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AN ITERATIVE METHOD FOR A FOURTH ORDER TRANSMISSION PROBLEM

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Abstract. We pursue a constructive solution to a fourth order transmission problem on a planar domain. We use an iterative technique that reduces the fourth order partial differential equations to second order Helmholtz-type equations. We use the layer potentials to solve the second order transmission problems. The methods that we use are suitable for numerical computations. This work is inspired by recent papers regarding the use of iterative methods for Neumann biharmonic problems, Robin problems and mixed problems.

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Key words. Iterative method, biharmonic equation, transmission problem, Helmholtz equation, single layer potential.

1. INTRODUCTION

The aim of this paper is to apply iterative methods to transmission problems associated with fourth order biharmonic-type equations. This work is inspired by several recent papers in which the authors used the iterative techniques for Neumann or Robin boundary value problems associated with fourth order partial differential equations.

In the article [2], the author Q.A. Dang studied the following Neumann boundary value problem associated with a biharmonic-type equation

$$\begin{aligned} \Delta^2 u - a\Delta u + bu = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g_0 \quad \text{on } \Gamma, \\ \frac{\partial \Delta u}{\partial n} = g_1 \quad \text{on } \Gamma. \end{aligned}$$

The author used an iterative technique that reduces the fourth order equations to second order equations, which are solved using numerical computations.

In the article [6], the authors A. Gomez-Polanco, J.M. Guevara-Jordan, B. Molina applied a mimetic method for the following Robin problem associated with a biharmonic-type equation

$$\begin{aligned} \Delta^2 u - a\Delta u + bu = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \sigma u = g_0 \quad \text{on } \Gamma, \end{aligned}$$

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$$\frac{\partial \Delta u}{\partial n} + \sigma \Delta u = g_1 \quad \text{on } \Gamma.$$

In the sequel we will consider the domain $D \subset \mathbb{R}^2$ to be an unbounded domain that has the Rellich property. Let $\Omega_1 \subset D$ be a bounded domain and $\Omega_2 = D \setminus \bar{\Omega}_1$. Denote by Γ the boundary of Ω_1. We will assume that Γ is sufficiently smooth.

Let $C_c^\infty(D)$ be the space of all infinitely differentiable functions in D with compact support. Denote by $H_0^k(D)$ the closure of $C_c^\infty(D)$ in $H^k(D)$.

In this paper, we will use an iterative technique for the following transmission problem associated with the more general biharmonic-type equation

$$\Delta^2 u_i - a\Delta u_i + bu_i = f \quad \text{in } \Omega_i, \tag{1}$$

$$u_1 - u_2 = g_1 \quad \text{on } \Gamma, \tag{2}$$

$$\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g_2 \quad \text{on } \Gamma, \tag{3}$$

$$\Delta u_1 - \Delta u_2 = g_3 \quad \text{on } \Gamma, \tag{4}$$

$$\frac{\partial \Delta u_1}{\partial n} - \frac{\partial \Delta u_2}{\partial n} = g_4 \quad \text{on } \Gamma, \tag{5}$$

where $a,\ b > 0$ are constants and the functions f, g_1, g_2, g_3, g_4 will be specified subsequently.

The solution u is searched in the space $H_0^4(D)$.

Let r and s be positive numbers such that $r \le s,\ r+s = a,\ rs = b-c,\ c \ge 0$. Note that these numbers always exist. For example, if $a^2 - 4b \ge 0$, then we can set $c = 0$ and the numbers r, s can be chosen to be the roots of the quadratic equation $x^2 - ax + b = 0$.

If the numbers r, s, c are defined as mentioned before, then the equations (1) can be factorized into the following equations

$$(\Delta - r) \circ (\Delta - s)u_i = f - cu_i \quad \text{in } \Omega_i. \tag{6}$$

If we denote $\Delta u_i - su_i = v_i$, then we can write the equations (6) as a system of four equations

$$\Delta v_i - rv_i = f - cu_i \quad \text{in } \Omega_i,$$
$$\Delta u_i - su_i = v_i \quad \text{in } \Omega_i.$$

Furthermore, from the boundary conditions (2),(4) and the definition of v_i, we deduce the following transmission condition for v_i

$$v_1 - v_2 = \Delta u_1 - su_1 - \Delta u_2 + su_2 = g_3 - sg_1 \quad \text{on } \Gamma.$$

