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ON HALF-SYNCHRONIZED SYSTEMS

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Abstract. A subclass of coded systems containing synchronized systems is the family of half-synchronized systems. In this note, we show that the property 'half-synchronized' lifts under hyperbolic maps. This enables us to define an equivalence relation on the set of half-synchronized systems. Also, when the domain is half-synchronized, we show that right-closing a.e., 1-1 a.e. factor maps have a strong decoding property.

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1. INTRODUCTION

Coded systems were defined by Blanchard and Hansel [1] as a generalization of sofic shifts. Amongst subshifts, a well-known subclass of coded systems is the family of synchronized systems. The purpose of this paper is to generalize parts of the synchronized theory to a certain subclass of coded systems, the so-called half-synchronized systems. The half-synchronized systems often serve as a landmark within the coded systems, showing the difficulties in extending the class of synchronized systems and keeping a satisfactory analog to the sofic shifts theory.

The study of conjugacy is of interest in symbolic dynamics. But there is no general algorithm for deciding whether two shift spaces are conjugate. Thus it makes sense to ask if there is an equivalence relation, weaker than conjugacy, on some subclasses of subshifts. The investigation for the existence of common extensions, where the factor maps have certain properties, is of interest in coded systems and has a long history. Amongst the properties, we are interested in hyperbolicity. Common extensions with hyperbolic maps lead to a version of almost conjugacy for coded systems. Fiebig showed in [3] that, having a common coded (or synchronized) hyperbolic extension, is an equivalence relation on the set of coded (or synchronized) systems. In Section 3, first we prove that 'being half-synchronized' is an invariant for hyperbolic maps (Theorem 3.3). Then, we show that a common half-synchronized hyperbolic extension is an equivalence relation on the set of half-synchronized systems (Theorem 3.4).

Closing maps are important in coding theory [6, 7]. They also have a natural description in hyperbolic dynamics [2]: right-closing (resp., left-closing) maps are injective on unstable (resp., stable) sets. When the domain is a synchronized system, right-closing a.e., 1-1 a.e. factor maps have a strong decoding property [4]. This property gives rise to finitary regular isomorphisms between general shift spaces. In section 4, we generalize this result to half-synchronized systems (Theorem 4.2).

2. BACKGROUND AND NOTATIONS

Let $\mathcal{A}$ be a non-empty finite set. The *full $\mathcal{A}$-shift*, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the set of all bi-infinite sequences of symbols from $\mathcal{A}$. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The *shift map* on $\mathcal{A}^{\mathbb{Z}}$ is the map σ where $\sigma(\{x_i\}) = \{y_i\}$ is defined by $y_i = x_{i+1}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is the *full shift* and any closed invariant subset of it is called a *shift space*.

Denote by $\mathcal{B}_n(X)$ the set of all admissible n-blocks and denote by $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$ the *language* of X. For $u \in \mathcal{B}(X)$, let the *cylinder* $[u]$ be the set $\{x \in X : x_{[l,l+|u|-1]} = u\}$.

Suppose X is a subshift over the alphabet $\mathcal{A}$. For m, $n \in \mathbb{Z}$ with $-m \leq n$, define the $(m + n + 1)$-*block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ by

$$y_i = \Phi(x_{i-m}x_{i-m+1}...x_{i+n}) = \Phi(x_{[i-m,i+n]}) \tag{1}$$

where y_i is a symbol in the alphabet $\mathcal{D}$. The map $\varphi = \Phi_{\infty}^{[-m,n]} : X \to \mathcal{D}^{\mathbb{Z}}$, defined by $y = \varphi(x)$ with y_i given by (1), is called the *sliding block code* (or *code*) induced by Φ. So there is $N \geq 0$ such that $\varphi(x)_0$ is determined by $x_{[-N,N]}$. We call $2N + 1$ a coding length for φ. If $m = n = 0$, then φ is an 1-*block code* and $\varphi = \Phi_{\infty}$. An onto (resp., invertible) code is called a *factor map* (resp., a *conjugacy*).

