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SOLVABILITY FOR A NONLINEAR FOURTH-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study the existence of a nontrivial solution for the fourth-order three-point boundary value problem having the following form

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = \alpha u'(\eta),$$

where $\eta \in (0, 1)$, $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. By using the Leray-Schauder nonlinear alternative, we prove the existence of at least one solution of the above problem. As an application, we also given some examples to illustrate the obtained results.

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Key words. Solvability, Green's function, Leray-Schauder nonlinear alternative, fixed point theorem, boundary value problem.

1. INTRODUCTION

The study of fourth-order three-point boundary value problems (BVP) for ordinary differential equations arises in a variety of different areas of applied mathematics and physics.

Many authors studied the existence of positive solutions for the nth-order m-point boundary value problems, using different methods such as fixed point theorems in cones, nonlinear alternative of Leray-Schauder and the Krasnosel'skii fixed point theorem (see [1, 4, 5, 12] and the references therein).

In 2003, by using the Leray-Schauder degree theory, Yuji Liu and Weigao Ge [8] proved the existence of positive solutions for the $(n-1, 1)$ three-point boundary value problems with a coefficient that changes sign, given as follows:

$$u^{(n)}(t) + \lambda a(t) f(u(t)) = 0, \quad \mathrm{t} \in (0, 1),$$
$$u(0) = \alpha u(\eta), u(1) = \beta u(\eta),\ u^{(i)}(0) = 0,\ \text{for } i = 1, 2, \dots, n-2, u^{(n-2)}(0)$$
$$= \alpha u^{(n-2)}(\eta),\ u^{(n-2)}(1) = \beta u^{(n-2)}(\eta),\ u^{(i)}(0) = 0,\ \text{for } i = 1, 2, \dots, n-3,$$

where $\eta \in (0, 1)$, $\alpha \geq 0$, $\beta \geq 0$, $\mathbb{R} = (-\infty, \infty)$, $f(0) > 0$, $\lambda > 0$ is a parameter and $a : (0, 1) \to \mathbb{R}$ may change sign.

In 2005, Paul W. Eloea and Bashir Ahmad [3] studied the existence of positive solutions of a nonlinear nth-order boundary value problem with nonlocal

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conditions, given as follows:

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad \mathrm{t} \in (0,1),$$
$$u(0) = 0, \quad u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$, f is either superlinear or sublinear. They used the fixed point theorem in cones due to Krasnosel'skii and Guo.

Then, in 2009, Xie, Liu and Bai [11] used the fixed-point theory to study the existence of positive solutions of a singular nth-order three-point boundary value problem on time scales, which is given in the following:

$$u^{(n)}(t) + h(t)f(u(t)) = 0, \quad \mathrm{t} \in (0,1),$$
$$u(a) = \alpha u(\eta), \quad u'(a) = 0, \ldots, u^{(n-2)}(a) = 0, \quad u(b) = \beta u(\eta),$$

where $a < \eta < b$, $0 \le \alpha < 1$, $0 < \beta(\eta - a)^{n-1} < (1-\alpha)(b-a)^{n-1} + \alpha(\eta - a)^{n-1}$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and $h \in C([a,b],[0,\infty))$ may be singular at $t = a$ and $t = b$.

In 2004, Yong-Ping Sun [9] studied the existence of a nontrivial solution for the three-point boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1,$$
$$u'(0) = 0, \quad u(1) = \alpha u'(\eta),$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, using the Leray-Schauder nonlinear alternative. In [10], the same author used a similar method to study nontrivial symmetric solutions of the nonlinear second-order three-point boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1,$$
$$u(0) = u(1) = \alpha u(\eta),$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $f(\cdot, x)$ is symmetric on $[0,1]$, for every $x \in \mathbb{R}$.

Moreover, Li and Sun [7] used the above mentioned method to study the nontrivial solutions of the nonlinear second-order three-point boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1,$$
$$au(0) - bu'(0) = 0, \quad u(1) - \alpha u(\eta) = 0,$$

where $\eta \in (0,1)$, $a, b, \alpha \in \mathbb{R}$, with $a^2 + b^2 > 0$.

Motivated by the above works, we extend the results obtained for the second-order boundary value problems to the fourth-order boundary value problems with new boundary conditions (see (2) below), by using a different approach than that in the above mentioned papers. More precisely, we prove the existence of a nontrivial solution for the fourth-order three-point boundary value problem (BVP)

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1. \tag{1}$$

$$u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = \alpha u'(\eta), \tag{2}$$

where $0 < \eta < 1$, $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\mathbb{R} = (-\infty, \infty)$.