The other transmission condition for v_i is obtained in a similar way from conditions (3) and (5)

$$\frac{\partial v_1}{\partial n} - \frac{\partial v_2}{\partial n} = \frac{\partial \Delta u_1}{\partial n} - s\frac{\partial u_1}{\partial n} - \frac{\partial \Delta u_2}{\partial n} + s\frac{\partial u_2}{\partial n} = g_4 - sg_2 \quad \text{on } \Gamma.$$

Therefore the transmission problem (1)–(5), that is associated with the biharmonic-type operator, can be replaced with the following equivalent system of equations associated with the Helmholtz operator

$$\begin{gathered}\Delta v_i - r v_i = f - c u_i \quad \text{in } \Omega_i,\\ v_1 - v_2 = g_3 - s g_1 \quad \text{on } \Gamma,\\ \frac{\partial v_1}{\partial n} - \frac{\partial v_2}{\partial n} = g_4 - s g_2 \quad \text{on } \Gamma,\\ \Delta u_i - s u_i = v_i \quad \text{in } \Omega_i,\\ u_1 - u_2 = g_1 \quad \text{on } \Gamma,\\ \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g_2 \quad \text{on } \Gamma.\end{gathered}$$

If we denote $\phi_i = -c u_i$, then the equations above become

$$\Delta v_i - r v_i = f + \phi_i \quad \text{in } \Omega_i, \tag{7}$$

$$v_1 - v_2 = g_3 - s g_1 \quad \text{and} \quad \frac{\partial v_1}{\partial n} - \frac{\partial v_2}{\partial n} = g_4 - s g_2 \quad \text{on } \Gamma, \tag{8}$$

$$\Delta u_i - s u_i = v_i \quad \text{in } \Omega_i, \tag{9}$$

$$u_1 - u_2 = g_1 \quad \text{and} \quad \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g_2 \quad \text{on } \Gamma. \tag{10}$$

The equations (7)–(10) can be regarded as transmission problems associated with second order Helmholtz-type equations, that could be solved using techniques based on boundary element computations. But the Helmholtz equations cannot be separated, because the functions ϕ_i are not determined and they depend on the functions u_i. For this reason we will use an iteration process that manages to reduce the fourth order equation to second order equations.

2. MAIN RESULTS

We pursue a constructive solution $u \in H_0^4(D)$ of the transmission problem (7)–(10), using the following iteration process IP that requires solving two second order Helmholtz equations at each step.

1. Let $\phi_1^{(0)} \in H_0(\Omega_1)$ and $\phi_2^{(0)} \in H_0(\Omega_2)$.

2. Given $\phi_i^{(k)}$, solve the transmission problems associated with the second order Helmholtz equations

$$\Delta v_i^{(k)} - r v_i^{(k)} = f + \phi_i^{(k)} \quad \text{in } \Omega_i, \tag{11}$$

$$v_1^{(k)} - v_2^{(k)} = g_3 - s g_1 \quad \text{and} \quad \frac{\partial v_1^{(k)}}{\partial n} - \frac{\partial v_2^{(k)}}{\partial n} = g_4 - s g_2 \quad \text{on } \Gamma, \tag{12}$$

$$\Delta u_i^{(k)} - s u_i^{(k)} = v_i^{(k)} \quad \text{in } \Omega_i, \tag{13}$$

(14) $$u_1^{(k)} - u_2^{(k)} = g_1 \quad \text{and} \quad \frac{\partial u_1^{(k)}}{\partial n} - \frac{\partial u_2^{(k)}}{\partial n} = g_2 \quad \text{on } \Gamma.$$

3. Compute the functions $\phi_i^{(k+1)}$ for the next step

$$\phi_i^{(k+1)} = (1-\tau)\phi_i^{(k)} - c\tau u_i^{(k)},$$

where τ is a parameter that will be defined subsequently.

In the sequel, we will present the convergence of the sequence $\left(u_i^{(k)}\right)$ defined by the iterative process given before, to the solution of the system of equations (7)–(10), that are equivalent to the transmission problem (1)–(5).

