MATHEMATICA, 58 (81), N° 1–2, 2016, pp. 39–50

FAVARD'S INEQUALITY FOR SEMINORMED FUZZY INTEGRAL AND SEMICONORMED FUZZY INTEGRAL

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Abstract. The purpose of this paper is to generalize Favard's inequality for seminormed and semiconormed fuzzy integrals of non-negative concave (convex) functions on a fuzzy measure space (X, Σ, μ), where μ is the Lebesgue measure. Moreover, for illustrating the theorems, several examples are given.

MSC 2010. 28E10, 21C05, 26D15.

Key words. Favard's inequality, seminormed fuzzy integral, semiconormed fuzzy integral, t-norm, s-norm, fuzzy measure.

1. INTRODUCTION

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno, as a tool for modeling non-deterministic problems [6]. Many authors generalized the Sugeno integral, by using some other operators instead of the operators $\vee$(sup) and $\wedge$(inf). Suárez and Gil presented two generalizations of the Sugeno integral: the seminormed fuzzy integral and the semiconormed fuzzy integral [5]. Recently, some classical integral inequalities have been worked for these integrals; for example, Ouyang et al. proved Chebyshev's inequality for the seminormed fuzzy integral [3] and Caballero et al. proved Markov's inequality for the seminormed fuzzy integral [1].

The following theorem expresses the classical Favard's inequality.

THEOREM 1.1. ([2]) *Let f be a concave non-negative function on $[a,b] \subseteq \mathbb{R}$. If $q > 1$, then*

$$\frac{1}{b-a}\int_a^b f^q(x)\mathrm{d}x \le \frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)^q. \tag{1}$$

If $0 < q < 1$, then the reverse inequality holds in (1).

The aim of this paper is to present the above theorem for seminormed and semiconormed fuzzy integrals. Moreover, we present some appropriate examples to illustrate our results.

This paper is organized as follows. In Section 2, we introduce some notations and concepts. In Section 3, we prove the main results and give some examples. In Section 4, we express our conclusions.

2. PRELIMINARY RESULTS

As usual, denote by $\mathbb{R}$, the set of real numbers. Let X be a non-empty set and let Σ be a σ-algebra of subsets of X. Throughout this paper, all the considered subsets are supposed to belong to Σ.

DEFINITION 2.1. ([4]) A set function $\mu : \Sigma \to [0, +\infty]$ is called a *fuzzy measure*, if the following properties are satisfied:

(F1) $\mu(\emptyset) = 0$;
(F2) $A, B \in \Sigma$ and $A \subseteq B$ imply $\mu(A) \leq \mu(B)$;
(F3) $\{A_i\} \subseteq \Sigma$, $A_1 \subseteq A_2 \subseteq \dots$ and $\bigcup_{i=1}^{\infty} A_i \in \Sigma$ imply $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim \mu(A_i)$;
(F4) $\{A_i\} \subseteq \Sigma$, $A_1 \supseteq A_2 \supseteq \dots$, $\mu(A_1) < \infty$ and $\bigcap_{i=1}^{\infty} A_i \in \Sigma$ imply $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim \mu(A_i)$.

The triplet (X, Σ, μ) is called a *fuzzy measure space*.

We denote the set of all measurable functions from X to $[0, 1]$, with respect to Σ, by $\mathcal{F}_+(X)$. Let f be a non-negative real-valued function defined on X. Denote the set $\{x \in X \mid f(x) \geq \alpha\}$ by F_α, for $\alpha \geq 0$.

DEFINITION 2.2. ([5]) A *t-norm* is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions, for any $x, y, z \in [0, 1]$:

(i) $T(x, 1) = T(1, x) = x$;
(ii) $y \leq z \Rightarrow T(x, y) \leq T(x, z)$;
(iii) $T(x, T(y, z)) = T(T(x, y), z)$;
(iv) $T(x, y) = T(y, x)$.

