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ON OPERATORS IN IDEAL MINIMAL SPACES

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Abstract. A collection m_X of subsets of a nonempty set X is called a minimal structure [6] on X if $\phi \in m_X$ and $X \in m_X$. As a generalization of the local closure function $\Gamma(A)$ [1] in an ideal topological space $(X, \tau, \mathcal{I})$, we introduce and investigate an operator $A^{\overline{*}}_m(\mathcal{I}, m_X)$ in an ideal minimal space $(X, m_X, \mathcal{I})$, where $\mathcal{I}$ is an ideal.

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1. INTRODUCTION AND PRELIMINARIES

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$, for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [2]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$, in case there is no reason for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is shown in Example 3.6 of [2] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*. Recall that A is said to be $*$-dense in itself (resp., τ^*-closed, $*$-perfect) if $A \subseteq A^*$ (resp., $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X, τ^*), respectively. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure [6] on X if $\phi \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it a minimal space. Set $m_X(x) = \{U \in m_X : x \in U\}$. For a subset A of X, the m-closure of A and the m-interior of A in (X, m_X) are defined in [7] as follows:

$$m - Int(A) = \cup\{U : U \subseteq A, U \in m_X\},$$
$$m - Cl(A) = \cap\{U : A \subseteq F, X - F \in m_X\}.$$

THEOREM 1.1. ([3]) *Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then:*

(1) $X = m - Int(X)$ *and* $\phi = m - Cl(\phi)$.
(2) $m - Int(A) \subseteq A$ *and* $A \subseteq m - Cl(A)$.
(3) *If* $A \in m_X$, *then* $m - Int(A) = A$ *and, if* $X - F \in m_X$, *then* $m - Cl(F) = F$.
(4) *If* $A \subseteq B$, *then* $m-Int(A) \subseteq m-Int(B)$ *and* $m-Cl(A) \subseteq m-Cl(B)$.
(5) $m-Int(m-Int(A)) = m-Int(A)$ *and* $m-Cl(m-Cl(A)) = m-Cl(A)$.
(6) $m-Cl(X-A) = X-m-Int(A)$ *and* $m-Int(X-A) = X-m-Cl(A)$.

DEFINITION 1.2. A minimal structure m_X on X is said to have

(1) property (B), if m_X is closed under arbitrary unions,
(2) property [I], if m_X is closed under finite intersections.

LEMMA 1.3. ([7]) *Let m_X have property B. Then the following properties hold:*

(1) $A \in m_X$ *if and only if* $m_X - Int(A) = A$,
(2) A *is* m_X*-closed if and only if* $m_X - Cl(A) = A$,
(3) $m_X - Int(A) \in m_X$ *and* $m_X - Cl(A)$ *is* m_X*-closed.*

2. LOCAL OPERATOR FUNCTIONS IN IDEAL MINIMAL SPACES

DEFINITION 2.1. Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. For a subset A of X, we define the following set operators: $A^*_m(\mathcal{I}, m_X) = \{x \in X : A \cap U \notin \mathcal{I}$, for every $U \in m_X(x)\}$ (see [8]), $A^{\overline{*}}_m(\mathcal{I}, m_X) = \{x \in X : A \cap m - Cl(U) \notin \mathcal{I}$, for every $U \in m_X(x)\}$. In the case there is no confusion, $A^{\overline{*}}_m(\mathcal{I}, \tau)$ (resp., $A^*_m(\mathcal{I}, \tau)$) is briefly denoted by $A^{\overline{*}}_m$ (resp. A^*_m) and is called the minimal local closure (resp., minimal local) function of A with respect to $\mathcal{I}$ and m_X.

REMARK 2.2. If an m_X-structure m_X is a topology τ, then $A^*_m = A^*$ and $A^{\overline{*}}_m = \Gamma(A)$ (see [1]).

LEMMA 2.3. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then $A^*_m(\mathcal{I}, m_X) \subseteq A^{\overline{*}}_m(\mathcal{I}, m_X)$, for every subset A of X.*

Proof. Let $x \in A^*_m(\mathcal{I}, m_X)$. Then, $A \cap U \notin \mathcal{I}$, for every m-open set U containing x. Since $A \cap U \subseteq A \cap m - cl(U)$, we have $A \cap m - cl(U) \notin \mathcal{I}$, therefore $x \in A^{\overline{*}}_m(\mathcal{I}, m_X)$.