We write the solutions u_i, v_i of the system (7)–(10) in the form

(15) $$u_i = \mu_i + U_i, \quad v_i = \nu_i + V_i,$$

where μ_i, U_i, ν_i, V_i are the solutions of the system of equations

(16) $$\Delta\nu_i - r\nu_i = \phi_i \quad \text{in } \Omega_i,$$

(17) $$\nu_1 - \nu_2 = 0 \quad \text{on} \quad \text{and} \quad \frac{\partial \nu_1}{\partial n} - \frac{\partial \nu_2}{\partial n} = 0 \quad \text{on } \Gamma,$$

(18) $$\Delta\mu_i - s\mu_i = \nu_i \quad \text{in } \Omega_i,$$

(19) $$\mu_1 - \mu_2 = 0 \quad \text{and} \quad \frac{\partial \mu_1}{\partial n} - \frac{\partial \mu_2}{\partial n} = 0 \quad \text{on } \Gamma,$$

(20) $$\Delta V_i - rV_i = f \quad \text{in } \Omega_i,$$

(21) $$V_1 - V_2 = g_3 - sg_1 \quad \text{and} \quad \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = g_4 - sg_2 \quad \text{on } \Gamma,$$

(22) $$\Delta U_i - sU_i = V_i \quad \text{in } \Omega_i,$$

(23) $$U_1 - U_2 = g_1 \quad \text{and} \quad \frac{\partial U_1}{\partial n} - \frac{\partial U_2}{\partial n} = g_2 \quad \text{on } \Gamma.$$

We need to have some simpler relations between the functions defined by sequences in the iterative process IP. We introduce the operator A that is defined by the relation

$$A(\phi_1 \oplus \phi_2) = u_1 \oplus u_2,$$

where u_i is the solution of the following system denoted by HS

$$\Delta v_i - rv_i = \phi_i \quad \text{in } \Omega_i,$$

$$v_1 - v_2 = 0 \quad \text{and} \quad \frac{\partial v_1}{\partial n} - \frac{\partial v_2}{\partial n} = 0 \quad \text{on } \Gamma,$$

$$\Delta u_i - su_i = v_i \quad \text{in } \Omega_i,$$

$$u_1 - u_2 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = 0 \quad \text{on } \Gamma,$$

and $\oplus$ is the concatenation operator.

We also write $A\phi_i = u_i$ instead of $A(\phi_1 \oplus \phi_2) = u_1 \oplus u_2$.

If we return to the system (16)–(23), then the definition of the operator A implies

$$A\phi_i = \mu_i.$$

From the definitions of $\phi_i,\ \mu_i,\ U_i$ given before, we deduce

$$\phi_i = -cu_i = -c\mu_i - cU_i = -cA\phi_i - cU_i,$$

and consequently we obtain the relation

$$\phi_i + cA\phi_i = -cU_i,$$

that can also be written as

$$(I + cA)\phi_i = -cU_i. \tag{24}$$

The purpose of introducing the operator A was to find a more succinct form for the sequences of functions defined by the 3-step iteration process IP. The succinct relation between the elements of the sequences is given by the following lemma.

Lemma 2.1. *For a given* $\phi_i^{(0)} \in H_0(D \setminus \Gamma)$, *the functions* $\phi_i^{(k)}$ *defined by the iterative process IP coincide with the functions* $\phi_i^{(k)}$ *defined by the iterative scheme*

$$\frac{\phi_i^{(k+1)} - \phi_i^{(k)}}{\tau} + (I + cA)\phi_i^{(k)} = -cU_i.$$

Proof. The relation given in the third step of the iterative process IP can be written as

$$\frac{\phi_i^{(k+1)} - \phi_i^{(k)}}{\tau} + \phi_i^{(k)} + cu_i^{(k)} = 0. \tag{25}$$

Let U_i, V_i be the solutions of the system (20)–(23), and let $\left(u_i^{(k)}\right), \left(v_i^{(k)}\right)$ be the sequences of functions given by the iterative process IP.