A point $x \in X$ is *doubly transitive* if every block in X appears in x infinitely often to the left and to the right. Let $\varphi : X \to Y$ be a factor map. If there is a positive integer d such that every doubly transitive point of Y has d pre-images, then we call d the *degree* of φ.

Let G be a directed graph and $\mathcal{V}$ (resp., $\mathcal{E}$) the set of its vertices (resp., edges) which is supposed to be countable. An *edge shift*, denoted by X_G, is a shift space which consist of all bi-infinite sequences of edges from $\mathcal{E}$.

A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where G is a graph and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ is its labeling. Associated to $\mathcal{G}$, a space

$$X_{\mathcal{G}} = \text{closure}\{\mathcal{L}_{\infty}(\xi) : \xi \in X_G\} = \overline{\mathcal{L}_{\infty}(X_G)}$$

is defined and $\mathcal{G}$ is called a *cover* of $X_{\mathcal{G}}$. When G is a finite graph, $X_{\mathcal{G}} = \mathcal{L}_{\infty}(X_G)$ is a *sofic shift*.

A block $v \in \mathcal{B}(X)$ is called *synchronizing* if, whenever uv, $vw \in \mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible subshift X is a *synchronized system* if it has a synchronizing block.

A *coded* system is the closure of the set of sequences obtained by freely concatenating the blocks in a list of blocks.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is called *right-resolving* if, for each vertex I of G, the edges starting at I carry different labels. A *minimal right-resolving cover* of a sofic shift X is a right-resolving cover having the fewest vertices among all right-resolving covers of X. It is unique up to isomorphism [7, Theorem 3.3.18] and is called the *Fischer cover* of X.

For $x \in \mathcal{B}(X)$, call $x_- = (x_i)_{i<0}$ (resp., $x_+ = (x_i)_{i\in\mathbb{Z}^+}$) the *left (resp., right) infinite* X*-ray* and let $X^+ = \{x_+ : x \in X\}$. The follower set of x_- is defined as $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+ \text{ is a point in } X\}$.

3. ONE EQUIVALENCE RELATION FOR HALF-SYNCHRONIZED SYSTEMS

We prove that hyperbolic maps lift the property of being half-synchronized. We use this to show that common half-synchronized hyperbolic extensions define an equivalence relation on the set of half-synchronized systems.

Definition 3.1. An irreducible subshift X is *half-synchronized* if there is a block $m \in \mathcal{B}(X)$ and a left-transitive point $x \in X$ such that $x_{[-|m|+1,0]} = m$ and $\omega_+(x_{(-\infty,0]}) = \omega_+(m)$. In this case m is called a *half-synchronizing block* for X.

Dyck shift and synchronized systems are half-synchronized. Now we review the concept of the Fischer cover for a half-synchronized system [5]. Let the collection of all follower sets $\omega_+(x_-)$ be the set of vertices of a graph X^+. There is an edge from I_1 to I_2 labeled a if and only if there is a X-ray x_- such that x_-a is a X-ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_-a)$. This labeled graph is called the *Krieger cover* for X. If X is a half-synchronized system with half-synchronizing block α, the irreducible component of the Krieger cover containing the vertex $\omega_+(\alpha)$ is called the Fischer cover of X.

The following definition is motivated by the notion of hyperbolic homeomorphism between the sets of doubly transitive points introduced in [8].

Definition 3.2. Let X and Y be irreducible subshifts. The factor map $\varphi : X \to Y$ is *hyperbolic* if there are a $d \in \mathbb{N}$, a block w, and d blocks

$$m^{(1)}, m^{(2)}, \dots, m^{(d)} \in \mathcal{B}_{2k+1}(X),$$

such that

(1) If $y \in Y$ with $y_{[-n,n]} = w$, then
$$\varphi^{-1}(y)_{[-k,k]} = \{m^{(1)}, m^{(2)}, \cdots, m^{(d)}\}.$$

(2) If $w' \in \mathcal{B}(Y)$ begins and ends with w, then, for each $1 \le i \le d$, there is a unique block $a^i \in \mathcal{B}(X)$, such that, for any $x \in X$ with $\varphi(x)_{[-n,n+p]} = w'$ and $x_{[-k,k]} = m^(i)$, it holds $x_{[-k,k+p]} = a^i$.