This paper is organized as follows. In Section 2, we present two lemmas that will be used to prove the some results. In Section 3, we present and prove our main results, which are given by some existence theorems and a corollary for the nontrivial solution of the BVP (1)–(2). Then we establish some criteria for the existence of at least one solution, by using the Leray-Schauder nonlinear alternative. Finally, in Section 4, as an application, we give some examples to illustrate the results that we have obtained.

2. PRELIMINARIES

Let $E = C([0,1])$ with the norm given by $\|y\| = \sup_{t\in[0,1]} |y(t)|$, for any $y \in E$. A solution $u(t)$ of the BVP (1)–(2) is called a nontrivial solution if $u(t) \neq 0$. To get our results, we need the following lemma.

LEMMA 2.1. *Let $y \in C([0,1])$, $\alpha \neq 1$, then the boundary value problem*

$$u^{(4)}(t) + y(t) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = \alpha u'(\eta),$$

has a unique solution

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta - s)^2 y(s)\mathrm{d}s.$$

Proof. Rewriting the differential equation as $u^{(4)}(t) = -y(t)$ and integrating four times from 0 to t, we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t^3}{6}c + \frac{t^2}{2}c_1 + tc_2 + c_3. \tag{3}$$

By the boundary condition (2), we have $u(0) = 0, u''(0) = u'''(0) = 0$, i.e., $c_1 = c_3 = c = 0$ and $u'(1) = \alpha u'(\eta)$, and thus we get

$$c_2 = \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha}{2(1-\alpha)}\int_0^\eta (\eta - s)^2 y(s)\mathrm{d}s. \tag{4}$$

Using the equations (3) and (4), we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 y(s)\mathrm{d}s.$$

This completes the proof. □

Define the integral operator $T : E \to E$, by

$$\begin{aligned} Tu(t) = &-\frac{1}{6}\int_0^t (t-s)^3 f(s,u(s))\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 f(s,u(s))\mathrm{d}s \\ &- \frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 f(s,u(s))\mathrm{d}s. \end{aligned} \tag{5}$$

By Lemma 2.1, the BVP (1)–(2) has a solution if and only if the operator T has a fixed point in E. So we only need to find a fixed point of T in E. By the Ascoli-Arzela theorem, we can prove that T is a completely continuous operator. Next, we present the Leray-Schauder nonlinear alternative.

Lemma 2.2 ([2]). *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. Let $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either* (i) *there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or* (ii) *there exists a fixed point $u^* \in \overline{\Omega}$ of T.*

3. EXISTENCE OF NONTRIVIAL SOLUTIONS

In this section, we prove the existence of a nontrivial solution for the BVP (1)–(2). Suppose that $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Theorem 3.1. *Suppose that $f(t,0) \neq 0$, $\alpha \neq 1$ and that there exist non-negative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad \textit{a.e. } (t,x) \in [0,1] \times \mathbb{R},$$

$$\frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s < 1.$$

Then the BVP (1)–(2) *has at least one nontrivial solution $u^* \in C([0,1])$.*

Proof. Let

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s,$$

$$N = \frac{1}{6}\int_0^1 (1-s)^3 h(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 h(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 h(s)\mathrm{d}s.$$

Then $M < 1$. Since $f(t,0) \neq 0$, there exists an interval $[a,b] \subset [0,1]$ such that $\min_{a\le t\le b} |f(t,0)| > 0$. As $h(t) \ge |f(t,0)|$, a.e. $t \in [0,1]$, we know that $N > 0$.

Let $A = N(1-M)^{-1}$ and $\Omega = \{u \in E : \|u\| < A\}$. Let $u \in \partial\Omega$ and $\lambda > 1$ be such that $Tu = \lambda u$. Then