We introduce the sequences $\mu_i^{(k)}, \nu_i^{(k)}$ defined by

$$u_i^{(k)} = \mu_i^{(k)} + U_i \quad \text{and} \quad v_i^{(k)} = \nu_i^{(k)} + V_i.$$

From the equalities (11), (12), (20), (21) we deduce

$$\Delta\nu_i^{(k)} - r\nu_i^{(k)} = \phi_i^{(k)} \quad \text{in } \Omega_i, \tag{26}$$

$$\nu_1^{(k)} - \nu_2^{(k)} = 0 \quad \text{on} \quad \text{and} \quad \frac{\partial\nu_1^{(k)}}{\partial n} - \frac{\partial\nu_2^{(k)}}{\partial n} = 0 \quad \text{on } \Gamma, \tag{27}$$

and, from the equalities (13), (14), (22), (23) we deduce

$$\Delta\mu_i^{(k)} - s\mu_i^{(k)} = \nu_i^{(k)} \quad \text{in } \Omega_i, \tag{28}$$

$$\mu_1^{(k)} - \mu_2^{(k)} = 0 \quad \text{and} \quad \frac{\partial\mu_1^{(k)}}{\partial n} - \frac{\partial\mu_2^{(k)}}{\partial n} = 0 \quad \text{on } \Gamma, \tag{29}$$

Therefore, from (26)-(29) and the definition of A, it follows that

$$A\phi_i^{(k)} = \mu_i^{(k)}.$$

Using the equalities above, we obtain successively

$$\phi_i^{(k)} + cu_i^{(k)} = \phi_i^{(k)} + c\mu_i^{(k)} + cU_i = (I + cA)\phi_i^{(k)} + cU_i,$$

and, if we substitute in relation (25), we get

$$\frac{\phi_i^{(k+1)} - \phi_i^{(k)}}{\tau} + (I + cA)\phi_i^{(k)} = -cU_i.$$

This ends the proof. □

In order to prove the convergence of the sequence $u_i^{(k)}$, defined by the iterative process IP, to the solution of the system of equations (7)–(10), we need the following lemma from the article [4]. We simply state the lemma without proof. The proof can be found in [4].

LEMMA 2.2. *Suppose that A is a linear, symmetric, positive and compact operator in a Hilbert space H and y is the solution of the equation*

$$Ay = f, \; f \in R(A).$$

Then the iterative method

$$\frac{y_{k+1} - y_k}{\tau} + Ay_k = f, \; \textit{with } \; y_0 \textit{ given},$$

converges if

$$0 < \tau < \frac{2}{\|A\|}.$$

We will apply Lemma 2.2 to the sequence $\phi_i^{(k)}$, using the relation that we have already proved in Lemma 2.1. First we need to show that the operator A is linear, symmetric, positive and compact.

We assume that the problem HS is well-posed and that it has a unique solution $u_i \in H_0^4(D)$.

LEMMA 2.3. *The operator A is linear, symmetric, positive and compact on $H_0(D)$.*

Proof. From the definition of the operator A we have

$$A\phi_i = u_i, \;\; \Delta v_i - rv_i = \phi_i \;\text{ and }\; \Delta u_i - su_i = v_i,$$

where u_i, v_i are the solutions of the transmission system HS consisting of the homogeneous boundary equations as specified in the definition of the operator

A. We deduce the following

$$
\begin{aligned}
(A\phi_i, \bar{\phi}_i) &= \int_{\Omega_1} u_1(\Delta \bar{v}_1 - r\bar{v}_1)\mathrm{d}x + \int_{D\setminus\bar{\Omega}_1} u_2(\Delta \bar{v}_2 - r\bar{v}_2)\mathrm{d}x \\
&= -\int_{\Omega_1} (\nabla u_1 \nabla \bar{v}_1 + r u_1 \bar{v}_1)\mathrm{d}x - \int_{D\setminus\bar{\Omega}_1} (\nabla u_2 \nabla \bar{v}_2 + r u_2 \bar{v}_2)\mathrm{d}x \\
&= \int_{\Omega_1} (\bar{v}_1 \Delta u_1 - r u_1 \bar{v}_1)\mathrm{d}x + \int_{D\setminus\bar{\Omega}_1} (\bar{v}_2 \Delta u_2 - r u_2 \bar{v}_2)\mathrm{d}x \\
&= \int_{\Omega_1} \bar{v}_1 v_1 \mathrm{d}x + (s-r)\int_{\Omega_1} u_1 \bar{v}_1 \mathrm{d}x \\
&+ \int_{D\setminus\bar{\Omega}_1} \bar{v}_2 v_2 \mathrm{d}x + (s-r)\int_{D\setminus\bar{\Omega}_1} u_2 \bar{v}_2 \mathrm{d}x.
\end{aligned}
$$