□

DEFINITION 2.4. ([7]) Let A a subset of (X, m_X). A point $x \in X$ is called

(1) an *m_θ-adherent point* of A, if $m-Cl(U) \cap A \neq \phi$, for every $U \in m_X(x)$.
(2) an *m_θ-interior point* of A, if $m - Cl(U) \subseteq A$, for every $U \in m_X(x)$.

The set of all m_θ-adherent points of A is called the *m_θ-closure* of A and is denoted by $m - Cl_\theta(A)$. If $A = m - Cl_\theta(A)$, then A is said to be *m_θ-closed*. The complement of an m_θ-closed set is said to be *m_θ-open*. The set of all m_θ-interior points of A is called the *m_θ-interior* of A and is denoted by $m - Int_\theta(A)$.

LEMMA 2.5. ([7]) *Let (X, m_X) be a minimal space and A be a subset of X. Then:*

(1) *If A is m-open, then $m - cl(A) = m - cl_\theta(A)$.*
(2) *If A is m-closed, then $m - Int(A) = m - Int_\theta(A)$.*

THEOREM 2.6. *Let (X, m_X) be a minimal space, $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $A^*_m \subseteq B^*_m$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $A^*_m(\mathcal{I}) \supseteq A^*_m(\mathcal{J})$.*
(3) *$A^*_m = m - cl(A^*_m) \subseteq m - cl_\theta(A)$ and A^*_m is m-closed, if m_X has property (B).*
(4) *If $A \subseteq A^*_m$ and A^*_m is m-open, then $A^*_m = m - cl_\theta(A)$.*
(5) *If $A \in \mathcal{I}$, then $A^*_m = \emptyset$.*

Proof. (1) Suppose that $x \notin B^*_m$. Then there exists $U \in m_X(x)$ such that $B \cap m - cl(U) \in \mathcal{I}$. Since $A \cap m - cl(U) \subseteq B \cap m - cl(U)$, $A \cap m - cl(U) \in \mathcal{I}$. Hence $x \notin A^*_m$. Thus $X \setminus B^*_m \subseteq X \setminus A^*_m$ or $A^*_m \subseteq B^*_m$.

(2) Suppose that $x \notin A^*_m(\mathcal{I})$. There exists $U \in m_X(x)$ such that $A \cap m - cl(U) \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $A \cap m - cl(U) \in \mathcal{J}$ and $x \notin A^*_m(\mathcal{J})$. Therefore, $A^*_m(\mathcal{J}) \subseteq A^*_m(\mathcal{I})$.

(3) We have $A^*_m \subseteq m - cl(A^*_m)$ in general. Let $x \in m - cl(A^*_m)$. Then $A^*_m \cap U \neq \emptyset$, for every $U \in m_X(x)$. Therefore, there exists some $y \in A^*_m \cap U$ and $U \in m_X(y)$. Since $y \in A^*_m$, $A \cap m - cl(U) \notin \mathcal{I}$ and hence $x \in A^*_m$. Hence we have $m - cl(A^*_m) \subseteq A^*_m$ and thus $A^*_m = m - cl(A^*_m)$. Again, let $x \in m - cl(A^*_m) = A^*_m$. Then $A \cap m - cl(U) \notin \mathcal{I}$, for every $U \in m_X(x)$. This implies $A \cap m - cl(U) \neq \emptyset$, for every $U \in m_X(x)$. Therefore, $x \in m - cl_\theta(A)$. This shows that $A^*_m(\mathcal{I}) = m - cl(A^*_m) \subseteq m - cl_\theta(A)$.

(4) For any subset A of X, by (3) we have $A^*_m = m - cl(A^*_m) \subseteq m - cl_\theta(A)$. Since $A \subseteq A^*_m$ and A^*_m is m-open, by Lemma 2.5, we have $m - cl_\theta(A) \subseteq m - cl_\theta(A^*_m) = m - cl(A^*_m) = A^*_m \subseteq m - cl_\theta(A)$ and hence $A^*_m = m - cl_\theta(A)$.