$$\begin{aligned}
\lambda A = \lambda\|u\| = \|Tu\| &= \max_{0\le t\le 1} |(Tu)(t)| \\
&\le \frac{1}{6}\max_{0\le t\le 1}\int_0^t (t-s)^3 |f(s,u(s))|\mathrm{d}s \\
&+ \max_{0\le t\le 1}\left|\frac{t}{2(1-\alpha)}\right|\int_0^1 (1-s)^2|f(s,u(s))|\mathrm{d}s \\
&+ \max_{0\le t\le 1}\left|\frac{\alpha t}{2(1-\alpha)}\right|\int_0^\eta (\eta-s)^2|f(s,u(s))|\mathrm{d}s \\
&\le \frac{1}{6}\int_0^1 (1-s)^3|f(s,u(s))|\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2|f(s,u(s))|\mathrm{d}s \\
&+ \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2|f(s,u(s))|\mathrm{d}s \\
&\le \left[\frac{1}{6}\int_0^1 (1-s)^3 k(s)|u(s)|\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)|u(s)|\mathrm{d}s \right.\\
&\left. + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)|u(s)|\mathrm{d}s\right] + \left[\frac{1}{6}\int_0^1 (1-s)^3 h(s)\mathrm{d}s \right.\\
&\left. + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 h(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 h(s)\mathrm{d}s\right] \\
&= M\|u\| + N
\end{aligned}$$

Therefore,

$$\lambda \le M + \frac{N}{A} = M + \frac{N}{N(1-M)^{-1}} = M + (1-M) = 1.$$

This contradicts $\lambda > 1$. By Lemma 2.2, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0) \neq 0$, the BVP (1)–(2) has a nontrivial solution $u^* \in E$.

This completes the proof. □

THEOREM 3.2. *Suppose that $f(t,0) \neq 0$, $\alpha < 1$ and that there exist non-negative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is fulfilled

(1) *there exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)\mathrm{d}s < \left[\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}\right]^p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$,

(2) *there exists a constant $\mu > -1$ such that*

$$k(s) \leq \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})} s^\mu, \quad a.e.\ s \in [0,1],$$

$$meas\left\{s \in [0,1] : k(s) < \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})} s^\mu\right\} > 0,$$

(3) *there exists a constant $\mu > -3$ such that*

$$k(s) \leq \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^\mu, \quad a.e.\ s \in [0,1],$$

$$meas\left\{s \in [0,1] : k(s) < \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^\mu\right\} > 0,$$

(4) *k satisfies*

$$k(s) \leq \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}, \quad a.e.\ s \in [0,1],$$

$$meas\left\{s \in [0,1] : k(s) < \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}\right\} > 0,$$

then the BVP (1)–(2) has at least one nontrivial solution $u^* \in E$.

Proof. Let M be defined as in the proof of Theorem 3.1. To prove Theorem 3.2, we only need to prove that $M < 1$. Since $\alpha < 1$, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s.$$

(1) Using the Hölder inequality, we have

$$
\begin{aligned}
M \leq & \left[\int_0^1 k^p(s)\mathrm{d}s\right]^{1/p} \left\{\frac{1}{6}\left[\int_0^1 (1-s)^{3q}\mathrm{d}s\right]^{1/q}\right. \\
& \left. + \frac{1}{2(1-\alpha)}\left[\int_0^1 (1-s)^{2q}\mathrm{d}s\right]^{1/q} + \frac{|\alpha|}{2(1-\alpha)}\left[\int_0^\eta (\eta-s)^{2q}\mathrm{d}s\right]^{1/q}\right\} \\
& \leq \left[\int_0^1 k^p(s)\mathrm{d}s\right]^{1/p} \left[\frac{1}{6}\left(\frac{1}{1+3q}\right)^{1/q} + \frac{1}{2(1-\alpha)}\left(\frac{1}{1+2q}\right)^{1/q}\right. \\
& \qquad \left. + \frac{|\alpha|}{2(1-\alpha)}\left(\frac{\eta^{1+2q}}{1+2q}\right)^{1/q}\right] \\
& < \frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})} \\
& \qquad \times \frac{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}} = 1.
\end{aligned}
$$

(2) In this case, we have

$$
\begin{aligned}
M < & \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}\left[\frac{1}{6}\int_0^1 (1-s)^3 s^\mu \mathrm{d}s + \frac{1}{2(1-\alpha)}\right. \\
& \left. \times \int_0^1 (1-s)^2 s^\mu \mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 s^\mu \mathrm{d}s\right] \\
\leq & \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}\left[\frac{1}{(1+\mu)(2+\mu)(3+\mu)(4+\mu)}\right. \\
& \left. + \frac{1}{(1-\alpha)}\frac{1}{(1+\mu)(2+\mu)(3+\mu)} + \frac{|\alpha|}{(1-\alpha)}\frac{\eta^{3+\mu}}{(1+\mu)(2+\mu)(3+\mu)}\right] \\
= & \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})} \\
& \times \frac{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)} = 1.
\end{aligned}
$$