We also have

$$
\int_{\Omega_1} u_1 \bar{v}_1 \mathrm{d}x + \int_{D\setminus\bar{\Omega}_1} u_2 \bar{v}_2 \mathrm{d}x = \int_{\Omega_1} u_1(\Delta \bar{u}_1 - s\bar{u}_1)\mathrm{d}x + \int_{\Omega_2} u_2(\Delta \bar{u}_2 - s\bar{u}_2)\mathrm{d}x
$$

$$
= -\int_{\Omega_1} (\nabla u_1 \nabla \bar{u}_1 + s u_1 \bar{u}_1)\mathrm{d}x - \int_{\Omega_2} (\nabla u_2 \nabla \bar{u}_2 + s u_2 \bar{u}_2)\mathrm{d}x.
$$

Therefore we obtain $(A\phi_i, \bar{\phi}_i) = (A\bar{\phi}_i, \phi_i)$. Thus the operator A is a symmetric operator. If we write $(A\phi, \phi)$, we get $(A\phi, \phi) \geq 0$. Therefore A is positive.

Obviously the operator A is linear.

Since the problem HS has a unique solution, it follows that the operator A maps $H_0(D)$ into $H_0^4(D)$. But the space $H_0^4(D)$ is compactly embedded into $H_0(D)$, because the domain D has the Rellich property. Thus the operator A is compact. This finishes the proof. □

THEOREM 2.4. *Consider the functions*

$$
f \in H_0(\Omega),\ g_1 \in H^{7/2}(\Gamma),\ g_2 \in H^{5/2}(\Gamma),\ g_3 \in H^{3/2}(\Gamma),\ g_4 \in H^{1/2}(\Gamma).
$$

Suppose that u_i is the solution of the problem (1)–(5), *and let τ satisfy the condition in Lemma* 2.2. *Then the sequence $u_i^{(k)}$, defined by the iterative process IP, converges to u_i.*

Proof. From Lemma 2.1 and Lemma 2.2, we deduce that the sequence $\phi_i^{(k)}$ is convergent. Then we have

$$
\Delta\left(v_i^{(k+1)} - v_i^{(k)}\right) - r\left(v_i^{(k+1)} - v_i^{(k)}\right) = \phi_i^{(k+1)} - \phi_i^{(k)} \quad \text{in } \Omega_i,
$$

$$
\left(v_1^{(k+1)} - v_1^{(k)}\right) - \left(v_2^{(k+1)} - v_2^{(k)}\right) = 0 \quad \text{on } \Gamma,
$$

$$
\frac{\partial\left(v_1^{(k+1)} - v_1^{(k)}\right)}{\partial n} - \frac{\partial\left(v_2^{(k+1)} - v_2^{(k)}\right)}{\partial n} = 0 \quad \text{on } \Gamma,
$$

$$
\Delta\left(u_i^{(k+1)} - u_i^{(k)}\right) - s\left(u_i^{(k+1)} - u_i^{(k)}\right) = v_i^{(k+1)} - v_i^{(k)} \quad \text{in } \Omega_i,
$$

$$\left(u_1^{(k+1)} - u_1^{(k)}\right) - \left(u_2^{(k+1)} - u_2^{(k)}\right) = 0 \quad \text{on } \Gamma,$$

$$\frac{\partial\left(u_1^{(k+1)} - u_1^{(k)}\right)}{\partial n} - \frac{\partial\left(u_2^{(k+1)} - u_2^{(k)}\right)}{\partial n} = 0 \quad \text{on } \Gamma.$$

Since the system HS, consisting of homogeneous boundary conditions, is uniquely solvable, it follows that

$$\|u_i^{(k+1)} - u_i^{(k)}\| \le C_1 \|v_i^{(k+1)} - v_i^{(k)}\| \le C_2 \|\phi_i^{(k+1)} - \phi_i^{(k)}\|.$$

Thus $u_i^{(k)}$ is a Cauchy sequence that converges to the solution u_i of the transmission system (1)–(5). □

3. SOLVING THE SECOND ORDER TRANSMISSION PROBLEMS ASSOCIATED WITH THE HELMHOLTZ OPERATOR

Now consider the transmission problem associated with the second order Helmholtz equation

$$\Delta u_i - s u_i = f \quad \text{in } \Omega_i,$$

$$u_1 - u_2 = g_1 \quad \text{on } \Gamma,$$

$$\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g_2 \quad \text{on } \Gamma,$$

with $f \in H_0(D \setminus \Gamma)$.