(5) Suppose that $x \in A^*_m$. Then, for any $U \in m_X(x)$, $A \cap m - cl(U) \notin \mathcal{I}$. But $A \cap m - cl(U) \subseteq A$ and $A \notin \mathcal{I}$. This is a contradiction. Hence $A^*_m = \emptyset$. □

LEMMA 2.7. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. If m_X has property [I] and U is m_θ-open, then $U \cap A^*_m = U \cap (U \cap A)^*_m \subseteq (U \cap A)^*_m$, for any subset A of X.*

Proof. Suppose that U is m_θ-open and $x \in U \cap A^*_m$. Then $x \in U$ and $x \in A^*_m$. Since U is m_θ-open, then there exists $W \in m_X$ such that $x \in W \subseteq m - cl(W) \subseteq U$. Let V be any m-open set containing x. Then $V \cap W \in m_X(x)$ and $m - cl(V \cap W) \cap A \notin \mathcal{I}$ and hence $m - cl(V) \cap (U \cap A) \notin \mathcal{I}$. This shows that $x \in (U \cap A)^*_m$ and hence we obtain $U \cap A^*_m \subseteq (U \cap A)^*_m$.

Moreover, $U \cap A^*_m \subseteq U \cap (U \cap A)^*_m$ and, by Theorem 2.6, $(U \cap A)^*_m \subseteq A^*_m$ and $U \cap (U \cap A)^*_m \subseteq U \cap A^*_m$. Therefore, $U \cap A^*_m = U \cap (U \cap A)^*_m$. □

THEOREM 2.8. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. If m_X has property [I] and A, B are subsets of X, then the following properties hold:*

(1) $(\emptyset)^*_m = \emptyset$.
(2) $A^*_m \cup B^*_m = (A \cup B)^*_m$.

Proof. (1) The proof is obvious.

(2) It follows from Theorem 2.6 that $(A \cup B)^*_m \supseteq A^*_m \cup B^*_m$. To prove the reverse inclusion, let $x \notin A^*_m \cup B^*_m$. Then x belongs neither to A^*_m nor to B^*_m. Therefore there exist U_x, $V_x \in m_X(x)$ such that $m - cl(U_x) \cap A \in \mathcal{I}$ and $m - cl(V_x) \cap B \in \mathcal{I}$. Since $\mathcal{I}$ is additive, $(m - cl(U_x) \cap A) \cup (m - cl(V_x) \cap B) \in \mathcal{I}$. Moreover, since $\mathcal{I}$ is hereditary and

$$\begin{aligned} m - cl(U_x \cap V_x) \cap (A \cup B) =& (mCl(U_x \cap V_x) \cap A) \cup (mCl(U_x \cap V_x) \cap B) \\ \subseteq& (m - cl(U_x) \cap A) \cup (m - cl(V_x) \cap B), \end{aligned}$$

$m - cl(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I}$. Since $U_x \cap V_x \in m_X(x)$, $x \notin (A \cup B)^*_m$. Hence $(X \setminus A^*_m) \cap (X \setminus B^*_m \subseteq X \setminus (A \cup B)^*_m$ or $(A \cup B)^*_m \subseteq A^*_m \cup B^*_m$. Hence, we obtain $A^*_m \cup B^*_m = (A \cup B)^*_m$. □

LEMMA 2.9. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Let m_X have property [I] and A, B be subsets of X. Then $A^*_m - B^*_m = (A - B)^*_m - B^*_m$.*

Proof. We have, by Theorem 2.8, $A^*_m = [(A - B) \cup (A \cap B)]^*_m = (A - B)^*_m \cup (A \cap B)^*_m \subseteq (A - B)^*_m \cup B^*_m$. Thus $A^*_m - B^*_m \subseteq (A - B)^*_m - B^*_m$. By Theorem 2.6, we get $(A - B)^*_m \subseteq A^*_m$ and hence $(A - B)^*_m - B^*_m \subseteq A^*_m - B^*_m$. Hence $A^*_m - B^*_m = (A - B)^*_m - B^*_m$. □