The next theorem says that [3, Theorem 4.2], which was stated for synchronized systems, is actually valid for half-synchronized systems as well. In fact, it says that the property 'half-synchronized' is weak enough to lift under hyperbolic maps.

Theorem 3.3. *Let X and Y be irreducible subshifts and $\varphi : X \to Y$ a hyperbolic factor map. Then, X is half-synchronized if and only if Y is half-synchronized.*

Proof. First suppose that X is half-synchronized and that $\mathcal{G} = (G, \mathcal{L})$ is the Fischer cover of X. By [5, Theorem 1.4], $\mathcal{G}$ has residual image in the one-sided shift. Since $\varphi = \Phi_\infty$ has a degree [3, Theorem 3.2], $(G, \Phi \circ \mathcal{L})$ has residual image in Y^+. So Y is half-synchronized.

Now let Y be half-synchronized. Since φ is hyperbolic, it has a degree ℓ [3, Theorem 3.2]. Hyperbolicity of φ implies that the transitive left ray y_- does not have less than ℓ preimages. On the other hand, if it has more than ℓ preimages, then a prolongation of y_- gives a point in Y with more than ℓ preimages. So y_- has exactly ℓ preimages.

Let $w \in \mathcal{B}(Y)$ and $m^{(1)}, m^{(2)}, \cdots, m^{(d)} \in \mathcal{B}(X)$ satisfying the conditions stated in Definition 3.2. Without loss of generality, we may assume that φ is an 1-block and that w is a half-synchronizing block. So there is a transitive left ray y_- terminating at w such that $\omega_+(w) = \omega_+(y_-)$. Let $m \in \mathcal{B}_{2n+1}(X)$ be such that $\Phi(m) = w$ and let $x_{1\_}, x_{2\_}, \cdots, x_{\ell\_}$ be ℓ preimages for y_-. Then, since φ has a degree, $x_{1\_}$ is transitive [3, Theorem 3.2]. We may assume that $m = m^{(1)}$ and that $x_{1\_}$ terminates at m. Now we show that m is a half-synchronizing block for X.

Suppose that $mu \in \mathcal{B}(X)$ and prolongate mu such that $muvm \in \mathcal{B}(X)$. Then, $w\Phi(uv)w \in \mathcal{B}(Y)$ and, since w is half-synchronizing, $\Phi(uv)w \in \omega_+(y_-)$. Now $y_-\Phi(uv)w$ also has ℓ preimages $x_{i\_}u_i$ for $1 \leq i \leq \ell$. Condition (2) of Definition 3.2 implies that $u_1 = uv$. So $u \in \omega_+(x_{1\_})$. $\square$

Theorem 3.3 enables us to introduce an equivalence relation using hyperbolic factor codes on the set of half-synchronized systems.

THEOREM 3.4. *Having a common hyperbolic extension is an equivalence relation on the half-synchronized systems.*

Proof. Let X and Y (resp., Y and Z) have a common half-synchronized hyperbolic extension $(V, \varphi_X, \varphi_Y)$ (resp., $(W, \varphi'_Y, \varphi'_Z)$). Suppose that $(\Sigma, \psi_V, \psi'_W)$ is the fiber product of (φ_Y, φ'_Y) and that Γ is an irreducible component of Σ such that the restrictions of ψ_V and ψ'_W to Γ are onto. Since φ_Y is hyperbolic, $\psi'_W : \Gamma \to W$ is hyperbolic and so, by Theorem 3.3, Γ is half-synchronized. The result then follows from the fact that the composition of hyperbolic maps is hyperbolic. $\square$