EXAMPLE 2.3. The following are the most important *t-norms*:

$$\begin{aligned} T_M(x, y) &= \min(x, y), \\ T_P(x, y) &= xy, \\ T_L(x, y) &= \max(x + y - 1, 0), \end{aligned}$$

$$T_D(x, y) = \begin{cases} 0 & (x, y) \in [0, 1)^2 \\ \min(x, y) & (x, y) \notin [0, 1)^2 \end{cases}.$$

DEFINITION 2.4. ([5]) A function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-conorm* (*s-norm*), if there is a *t-norm* T such that $S(x, y) = 1 - T(1 - x, 1 - y)$. A *t-conorm* satisfies:

(i′) $S(x, 0) = S(0, x) = x$, for any $x \in [0, 1]$,

and the conditions (ii)-(iv) of the *t-norms*.

EXAMPLE 2.5. The following are four important *t-conorms*.

$$\begin{aligned} S_M(x, y) &= \max(x, y), \\ S_P(x, y) &= x + y - xy, \\ S_L(x, y) &= \min(x + y - 1, 0), \end{aligned}$$

$$S_D(x, y) = \begin{cases} 1 & (x, y) \in (0, 1]^2 \\ \max(x, y) & (x, y) \notin (0, 1]^2 \end{cases}.$$

REMARK 2.6. If a binary operator $T(S)$ satisfies the conditions (i) and (ii) of Definition 2.2 ((i′) and (ii) of Definition 2.4), then it is called a *t-seminorm* (*t-semiconorm*).

Suárez and Gil proposed the following two families of fuzzy integrals.

DEFINITION 2.7. Let T be a t-seminorm. The *seminormed fuzzy integral* of a function $f \in \mathcal{F}_+(X)$ over $A \in \Sigma$, with respect to a fuzzy measure μ, is defined as

$$\int_{T,A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)).$$

DEFINITION 2.8. Let S be a t-semiconorm. The *semiconormed fuzzy integral* of a function $f \in \mathcal{F}_+(X)$ over $A \in \Sigma$, with respect to a fuzzy measure μ, is defined as

$$\int_{S,A} f \mathrm{d}\mu = \bigwedge_{\alpha \in [0,1]} S(\alpha, \mu(A \cap F_\alpha)).$$

The following properties for the seminormed fuzzy integral can be found in [5].

PROPOSITION 2.9. *Let $f, g \in \mathcal{F}_+(X)$ and $A, B \in \Sigma$. Then we have:*

(a) $f \le g \Rightarrow \int_{T,A} f \mathrm{d}\mu \le \int_{T,A} g \mathrm{d}\mu$,

(b) $A \subseteq B \Rightarrow \int_{T,A} f \mathrm{d}\mu \le \int_{T,B} f \mathrm{d}\mu$,

(c) $\forall k, 0 < k < 1$, $\int_{T,A} k \mathrm{d}\mu = T(k, \mu(A))$,

(d) $\mu(A) = 0 \Rightarrow \int_{T,A} f \mathrm{d}\mu = 0$,

(e) $\int_{T,A} f \wedge g \mathrm{d}\mu \le \int_{T,A} f \mathrm{d}\mu \wedge \int_{T,A} g \mathrm{d}\mu$,

(f) $\int_{T,A} f \vee g \mathrm{d}\mu \ge \int_{T,A} f \mathrm{d}\mu \vee \int_{T,A} g \mathrm{d}\mu$,

(g) $\int_{T,A\cup B} f \mathrm{d}\mu \ge \int_{T,A} f \mathrm{d}\mu \vee \int_{T,B} f \mathrm{d}\mu$,

(h) $\int_{T,A\cap B} f \mathrm{d}\mu \le \int_{T,A} f \mathrm{d}\mu \wedge \int_{T,B} g \mathrm{d}\mu$.

