3.4 it follows that

$$F(y_1) \le \frac{d-c}{4}\bigvee_c^{d-0}(F_+'). \tag{3.13}$$

Combining (3.12) and (3.13) we deduce that

$$|u(x_0,y_0)| \le \frac{(b-a)(d-c)}{16}\bigvee_c^{d-0}(F_+'). \tag{3.14}$$

By applying once again Lemma 3.3 it follows that for all $x\in[a,b]$ the function $\left(f_+'\right)_y(x,\cdot)$ is right differentiable on $[c,d)$ and

$$\left|\left(f_+''\right)_{yy}(x,y)\right| = |u_{xxyy}(x,y)| \quad \text{for all } (x,y)\in[a,b]\times[c,d).$$

In the above equality we use the notation $\left(f_+''\right)_{yy} := \left(\left(\left(f_+'\right)_y\right)_+'\right)_y$. The continuity of u_{xxyy} on A together with the dominated convergence theorem ensure that F_+' is right differentiable on $[c,d)$ and $F_+''(y)=\int_a^b \left(f_+''\right)_{yy}(x,y)\mathrm{d}x$, for all $y\in[c,d)$.

Now we have

$$\begin{aligned}\bigvee_{c}^{d-0}(F'_+) &\leq \int_c^d |F''_+(y)|\mathrm{d}y = \int_c^d \left|\int_a^b \left(f''_+\right)_{yy}(x,y)dx\right| \mathrm{d}y \\ &\leq \int_c^d \left(\int_a^b \left|\left(f''_+\right)_{yy}(x,y)\right| \mathrm{d}x\right) \mathrm{d}y \\ &= \int_c^d \left(\int_a^b |u_{xxyy}(x,y)|\,\mathrm{d}x\right) \mathrm{d}y,\end{aligned}$$

hence

$$\bigvee_{c}^{d-0}(F'_+) \leq \iint_A |u_{xxyy}(x,y)|\,\mathrm{d}x\mathrm{d}y. \tag{3.15}$$

By (3.14) and (3.15) we conclude that (3.9) holds. □

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Received September 8, 2016
Accepted January 25, 2017

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