COROLLARY 2.10. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Let m_X have property [I] and A, B be subsets of X with $B \in \mathcal{I}$. Then $(A \cup B)^*_m = A^*_m = (A - B)^*_m$.*

Proof. Since $B \in \mathcal{I}$, by Theorem 2.6, we have $B^*_m = \emptyset$. By Lemma 2.9, we have $A^*_m = (A - B)^*_m$ and, by Theorem 2.8, $(A \cup B)^*_m = A^*_m \cup B^*_m = A^*_m$. □

3. CLOSURE COMPATIBILITY OF MINIMAL SPACES

DEFINITION 3.1. Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. We say the m_X is closure *m-compatible* with the ideal $\mathcal{I}$ and we denote $m_X \overline{\sim} \mathcal{I}$, if the following holds, for every $A \subseteq X$: if, for every $x \in A$, there exists $U \in m_X(x)$ such that $m - cl(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

REMARK 3.2. If m_X is m-compatible with $\mathcal{I}$, then m_X is closure m-compatible with $\mathcal{I}$.

THEOREM 3.3. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) and (5) ⇒ (1) hold. If m_X has property [I], then the following properties are equivalent:*

(1) $m_X \widetilde{\sim} \mathcal{I}$.
(2) *If a subset A of X has a cover of m-open sets each of whose m-closure intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$.*
(3) *For every $A \subseteq X$, $A \cap A^*_m = \emptyset$ implies that $A \in \mathcal{I}$.*
(4) *For every $A \subseteq X$, $A - A^*_m \in \mathcal{I}$.*
(5) *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^*_m$, then $A \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin A^*_m$ and there exists $V_x \in m_X(x)$ such that $m - cl(V_x) \cap A \in \mathcal{I}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in m_X(x)$ and, by (2), $A \in \mathcal{I}$.

(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A - A^*_m \subseteq A$ and $(A - A^*_m) \cap (A - A^*_m)^*_m \subseteq (A - A^*_m) \cap A^*_m = \emptyset$. By (3), $A - A^*_m \in \mathcal{I}$.

(4) $\Rightarrow$ (5): By (4), for every $A \subseteq X$, $A - A^*_m \in \mathcal{I}$. Let $A - A^*_m = J \in \mathcal{I}$. Then $A = J \cup (A \cap A^*_m)$ and, by Theorem 2.8 (2) and Theorem 2.6 (5), $A^*_m = J^*_m \cup (A \cap A^*_m)^*_m = (A \cap A^*_m)^*_m$. Therefore, we have $A \cap A^*_m = A \cap (A \cap A^*_m)^*_m \subseteq (A \cap A^*_m)^*_m$ and $A \cap A^*_m \subseteq A$. By the assumption $A \cap A^*_m = \emptyset$, we have $A = A - A^*_m \in \mathcal{I}$.

(5) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that, for every $x \in A$, there exists $U \in m_X(x)$ such that $m - cl(U) \cap A \in \mathcal{I}$. Then $A \cap A^*_m = \emptyset$. Suppose that A contains some B such that $B \subseteq B^*_m$. Then $B = B \cap B^*_m \subseteq A \cap A^*_m = \emptyset$. Therefore, A contains no nonempty subset B with $B \subseteq B^*_m$. Hence $A \in \mathcal{I}$. □

Theorem 3.4. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. If m_X is closure m-compatible with $\mathcal{I}$, then the implications* (1) $\Rightarrow$ (2) *and* (3) $\Rightarrow$ (1) *hold. If m_X has property $[I]$, then the following properties are equivalent:*

(1) *For every $A \subseteq X$, $A \cap A^*_m = \emptyset$ implies that $A^*_m = \emptyset$.*
(2) *For every $A \subseteq X$, $(A - A^*_m)^*_m = \emptyset$.*
(3) *For every $A \subseteq X$, $(A \cap A^*_m)^*_m = A^*_m$.*

Proof. First, we show that (1) holds, if m_X is closure compatible with $\mathcal{I}$. Let A be any subset of X such that $A \cap A^*_m = \emptyset$. By Theorem 3.3, $A \in \mathcal{I}$ and, by Theorem 2.6 (5), $A^*_m = \emptyset$.