4. THE EXISTENCE OF DECODER BLOCK

In what follows we use the abbreviation a.e. for the term *almost everywhere.* A factor code $\varphi : X \to Y$ is 1-1 a.e. if any doubly transitive point in Y has exactly one preimage. It is *right-closing almost everywhere* if there is $n \in \mathbb{N}$ such that for any two left-transitive points $x, y \in X$ with $x_{(-\infty,0]} = y_{(-\infty,0]}$ and $\varphi(x)_{(-\infty,n]} = \varphi(y)_{(-\infty,n]}$, it holds that $x_1 = y_1$. In the last case, φ is called *n-step right-closing a.e.*

DEFINITION 4.1. Let $\varphi : X \to Y$ be a factor map between arbitrary subshifts X and Y. A block $w \in \mathcal{B}(Y)$ is a *decoder block* for φ if there is

$k \in \mathbb{N}$, the *anticipation* of w, such that for all $n \in \mathbb{N}$ and all points $x, y \in X$ with $\varphi(x)_{[-|w|+1,0]} = \varphi(y)_{[-|w|+1,0]}$ and $\varphi(x)_{[1,n+k]} = \varphi(y)_{[1,n+k]}$, it holds that $x_{[1,n]} = y_{[1,n]}$.

Remind that, if $\varphi : X \to Y$ is a factor map with a decoder block and μ an ergodic measure on X with full support, then $\varphi : (X, \mu) \to (Y, \nu)$ is a finitary regular isomorphism, where $\nu = \mu \circ \varphi^{-1}$ (see [4]).

Fiebig showed that, when the domain is a synchronized system, then right-closing a.e., 1-1 a.e. factor maps have a strong decoding property [4]. We generalize this property to half-synchronized systems. The ingredients for the proof are almost similar to [4, Theorem 2.3].

THEOREM 4.2. *Let X be a half-synchronized system and $\varphi : X \to Y$ a factor map. Then, φ is right-closing a.e., 1-1 a.e. if and only if it has a decoder block.*

Proof. Suppose that φ is k-step right-closing a.e. and that $m \in \mathcal{B}(X)$ is a half-synchronizing block with length greater than a coding length for φ. Let $z \in X$ be a doubly transitive point satisfying the conditions stated in Definition 3.1. By compactness, combined with the fact that φ is 1-1 a.e., there is a $p \in \mathbb{N}$ such that, for $x \in X$ with $\varphi(x)_{[-p,p]} = \varphi(z)_{[-p,p]}$, it holds that $x_{[-|m|+1,0]} = m$. We claim that $w := \varphi(z)_{[-p,p]}$ is a decoder block with anticipation k.

Pick $x, y \in X$ with $\varphi(x)_{[-|w|+1,0]} = \varphi(y)_{[-|w|+1,0]} = w$ and $\varphi(x)_{[1,n+k]} = \varphi(y)_{[1,n+k]}$. Then, $x_{[-p-|m|+1,-p]} = y_{[-p-|m|+1,-p]} = m$. Since m is half-synchronizing, there are points $x', y' \in X$ such that

$$x'_i = \begin{cases} z_{i+p} & i \le -p, \\ x_i & i \ge -p+1, \end{cases} \qquad y'_i = \begin{cases} z_{i+p} & i \le -p, \\ y_i & i \ge -p+1. \end{cases}$$

Observe that x' and y' are left transitive and $\varphi(x')_{(-\infty,n+k]} = \varphi(y')_{(-\infty,n+k]}$. Since φ is right-closing a.e., $x'_{[-p+1,n]} = y'_{[-p+1,n]}$. So $x_{[1,n]} = y_{[1,n]}$.

Now suppose that w is a decoder block with anticipation k and consider $y \in D(Y)$. So w appears in y infinitely often to the left and to the right. Since w is a decoder block, φ is 1-1 a.e.. If $y_{[i-|w|+1,i]} = w$ for some $i \in \mathbb{Z}$, a coordinate $j > i$ of the preimages of y is determined by $y_{[i-|w|+1,j+k]}$. Therefore, φ is k-step right-closing a.e.. □

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