(3) In this case, we have

$$
\begin{aligned}
M < & \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}\left[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s\right. \\
& \left. + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2(1-s)^\mu \mathrm{d}s\right] \\
& \leq \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}\left[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s\right.
\end{aligned}
$$

$$+\frac{1}{2(1-\alpha)}\int_0^1(1-s)^{2+\mu}\mathrm{d}s+\frac{|\alpha|}{2(1-\alpha)}\int_0^1(1-s)^{2+\mu}\mathrm{d}s\Big]$$
$$=\frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}\,\frac{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}{6(1-\alpha)(3+\mu)(4+\mu)}$$
$$=1.$$

(4) In this case, we have

$$M<\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}\Big[\frac{1}{6}\int_0^1(1-s)^3\mathrm{d}s$$
$$+\frac{1}{2(1-\alpha)}\int_0^1(1-s)^2\mathrm{d}s+\frac{|\alpha|}{2(1-\alpha)}\int_0^\eta(\eta-s)^2\mathrm{d}s\Big]$$
$$=\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}\,\frac{(1-\alpha)+4(1+|\alpha|\eta^3)}{24(1-\alpha)}=1.$$

This completes the proof. □

THEOREM 3.3. *Suppose that $f(t,0)\neq 0$, $\alpha>1$ and that there exist non-negative functions $k,h\in L^1[0,1]$ such that*

$$|f(t,x)|\le k(t)|x|+h(t),\quad a.e.\ (t,x)\in[0,1]\times\mathbb{R}.$$

If one of the following conditions holds

(1) *there exists a constant $p>1$ such that*

$$\int_0^1 k^p(s)\mathrm{d}s<\left[\frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})}\right]^p,$$

where $\frac{1}{p}+\frac{1}{q}=1$,

(2) *there exists a constant $\mu>-1$ such that*

$$k(s)\le\frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}s^\mu,\quad a.e.\ s\in[0,1],$$
$$meas\left\{s\in[0,1]:k(s)<\frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}s^\mu\right\}>0,$$

(3) *there exists a constant $\mu>-3$ such that*

$$k(s)\le\frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}(1-s)^\mu,\quad a.e.\ s\in[0,1],$$
$$meas\left\{s\in[0,1]:k(s)<\frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}(1-s)^\mu\right\}>0,$$

(4) *k satisfies*

$$k(s)\le\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)},\quad a.e.\ s\in[0,1],$$
$$meas\left\{s\in[0,1]:k(s)<\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}\right\}>0.$$

then the BVP (1)–(2) *has at least one nontrivial solution* $u^* \in E$.

Proof. Let M be defined as in the proof of Theorem 3.1. To prove Theorem 3.3, we only need to prove that $M < 1$. Since $\alpha > 1$, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s.$$

(1) Using the Hölder inequality, we have

$$\begin{aligned} M &\le \left[\int_0^1 k^p(s)\mathrm{d}s\right]^{1/p} \left\{\frac{1}{6}\left[\int_0^1 (1-s)^{3q}\mathrm{d}s\right]^{1/q} + \frac{1}{2(\alpha-1)}\left[\int_0^1 (1-s)^{2q}\mathrm{d}s\right]^{1/q} + \frac{\alpha}{2(\alpha-1)}\left[\int_0^\eta (\eta-s)^{2q}\mathrm{d}s\right]^{1/q}\right\} \\ &\le \left[\int_0^1 k^p(s)\mathrm{d}s\right]^{1/p}\left[\frac{1}{6}\left(\frac{1}{1+3q}\right)^{1/q} + \frac{1}{2(\alpha-1)}\left(\frac{1}{1+2q}\right)^{1/q} + \frac{\alpha}{2(\alpha-1)}\left(\frac{\eta^{1+2q}}{1+2q}\right)^{1/q}\right] \\ &< \frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})} \\ &\quad \times \frac{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})}{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}} = 1. \end{aligned}$$

(2) In this case, we have

$$\begin{aligned} M &< \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}\left[\frac{1}{6}\int_0^1 (1-s)^3 s^\mu \mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 s^\mu\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2 s^\mu\mathrm{d}s\right] \\ &= \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})} \\ &\quad \times \frac{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)} = 1. \end{aligned}$$