(1) $\Rightarrow$ (2): Assume that, for every $A \subseteq X$, $A \cap A^*_m = \emptyset$ implies that $A^*_m = \emptyset$. Let $B = A - A^*_m$. Then

$$\begin{aligned} B \cap B^*_m &= (A - A^*_m) \cap (A - A^*_m)^*_m \\ &= (A \cap (X - A^*_m)) \cap (A \cap (X - A^*_m))^*_m \\ &\subseteq [A \cap (X - A^*_m)] \cap [A^*_m \cap ((X - A^*_m)^*_m)] = \emptyset. \end{aligned}$$

By (1), we have $B^*_m = \emptyset$. Hence $(A - A^*_m)^*_m = \emptyset$.

(2) $\Rightarrow$ (3): Assume that, for every $A \subseteq X$, $(A - \overline{A^*_m})^{\overline{*}}_m = \emptyset$.

$$\begin{aligned} A &= (A - \overline{A^*_m}) \cup (A \cap \overline{A^*_m}) \\ \overline{A^*_m} &= [(A - \overline{A^*_m}) \cup (A \cap \overline{A^*_m})]^{\overline{*}}_m \\ &= (A - \overline{A^*_m})^{\overline{*}}_m \cup (A \cap \overline{A^*_m})^{\overline{*}}_m \quad \text{by Theorem 2.8} \\ &= (A \cap \overline{A^*_m})^{\overline{*}}_m. \end{aligned}$$

(3) $\Rightarrow$ (1): Assume that, for every $A \subseteq X$, $A \cap \overline{A^*_m} = \emptyset$ and $(A \cap \overline{A^*_m})^{\overline{*}}_m = \overline{A^*_m}$. This implies that $\emptyset = (\emptyset)^{\overline{*}}_m = \overline{A^*_m}$. □

THEOREM 3.5. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold. If m_X has property (B), then the following properties are equivalent:*

(1) *For every m-clopen G, $G \subseteq \overline{G^*_m}$.*
(2) $X = \overline{X^*_m}$.
(3) $m - cl(m_X) \cap \mathcal{I} = \emptyset$, *where* $m - cl(m_X) = \{m - cl(V) : V \in m_X\}$.
(4) *If $I \in \mathcal{I}$, then $m - Int_\theta(I) = \emptyset$.*

Proof. (1)$\Rightarrow$ (2): Since X is m-clopen, then $X = \overline{X^*_m}$.

(2)$\Rightarrow$ (3): $X = \overline{X^*_m} = \{x \in X : m - cl(U) \cap X = m - cl(U) \notin \mathcal{I}$, for each m-open set U containing $x\}$. Hence $m - cl(m_X) \cap \mathcal{I} = \emptyset$.

(3) $\Rightarrow$ (4): Let $m - cl(m_X) \cap \mathcal{I} = \emptyset$ and $I \in \mathcal{I}$. Suppose that $x \in m - Int_\theta(I)$. Then there exists an m-open set U such that $x \in U \subseteq m - cl(U) \subseteq I$. Since $I \in \mathcal{I}$, $\emptyset \neq \{x\} \subseteq m - cl(U) \in m - cl(m_X) \cap \mathcal{I}$. This is in contradiction with $m - cl(m_X) \cap \mathcal{I} = \emptyset$. Therefore, $m - Int_\theta(I) = \emptyset$.

(4)$\Rightarrow$ (1): Let $x \in G$. Assume $x \notin \overline{G^*_m}$. Then there exists $U_x \in m_X(x)$ such that $G \cap m - cl(U_x) \in \mathcal{I}$ and hence $G \cap U_x \in \mathcal{I}$. Since G is m-clopen, by (4) and Lemma 2.5, $x \in G \cap U_x = m - Int(G \cap U_x) \subseteq m - Int(G \cap m - cl(U_x)) = m - Int_\theta(G \cap m - cl(U_x)) = \emptyset$. This is a contradiction. Hence $x \in \overline{G^*_m}$ and $G \subseteq \overline{G^*_m}$. □