REMARK 2.10. All the above properties, except (c), also hold for the semiconormed fuzzy integral. The following property replaces (c):

(c′) $\forall k, 0 < k < 1$, $\int_{S,A} k \mathrm{d}\mu = S(k, \mu(A'))$, where A' is the complement of A.

3. MAIN RESULT

In this section, we prove Favard's inequality for the seminormed and semiconormed fuzzy integrals. We present this inequality for the seminormed fuzzy integral of the concave functions and the semiconormed fuzzy integral of the convex functions. Throughout this section, we suppose that $X \subseteq [0, 1]$.

THEOREM 3.1. *Let $f : [a, b] \to [0, 1]$ be a concave function and μ be the Lebesgue measure on $\mathbb{R}$. Then for any $q > 0$, we have:*

(a) *if* $f(b) > f(a)$, *then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\bigg(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu + af(b) - bf(a)}{f(b) - f(a)}\bigg).$$

(b) *if* $f(b) = f(a)$, *then*

$$f^q \mathrm{d}\mu \geq T(f^q(a), b-a).$$

(c) *if* $f(a) > f(b)$, *then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\bigg(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, \frac{bf(a) - af(b) - 2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu}{f(a) - f(b)} - a\bigg).$$

Proof. Assume that $x \in [a,b]$. Set $x = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b$, then, by the concavity of f, we have

$$f(x) \geq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) = h(x).$$

By Proposition 2.9 (a), we obtain $f^q d\mu \geq h^q d\mu$.

(a) If $f(b) > f(a)$, then

$$\begin{aligned} f^q \mathrm{d}\mu &\geq h^q \mathrm{d}\mu \\ &= \bigvee_{\alpha \in [0,1]} T\bigg(\alpha, \mu([a,b] \cap \{x | h(x) \geq \alpha^{\frac{1}{q}}\})\bigg) \\ &= \bigvee_{\alpha \in [0,1]} T\bigg(\alpha, \mu([a,b] \cap \{x | x \geq \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b) - f(a)}\})\bigg) \\ &= \bigvee_{\alpha \in [0,1]} T\bigg(\alpha, (b - \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b) - f(a)})\bigg). \end{aligned}$$

Assume that $\alpha = \frac{2^q}{(q+1)}(\frac{1}{b-a} f \mathrm{d}\mu)^q$, then $\alpha \in [0,1]$ and thus

$$\bigvee_{\alpha \in [0,1]} T\bigg(\alpha, (b - \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b) - f(a)}\})\bigg) \geq$$
$$T\bigg(\frac{2^q}{q+1}(\frac{1}{b-a} f \mathrm{d}\mu)^q, (b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu + af(b) - bf(a)}{f(b) - f(a)})\bigg).$$

It follows that

$f^q \mathrm{d}\mu \geq T\bigg(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu + af(b) - bf(a)}{f(b) - f(a)}\bigg).$

On the other hand, since $b - a \leq 1$, we get

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq f^q \mathrm{d}\mu,$$

hence,

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\Big(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu + af(b) - bf(a)}{f(b)-f(a)}\Big).$$

(b) If $f(a) = f(b)$, then $h(x) = f(a) = f(b)$ and, using Proposition 2.9 (a), (c), one has

$$f^q \mathrm{d}\mu \geq h^q \mathrm{d}\mu = f^q(a) \mathrm{d}\mu = T(f^q(a), b-a).$$

(c) If $f(a) > f(b)$, then

$$\begin{aligned} f^q \mathrm{d}\mu &\geq h^q \mathrm{d}\mu \\ &= \bigvee_{\alpha\in[0,1]} T\Big(\alpha, \mu([a,b] \cap \{x | h(x) \geq \alpha^{\frac{1}{q}}\})\Big) \\ &= \bigvee_{\alpha\in[0,1]} T\left(\alpha, \mu\left([a,b] \cap \left\{x | x \leq \frac{bf(a) - af(b) - \alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)}\right\}\right)\right) \\ &= \bigvee_{\alpha\in[0,1]} T\left(\alpha, \left(\frac{bf(a) - af(b) - \alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)} - a\right)\right). \end{aligned}$$