(3) In this case, we have

$$M < \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}\left[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2(1-s)^\mu\mathrm{d}s\right]$$

$$\le \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}\bigg[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s$$
$$+\frac{1}{2(\alpha-1)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s+\frac{\alpha}{2(\alpha-1)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s\bigg]$$
$$=\frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}\,\frac{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}{6(\alpha-1)(3+\mu)(4+\mu)}=1.$$

(4) In this case, we have

$$M<\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}\bigg[\frac{1}{6}\int_0^1 (1-s)^3\mathrm{d}s+\frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2\mathrm{d}s$$
$$+\frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2\mathrm{d}s\bigg]$$
$$=\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}\,\frac{(\alpha-1)+4(1+\alpha\eta^3)}{24(\alpha-1)}=1.$$

This completes the proof. □

COROLLARY 3.4. *Suppose $f(t,0)\neq 0$, $\alpha<1$ and that there exist non-negative functions $k,h\in L^1[0,1]$ such that*

$$|f(t,x)|\le k(t)|x|+h(t),\quad a.e.\ (t,x)\in[0,1]\times\mathbb{R}.$$

If one of following conditions holds

(1) *there exists a constant $p>1$ such that*

$$\int_0^1 k^p(s)ds<\left[\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+|\alpha|)}\right]^p,$$

where $\frac{1}{p}+\frac{1}{q}=1$,

(2) *there exists a constant $\mu>-1$ such that*

$$k(s)\le\frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|)}s^\mu,\quad a.e.\ s\in[0,1],$$

$$meas\left\{s\in[0,1]:k(s)<\frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|)}s^\mu\right\}>0,$$

(3) *k satisfies*

$$k(s)\le\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|)},\quad a.e.\ s\in[0,1],$$

$$meas\left\{s\in[0,1]:k(s)<\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|)}\right\}>0,$$

then the BVP (1)–(2) has at least one nontrivial solution $u^*\in E$.

Proof. In this case, we have

$$\begin{aligned}
M &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\
&\qquad + \frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s \\
&\le \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\
&\qquad + \frac{|\alpha|}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\
&= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1+|\alpha|}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s.
\end{aligned}$$

Now, the proof follows, by using the same method as the one used in the proof of Theorem 3.2. The proof is complete. □

Corollary 3.5. *Suppose that $f(t,0)\neq 0$, $\alpha>1$ and that there exist nonnegative functions $k,h\in L^1[0,1]$ such that*

$$|f(t,x)|\le k(t)|x|+h(t),\quad a.e.\ (t,x)\in[0,1]\times\mathbb{R}.$$

If one of the following conditions holds

(1) *there exists a constant $p>1$ such that*

$$\int_0^1 k^p(s)ds < [\frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha)}]^p,$$

where $\frac{1}{p}+\frac{1}{q}=1$,

(2) *there exists a constant $\mu>-1$ such that*

$$k(s)\le \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha)}s^\mu,\quad a.e.\ s\in[0,1],$$

$$meas\left\{s\in[0,1]: k(s)<\frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha)}s^\mu\right\}>0,$$

(3) *k satisfies*

$$k(s)\le\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha)},\quad a.e.\ s\in[0,1],$$

$$meas\left\{s\in[0,1]: k(s)<\frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha)}\right\}>0,$$

then the BVP (1)–(2) has at least one nontrivial solution $u^*\in E$.

Proof. In this case, we have

$$\begin{aligned} M &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\ &\quad + \frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s \\ &\le \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\ &\quad + \frac{\alpha}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\ &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1+\alpha}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s. \end{aligned}$$

The rest of the proof follows in the same way as in the proof of Theorem 3.3. This completes the proof. □

4. EXAMPLES

In order to illustrate the above results, we consider some examples.

EXAMPLE 4.1. Consider the three-point boundary value problem

$$(6) \qquad \begin{aligned} & u^{(4)} + \frac{t}{\sqrt{2}} u \sin u - t - 2 = 0, \quad 0 < t < 1, \\ & u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -3u'(1/2). \end{aligned}$$

Set $\eta = 1/2$, $\alpha = -3 \neq 1$ and

$$f(t,x) = \frac{t}{\sqrt{2}} x \sin x - t - 2,$$

$$k(t) = t, \quad h(t) = t + 2.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

Moreover, we have

$$\begin{aligned} M &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s \\ &\quad + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s \\ &= \frac{1}{6}\int_0^1 (1-s)^3 s\mathrm{d}s + \frac{1}{8}\int_0^1 (1-s)^2 s\mathrm{d}s + \frac{3}{8}\int_0^{1/2} (\frac{1}{2}-s)^2 s\mathrm{d}s = 0.019 < 1. \end{aligned}$$

Hence, by Theorem 3.1, the BVP (6) has at least one nontrivial solution u^* in E.