THEOREM 3.6. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space, m_X be closure m-compatible with $\mathcal{I}$. Then, for every m_θ-open set G and any subset A of X, $m - cl((G \cap A)^{\overline{*}}_m) = (G \cap A)^{\overline{*}}_m \subseteq (G \cap \overline{A^*_m})^{\overline{*}}_m \subseteq m - cl_\theta(G \cap \overline{A^*_m})$.*

Proof. By Theorem 3.4(3) and Theorem 2.6, we have $(G \cap A)^{\overline{*}}_m = ((G \cap A) \cap (G \cap A)^{\overline{*}}_m)^{\overline{*}}_m \subseteq (G \cap \overline{A^*_m})^{\overline{*}}_m$. Moreover, by Theorem 2.6, we have that $m - cl((G \cap A)^{\overline{*}}_m) = (G \cap A)^{\overline{*}}_m \subseteq (G \cap \overline{A^*_m})^{\overline{*}}_m \subseteq m - cl_\theta(G \cap \overline{A^*_m})$. □

4. THE $\overline{\Psi}$−OPERATOR

DEFINITION 4.1. Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. The operator $\overline{\Psi} : \mathcal{P}(X) \to m_X$ is defined as follows: for every $A \in X$, $\overline{\Psi}(A) = \{x \in X :$ there exists $U \in m_X(x)$ such that $m - cl(U) - A \in \mathcal{I}\}$. Observe that $\overline{\Psi}(A) = X - (X - A)^{\overline{*}}_m$.

Several basic facts concerning the behavior of the operator $\overline{\Psi}$ are included in the following theorem.

Theorem 4.2. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then the following properties hold:*

(1) *If $A \subseteq X$ and m_X has property (B), then $\overline{\Psi}(A)$ is m-open.*
(2) *If $A \subseteq B$, then $\overline{\Psi}(A) \subseteq \overline{\Psi}(B)$.*
(3) *If $A \subseteq X$, then $\overline{\Psi}(A) = \overline{\Psi}(\overline{\Psi}(A))$ if and only if*

$$(X - A)^*_m = ((X - A)^*_m)^*_m.$$

Proof. (1) This follows from Theorem 2.6 (3).
(2) This follows from Theorem 2.6 (1).
(3) This follows from the below facts:

i) $\overline{\Psi}(A) = X - (X - A)^*_m$.
ii) $\overline{\Psi}(\overline{\Psi}(A)) = X - [X - (X - (X - A)^*_m)]^*_m = X - ((X - A)^*_m)^*_m$.

□

Theorem 4.3. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space and m_X have property $[I]$. Then the following properties hold:*

(1) *If $A, B \in \mathcal{P}(X)$, then $\overline{\Psi}(A \cap B) = \overline{\Psi}(A) \cap \overline{\Psi}(B)$.*
(2) *If $A \in \mathcal{I}$, then $\overline{\Psi}(A) = X - X^*_m$.*
(3) *If $A \subseteq X$, $I \in \mathcal{I}$, then $\overline{\Psi}(A - I) = \overline{\Psi}(A)$.*
(4) *If $A \subseteq X$, $I \in \mathcal{I}$, then $\overline{\Psi}(A \cup I) = \overline{\Psi}(A)$.*
(5) *If $(A - B) \cup (B - A) \in \mathcal{I}$, then $\overline{\Psi}(A) = \overline{\Psi}(B)$.*

Proof.

$$\begin{aligned}(1)\ \overline{\Psi}(A \cap B) =& X - (X - (A \cap B))^*_m = X - [(X - A) \cup (X - B)]^*_m \\ =& X - [(X - A)^*_m \cup (X - B)^*_m] \\ =& [X - (X - A)^*_m \cap [X - (X - B)^*_m] \\ =& \overline{\Psi}(A) \cap \overline{\Psi}(B).\end{aligned}$$