Again, if we assume that $\alpha = \frac{2^q}{(q+1)}(\frac{1}{b-a} f \mathrm{d}\mu)^q$, then

$$\bigvee_{\alpha\in[0,1]} T\left(\alpha, \left(\tfrac{bf(a)-af(b)-\alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)} - a\right)\right) \geq$$
$$T\left(\tfrac{2^q}{q+1}(\tfrac{1}{b-a} f \mathrm{d}\mu)^q, \left(\tfrac{bf(a)-af(b)-2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu}{f(a)-f(b)} - a\right)\right).$$

The latter implies that

$$f^q \mathrm{d}\mu \geq T\Big(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, \frac{bf(a) - af(b) - 2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu}{f(a)-f(b)} - a\Big).$$

Consequently,

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\Big(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, \frac{bf(a) - af(b) - 2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu}{f(a)-f(b)} - a\Big),$$

which completes the proof. □

The following examples illustrate the validity of Theorem 3.1.

Example 3.2. Suppose that $f(x) = \sqrt{x}$, $[a,b] = [0,1]$ and $q = 0.5$. Then, $f(1) > f(0)$.

i) If $T(x,y)=\min(x,y)$, then we have

$$\begin{aligned}
&\int_{T,[0,1]} x^{\frac{1}{4}}\mathrm{d}\mu \approx 0.7245,\\
&\frac{\sqrt{2}}{0.5+1}\left(\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right)^{0.5} \approx \frac{\sqrt{2}}{1.5}\sqrt{0.6180}\approx 0.7412,\\
&1-2\left(\frac{1}{1.5}\right)^{2}\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu \approx 1-\frac{2}{2.25}\times 0.6180\approx 0.4507.
\end{aligned}$$

And thus

$$\begin{aligned}
\int_{T,[0,1]} x^{\frac{1}{4}}\mathrm{d}\mu \approx& 0.7245 \geq 0.4507\\
\approx& T\left(\frac{\sqrt{2}}{0.5+1}\left(\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right)^{0.5}, 1-2\left(\frac{1}{1.5}\right)^{2}\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right).
\end{aligned}$$

ii) If $T(x,y)=xy$, then we have

$$\begin{aligned}
&\int_{T,[0,1]} x^{\frac{1}{4}}\mathrm{d}\mu = \bigvee_{\alpha\in[0,1]}\alpha(1-\alpha^{4})\approx 0.5350,\\
&\frac{\sqrt{2}}{0.5+1}\left(\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right)^{0.5} = \frac{\sqrt{2}}{1.5}\left(\bigvee_{\alpha\in[0,1]}\alpha(1-\alpha^{2})\right)^{0.5} \approx \frac{\sqrt{2}}{1.5}\sqrt{0.3850}\approx 0.5850,\\
&1-2\left(\frac{1}{1.5}\right)^{2}\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu \approx 1-\frac{2}{2.25}\times 0.3850\approx 0.3422.
\end{aligned}$$

And consequently

$$\begin{aligned}
\int_{T,[0,1]} x^{\frac{1}{4}}\mathrm{d}\mu \approx& 0.5350 \geq 0.2002\\
\approx& T\left(\frac{\sqrt{2}}{0.5+1}\left(\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right)^{0.5}, 1-2\left(\frac{1}{1.5}\right)^{2}\int_{T,[0,1]}\sqrt{x}\mathrm{d}\mu\right).
\end{aligned}$$