EXAMPLE 4.2. Consider the three-point boundary value problem

$$
(7) \qquad \begin{gathered}
u^{(4)} + \frac{3}{25}(7+t)u - e^t + 1 = 0, \quad 0 < t < 1, \\
u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -2u'(1/4).
\end{gathered}
$$

Set $\eta = 1/4$, $\alpha = -2 < 1$ and

$$f(t,x) = \frac{3}{25}(7+t)x - e^t + 1,$$

$$k(t) = \frac{1}{2}(7+t), \quad h(t) = e^t + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

Let $p = q = 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_0^1 k^p(s)\mathrm{d}s = \int_0^1 \frac{1}{4}(7+s)^2\mathrm{d}s = \frac{169}{12}.$$

Moreover, we have

$$\left[\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}\right]^p = 49.454.$$

Therefore,

$$\int_0^1 k^p(s)\mathrm{d}s < \left[\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}\right]^p.$$

Hence, by Theorem 3.2 (1), the BVP (7) has at least one nontrivial solution u^* in E.

EXAMPLE 4.3. Consider the three-point boundary value problem

$$
(8) \qquad \begin{gathered}
u^{(4)} + \tfrac{tu^2}{9(5+u)} \cos u - e^t - 1 = 0, \quad 0 < t < 1, \\
u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -4u'(1/3).
\end{gathered}
$$

Set $\eta = 1/3$, $\alpha = -4 < 1$ and

$$f(t,x) = \frac{tx^2}{9(5+x)} \cos x - e^t - 1,$$

$$k(t) = \frac{1}{9}t, \quad h(t) = e^t + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

Let $\mu = 1 > -1$. Then

$$\frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})} = 58.559.$$

Therefore,

$$k(s) = \frac{1}{9}s < 58.559.s,$$

$$meas\left\{s \in [0,1] : k(s) < \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}s^{\mu}\right\} > 0.$$

Hence, by Theorem 3.2 (2), the BVP (8) has at least one nontrivial solution u^* in E.

Example 4.4. Consider the three-point boundary value problem

$$\begin{aligned} &u^{(4)} + \frac{5u^3}{8(1+u)(1-t)^{-2}}\sin u + t^4 - 3 = 0, \quad 0 < t < 1, \\ &u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -6u'(1/2). \end{aligned} \tag{9}$$

Set $\eta = 1/2$, $\alpha = -6 < 1$ and

$$f(t,x) = \frac{5x^3}{8(1+x)(1-t)^{-2}}\sin x + t^4 - 3,$$

$$k(t) = \frac{5}{8(1-t)^{-2}}, \quad h(t) = t^4 + 3.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

Let $\mu = 2 > -3$. Then $\dfrac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)} = \dfrac{1260}{161}$. Therefore,

$$k(s) = \frac{5}{8}(1-s)^2 < \frac{1260}{161}(1-s)^2,$$

$$meas\left\{s \in [0,1] : k(s) < \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^{\mu}\right\} > 0.$$

Hence, by Theorem 3.2 (3), the BVP (9) has at least one nontrivial solution u^* in E.

Example 4.5. Consider the three-point boundary value problem

$$\begin{aligned} &u^{(4)} + \frac{tu^2}{5(3+u)} + e^t - 3 = 0, \quad 0 < t < 1, \\ &u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -5u'(1/5). \end{aligned} \tag{10}$$

Set $\eta = 1/5$, $\alpha = -5 < 1$, $f(t,x) = \dfrac{tx^2}{5(3+x)} + e^t - 3$, $k(t) = \dfrac{t}{5}$ and $h(t) = e^t + 3$. It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

Moreover, we have $\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)} = \frac{1800}{127}$. Therefore,

$$k(s) = \frac{s}{5} < \frac{1800}{127}, \quad s \in [0,1],$$

and

$$meas\left\{s \in [0,1] : k(s) < \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}\right\} > 0.$$

Hence, by Theorem 3.2 (4), the BVP (10) has at least one nontrivial solution u^* in E.

Remark 4.6. We can give similar examples for Theorem 3.3, Corollary 3.1 and Corollary 3.2.

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