The iterative process IP reduces the fourth order equations to this type of second order transmission equations.

We have two cases: $(g_1, g_2) \in H^{7/2}(\Gamma) \times H^{5/2}(\Gamma)$ and $(g_1, g_2) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. It suffices to consider just the second case.

If we use the domain potential, we can find solutions $u_{p,i} \in H_0^4(D)$ for the nonhomogeneous equations

$$\Delta u_i - s u_i = f \quad \text{in } \Omega_i.$$

Making adjustments for the traces of the solutions $u_{p,i}$ in the boundary conditions g_1, g_2, it will suffice to solve the homogeneous problem

$$\Delta u_i - s u_i = 0 \quad \text{in } \Omega_i,$$

$$u_1 - u_2 = g_1 \quad \text{on } \Gamma,$$

$$\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g_2 \quad \text{on } \Gamma.$$

We will use the boundary layer potentials to find a solution for the homogeneous second order transmission problem. First we recall the following well-known facts about the layer potentials and their boundary behaviour. Let $E(x, y)$ be the fundamental solution of the Helmholtz equation.

Definition 3.1. For $h \in H^{-1/2}(\Gamma)$ define the single layer potential S with density h by

$$Sh(x) = \int_\Gamma E(x,y)h(y)\,\mathrm{d}y,\ x \in \mathbb{R}^n \setminus \Gamma,$$

and the double layer potential D with density h by

$$Dh(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial n} h(y)\,\mathrm{d}y,\ x \in \mathbb{R}^n \setminus \Gamma.$$

Lemma 3.2. *The single layer potential operator* $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is given by*

$$Sh(x) = \int_\Gamma E(x,y)h(y)\,\mathrm{d}y = \lim_{z\to x}\int_\Gamma E(z,y)h(y)\,\mathrm{d}y,\ x \in \Gamma.$$

The double layer potential operator $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is given by*

$$Kh(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial n} h(y)\,\mathrm{d}y = \lim_{z\to x,\ z\in\Omega_1} Dh(z) + \frac{1}{2}h(x),\ x \in \Gamma.$$

The single layer potential operator satisfies the jump relation

$$\frac{\partial Sh(x)}{\partial n} = \frac{1}{2}h(x) + K'h(x),\ x \in \Gamma,$$

where K' *is the adjoint operator of* K.

We set $u_i = v_i + Dg_1$. Using the jump relation for the double layer potential, we deduce that (u_1, u_2) is a solution of the transmission problem stated before if and only if (v_1, v_2) is a solution of the following transmission problem

$$\begin{aligned} \Delta v_i - s v_i &= 0 \quad \text{in } \Omega_i, \\ v_1 - v_2 &= 0 \quad \text{on } \Gamma, \\ \frac{\partial v_1}{\partial n} - \frac{\partial v_2}{\partial n} &= F \quad \text{on } \Gamma, \end{aligned}$$

where F is given by

$$F = g_2 - \left(\frac{\partial Dg_1}{\partial n}\right)_+ + \left(\frac{\partial Dg_1}{\partial n}\right)_- .$$

From the properties of the layer potentials, we also have $\left(\frac{\partial Dg_1}{\partial n}\right)_+ = \left(\frac{\partial Dg_1}{\partial n}\right)_-$ and thus $F = g_2$.

Since $v_1 - v_2 = 0$, we search for a function v such that $v = v_1$ on Ω_1 and $v = v_2$ on Ω_2. We pursue the function v in the form of a single layer potential Sh with density $h \in H^{-1/2}(\Gamma)$.

From the jump relations for the layer potentials, we deduce that the second boundary condition of the transmission problem leads to the boundary integral equation $\left(\frac{1}{2}h - K'h\right) + \left(\frac{1}{2}h + K'h\right) = F$, which reduces to $h = F$. Thus

$u_i = Sg_2 + Dg_1 \in H_0^4(D)$ solves the transmission problem associated with the Helmholtz equation.

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