EXAMPLE 3.3. Let $f(x) = \sqrt{x}$, $[a, b] = [0, \frac{1}{4}]$, $q = 4$ and $T(x, y) = xy$. In this case, we have $f(\frac{1}{4}) > f(0)$ and

$$4 \int_{T,[0,\frac{1}{4}]} x^2 \mathrm{d}\mu = 4 \left(\bigvee_{\alpha\in[0,1]} \alpha \left(\frac{1}{4} - \sqrt{\alpha} \right) \right) \approx 4 \times 0.0023 = 0.0092,$$

$$\frac{16}{5} \left(4 \int_{T,[0,\frac{1}{4}]} \sqrt{x} \mathrm{d}\mu \right)^4 = \frac{4096}{5} \left(\bigvee_{\alpha\in[0,1]} \alpha \left(\frac{1}{4} - \alpha^2 \right) \right)^4 = \frac{4096}{5} \left(\frac{1}{12\sqrt{3}} \right)^4$$
$$\approx 0.0044,$$

$$\frac{1}{4} - \frac{2(\frac{1}{5})^{\frac{1}{4}} (\int_{T,[0,\frac{1}{4}]} \sqrt{x} \mathrm{d}\mu)}{\frac{1}{2}} = \frac{1}{4} - \frac{2(\frac{1}{5})^{\frac{1}{4}} (\frac{1}{12\sqrt{3}})}{\frac{1}{2}} \approx 0.1213.$$

Therefore,

$$4 \int_{T,[0,\frac{1}{4}]} x \mathrm{d}\mu = 0.0092 \geq 0.0005$$
$$\approx T \left(\frac{16}{5} \left(4 \int_{T,[0,\frac{1}{4}]} \sqrt{x} \mathrm{d}\mu \right)^4, \frac{1}{4} - \frac{2(\frac{1}{5})^{\frac{1}{4}} (\int_{T,[0,\frac{1}{4}]} \sqrt{x} \mathrm{d}\mu)}{\frac{1}{2}} \right).$$

In the next example, we show that the concavity of the function f in Theorem 3.1 is necessary.

EXAMPLE 3.4. Suppose that $f(x) = x^2$, $[a, b] = [0, 1]$, $q = \frac{1}{3}$ and $T(x, y) = xy$. In this case, we have: $f(1) > f(0)$ and

$$\int_{T,[0,1]} x^{\frac{2}{3}} \mathrm{d}\mu = \bigvee_{\alpha\in[0,1]} \alpha(1 - \alpha^{\frac{3}{2}}) \approx 0.3257,$$

$$\frac{\sqrt[3]{2}}{\frac{1}{3} + 1} \left(\int_{T,[0,1]} x^2 \mathrm{d}\mu \right)^{\frac{1}{3}} = \frac{3\sqrt[3]{2}}{4} \left(\bigvee_{\alpha\in[0,1]} \alpha(1 - \sqrt{\alpha}) \right)^{\frac{1}{3}} \approx \frac{3\sqrt[3]{2}}{4} (0.1481)^{\frac{1}{3}} \approx 0.5,$$

$$1 - 2 \left(\frac{3}{4} \right)^3 \int_{T,[0,1]} x^2 \mathrm{d}\mu \approx 1 - 2 \left(\frac{3}{4} \right)^3 \times 0.1481 \approx 0.8750.$$

Consequently,

$$\int_{T,[0,1]} x^{\frac{2}{3}} \mathrm{d}\mu \approx 0.3257 \ngeq 0.4375$$
$$\approx T \left(\frac{\sqrt[3]{2}}{\frac{1}{3} + 1} \left(\int_{T,[0,1]} x^2 \mathrm{d}\mu \right)^{\frac{1}{3}}, 1 - 2 \left(\frac{3}{4} \right)^3 \int_{T,[0,1]} x^2 \mathrm{d}\mu \right).$$

Theorem 3.5. *Let $f : [a,b] \to [0,1]$ be a concave function and μ be the Lebesgue measure on $\mathbb{R}$. Then, for any $q > 0$, the following implications hold:*

(a) *if $f(b) > f(a)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\bigg(\frac{q+1}{2^q}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, b - \frac{\frac{(q+1)^{\frac{1}{q}}}{2} f \mathrm{d}\mu + af(b) - bf(a)}{f(b) - f(a)}\bigg),$$

(b) *if $f(b) = f(a)$, then*

$$f^q \mathrm{d}\mu \geq T(f^q(a), b-a),$$

(c) *if $f(a) > f(b)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \geq T\bigg(\frac{q+1}{2^q}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, \frac{bf(a) - af(b) - \frac{(q+1)^{\frac{1}{q}}}{2} f \mathrm{d}\mu}{f(a) - f(b)} - a\bigg).$$

Proof. Considering $\alpha = \frac{q+1}{2^q}(\frac{1}{b-a} f \mathrm{d}\mu)^q$, the proof is the same as in the previous theorem. □

In the sequel, we prove Favard's inequality for the semiconormed fuzzy integral of a convex function.

Theorem 3.6. *Let $f : [a,b] \to [0,1]$ be a convex function and μ be the Lebesgue measure on $\mathbb{R}$. Then, for any $q > 0$, the following implications hold:*

(a) *if $f(b) > f(a)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \leq \frac{1}{b-a} S\bigg(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu + af(b) - bf(a)}{f(b) - f(a)}\bigg),$$

(b) *if $f(b) = f(a)$, then*

$$f^q \mathrm{d}\mu \leq S(f^q(a), \mu(X) - b + a),$$

(c) *if $f(a) > f(b)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \leq \frac{1}{b-a} S\bigg(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f \mathrm{d}\mu\Big)^q, \frac{bf(a) - af(b) - 2(\frac{1}{q+1})^{\frac{1}{q}} f \mathrm{d}\mu}{f(a) - f(b)} - a\bigg).$$

Proof. Assume that $x \in [a,b]$. Set $x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b$. Then, by the convexity of f, we have

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) = h(x).$$

By Proposition 2.9 (a), we have $f^q \mathrm{d}\mu \leq h^q \mathrm{d}\mu$.

(a) If $f(b) > f(a)$, then

$$\begin{aligned} f^q \mathrm{d}\mu &\leq h^q \mathrm{d}\mu \\ &= \bigwedge_{\alpha\in[0,1]} S\Big(\alpha, \mu([a,b] \cap \{x|h(x) \geq \alpha^{\frac{1}{q}}\})\Big) \\ &= \bigwedge_{\alpha\in[0,1]} S\left(\alpha, \mu\left([a,b] \cap \left\{x|x \geq \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b)-f(a)}\right\}\right)\right) \\ &= \bigwedge_{\alpha\in[0,1]} S\left(\alpha, \left(b - \frac{\alpha^{\frac{1}{q}}(b-a) + af(b) - bf(a)}{f(b)-f(a)}\right)\right). \end{aligned}$$

If we assume that $\alpha = \frac{2^q}{q+1}(\frac{1}{b-a} f\mathrm{d}\mu)^q$, then $\alpha \in [0,1]$ and thus

$$\bigwedge\nolimits_{\alpha\in[0,1]} S\left(\alpha, \left(b - \tfrac{\alpha^{\frac{1}{q}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right)\right) \leq$$
$$S\left(\tfrac{2^q}{q+1}\left(\tfrac{1}{b-a} f\mathrm{d}\mu\right)^q, \left(b - \tfrac{2(\frac{1}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu + af(b)-bf(a)}{f(b)-f(a)}\right)\right).$$

It follows that

$$f^q \mathrm{d}\mu \leq S\Big(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f\mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu + af(b) - bf(a)}{f(b)-f(a)}\Big).$$

Consequently,

$$\frac{1}{b-a} f^q \mathrm{d}\mu \leq \frac{1}{b-a} S\Big(\frac{2^q}{q+1}\Big(\frac{1}{b-a} f\mathrm{d}\mu\Big)^q, b - \frac{2(\frac{1}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu + af(b) - bf(a)}{f(b)-f(a)}\Big).$$

(b) If $f(a) = f(b)$, then $h(x) = f(a) = f(b)$ and, using Remark 2.10 (c′), we have

$$f^q \mathrm{d}\mu \leq h^q \mathrm{d}\mu = f^q(a)\mathrm{d}\mu = S(f^q(a), \mu(X) - (b-a)).$$

(c) If $f(a) > f(b)$, then

$$\begin{aligned} f^q \mathrm{d}\mu &\leq h^q \mathrm{d}\mu \\ &= \bigwedge_{\alpha\in[0,1]} S\Big(\alpha, \mu([a,b] \cap \{x|h(x) \geq \alpha^{\frac{1}{q}}\})\Big) \\ &= \bigwedge_{\alpha\in[0,1]} S\left(\alpha, \mu\left([a,b] \cap \left\{x|x \leq \frac{bf(a) - af(b) - \alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)}\right\}\right)\right) \\ &= \bigwedge_{\alpha\in[0,1]} S\left(\alpha, \left(\frac{bf(a) - af(b) - \alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)} - a\right)\right). \end{aligned}$$

Again, if we assume that $\alpha = \frac{2^q}{q+1}(\frac{1}{b-a} f \mathrm{d}\mu)^q$, then

$$\bigwedge_{\alpha\in[0,1]} S\left(\alpha, \left(\frac{bf(a)-af(b)-\alpha^{\frac{1}{q}}(b-a)}{f(a)-f(b)} - a\right)\right)$$
$$\leq S\left(\frac{2^q}{q+1}\left(\frac{1}{b-a} f \mathrm{d}\mu\right)^q, \left(\frac{bf(a)-af(b)-2(\frac{1}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu}{f(a)-f(b)} - a\right)\right),$$

It follows that

$$f^q \mathrm{d}\mu \leq S\Big(\frac{2^q(b-a)}{q+1}\Big(\frac{1}{b-a} f\mathrm{d}\mu\Big)^q, \frac{bf(a)-af(b)-2(\frac{b-a}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu}{f(a)-f(b)} - a\Big).$$

and, consequently,

$$\frac{1}{b-a} f^q \mathrm{d}\mu$$
$$\leq \frac{1}{b-a} S\Big(\frac{2^q(b-a)}{q+1}\Big(\frac{1}{b-a} f\mathrm{d}\mu\Big)^q, \frac{bf(a)-af(b)-2(\frac{b-a}{q+1})^{\frac{1}{q}} f\mathrm{d}\mu}{f(a)-f(b)} - a\Big).$$

This completes the proof. □

For illustrating the last theorem, we present an example.

EXAMPLE 3.7. Suppose that $f(x) = x^2$, $[a,b] = [0,1]$, $q = 0.5$ and $S(x,y) = x + y - xy$. Then $f(1) > f(0)$ and simple calculations show that

$$\int_{S,[0,1]} x\mathrm{d}\mu = \bigwedge_{\alpha\in[0,1]} S(\alpha, 1-\alpha) = \bigwedge_{\alpha\in[0,1]} (\alpha + 1 - \alpha - \alpha(1-\alpha)) = 0.75,$$
$$\frac{\sqrt{2}}{1.5}\Big(\int_{S,[0,1]} x^2\mathrm{d}\mu\Big)^{\frac{1}{2}} = \frac{\sqrt{2}}{1.5}\Big(\bigwedge_{\alpha\in[0,1]} (1 - \alpha^{\frac{1}{2}} + \alpha^{\frac{3}{2}})\Big)^{\frac{1}{2}} \approx \frac{\sqrt{2}}{1.5}(0.6151)^{\frac{1}{2}} \approx 0.7394,$$
$$1 - 2\left(\frac{1}{1.5}\right)^2 \int_{S,[0,1]} x^2\mathrm{d}\mu \approx 1 - \frac{8}{9} \times 0.6151 \approx 0.4532.$$

Therefore,

$$\int_{S,[0,1]} x\mathrm{d}\mu \approx 0.75 \leq 0.8575 \approx S(\frac{\sqrt{2}}{1.5}(\int_{S,[0,1]} x^2\mathrm{d}\mu)^{\frac{1}{2}}, 1 - 2(\frac{1}{1.5})^2 \int_{S,[0,1]} x^2\mathrm{d}\mu).$$

The following example shows that the convexity of the function f in Theorem 3.6 is necessary.

Example 3.8. Let $f(x) = \sqrt{x}$, $[a, b] = [0, 1]$, $q = 0.5$ and $S(x, y) = x + y - xy$. Then $f(1) > (0)$ and we have

$$\int_{S,[0,1]} x^{\frac{1}{4}} \mathrm{d}\mu = \bigwedge_{\alpha\in[0,1]} S(\alpha, 1-\alpha^4) = \bigwedge_{\alpha\in[0,1]} (1-\alpha^4+\alpha^5) \approx 0.9180,$$

$$\frac{\sqrt{2}}{1.5}\left(\int_{S,[0,1]} \sqrt{x}\mathrm{d}\mu\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{1.5}(\bigwedge_{\alpha\in[0,1]} (1-\alpha^2+\alpha^3))^{\frac{1}{2}} \approx \frac{\sqrt{2}}{1.5}(0.8519)^{\frac{1}{2}} \approx 0.8701,$$

$$1-2\left(\frac{1}{1.5}\right)^2 \int_{S,[0,1]} \sqrt{x}\mathrm{d}\mu \approx 1-\frac{8}{9}\times 0.8519 \approx 0.2427.$$

Consequently,

$$\int_{S,[0,1]} x^{\frac{1}{4}} \mathrm{d}\mu \approx 0.9180 \not\leq 0.9016$$
$$\approx S\left(\frac{\sqrt{2}}{1.5}(\int_{S,[0,1]} \sqrt{x}\mathrm{d}\mu)^{\frac{1}{2}}, 1-2(\frac{1}{1.5})^2 \int_{S,[0,1]} \sqrt{x}\mathrm{d}\mu\right).$$

Theorem 3.9. *Let $f : [a, b] \to [0, 1]$ be a convex function and μ be the Lebesgue measure on $\mathbb{R}$. Then, for any $q > 0$, we have:*

(a) *if $f(b) > f(a)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \leq \frac{1}{b-a} S\left(\frac{q+1}{2^q}\left(\frac{1}{b-a} f\mathrm{d}\mu\right)^q, b - \frac{\frac{(q+1)^{\frac{1}{q}}}{2} f\mathrm{d}\mu + af(b) - bf(a)}{f(b)-f(a)}\right),$$

(b) *if $f(b) = f(a)$, then*

$$f^q \mathrm{d}\mu \leq S(f^q(a), \mu(X) - b + a),$$

(c) *if $f(a) > f(b)$, then*

$$\frac{1}{b-a} f^q \mathrm{d}\mu \leq \frac{1}{b-a} S\left(\frac{q+1}{2^q}\left(\frac{1}{b-a} f\mathrm{d}\mu\right)^q, \frac{bf(a) - af(b) - \frac{(q+1)^{\frac{1}{q}}}{2} f\mathrm{d}\mu}{f(a)-f(b)} - a\right).$$

Proof. Take $\alpha = \frac{q+1}{2^q}(\frac{1}{b-a} f\mathrm{d}\mu)^q$ in the proof of the previous theorem. □

4. CONCLUSION

In this note, we have proved Favard's inequality for the seminormed and semiconormed fuzzy integrals. Moreover, by some examples, it is shown that the assumptions of concavity and convexity for the functions in the fuzzy version of Favard's inequality are necessary.

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Received June 23, 2016
Accepted April 20, 2017

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