(2) By Corollary 2.10, we obtain that $(X - A)^*_m = X^*_m$ if $A \in \mathcal{I}$.
(3) This follows from Corollary 2.10 and $\overline{\Psi}(A - I) = X - [X - (A - I)]^*_m = X - [(X - A) \cup I]^*_m = X - (X - A)^*_m = \overline{\Psi}(A)$.
(4) This follows from Corollary 2.10 and $\overline{\Psi}(A \cup I) = X - [X - (A \cup I)]^*_m = X - [(X - A) - I]^*_m = X - (X - A)^*_m = \overline{\Psi}(A)$.
(5) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{I}$, by heredity. Also observe that $B = (A - I) \cup J$. Thus $\overline{\Psi}(A) = \overline{\Psi}(A - I) = \Psi[(A - I) \cup J] = \overline{\Psi}(B)$, by (3) and (4). □

Corollary 4.4. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then $U \subseteq \overline{\Psi}(U)$, for every m_θ-open set $U \subseteq X$.*

Proof. We know that $\overline{\Psi}(U) = X - (X-U)^*_m$. Now $(X-U)^*_m \subseteq m-cl_\theta(X-U) = X-U$, since $X-U$ is m_θ-closed. Therefore, $U = X-(X-U) \subseteq X-(X-U)^*_m = \overline{\Psi}(U)$. □

THEOREM 4.5. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space and $A \subseteq X$. Then the following properties hold:*

(1) $\overline{\Psi}(A) = \cup\{U \in m_X : m - cl(U) - A \in \mathcal{I}\}$.
(2) $\overline{\Psi}(A) \supseteq \cup\{U \in m_X : (m - cl(U) - A) \cup (A - m - cl(U)) \in \mathcal{I}\}$.

Proof. (1) This follows immediately from the definition of the $\overline{\Psi}$-operator.
(2) By the heredity of $\mathcal{I}$, it is obvious that $\cup\{U \in m_X : (m - cl(U) - A) \cup (A - m - cl(U)) \in \mathcal{I}\} \subseteq \cup\{U \in m_X : m - cl(U) - A \in \mathcal{I}\} = \overline{\Psi}(A)$, for every $A \subseteq X$. □

THEOREM 4.6. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space and assume that m_X has property $[I]$. If $\sigma = \{A \subseteq X : A \subseteq \overline{\Psi}(A)\}$, then σ is a topology for X.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \overline{\Psi}(A)\}$. Since $\phi \in \mathcal{I}$, by Theorem 2.6 (5), $(\phi)^*_m = \phi$ and $\overline{\Psi}(X) = X - (X-X)^*_m = X - (\phi)^*_m = X$. Moreover, $\overline{\Psi}(\phi) = X - (X-\phi)^*_m \supseteq X - X = \phi$. Therefore, we obtain that $\phi \subseteq \overline{\Psi}(\phi)$ and $X \subseteq \overline{\Psi}(X) = X$, and thus ϕ and $X \in \sigma$. Now if $A, B \in \sigma$, then by Theorem 4.3 (1) $A \cap B \subseteq \overline{\Psi}(A) \cap \overline{\Psi}(B) = \overline{\Psi}(A \cap B)$, which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \overline{\Psi}(A_\alpha) \subseteq \overline{\Psi}(\cup A_\alpha)$, for every α, and hence $\cup A_\alpha \subseteq \overline{\Psi}(\cup A_\alpha)$. This shows that σ is a topology. □

By Theorem 4.3 and Corollary 4.4 the following relations hold:

$$\begin{array}{ccc} m_\theta\text{-open} & \longrightarrow & m\text{-open} \\ \downarrow & & \\ \sigma\text{-open} & & \end{array}$$

THEOREM 4.7. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then $m_X \sim \mathcal{I}$ if and only if $\overline{\Psi}(A) - A \in \mathcal{I}$, for every $A \subseteq X$.*

Proof. Necessity. Assume $m_X \sim \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \overline{\Psi}(A) - A$ if and only if $x \notin A$ and $x \notin (X-A)^*_m$ if and only if $x \notin A$ and there exists $U_x \in m_X(x)$ such that $m - cl(U_x) - A \in \mathcal{I}$ if and only if there exists $U_x \in m_X(x)$ such that $x \in m - cl(U_x) - A \in \mathcal{I}$. Now, for each $x \in \overline{\Psi}(A) - A$ and $U_x \in m_X(x)$, $m - cl(U_x) \cap (\overline{\Psi}(A) - A) \in \mathcal{I}$, by heredity, and hence $\overline{\Psi}(A) - A \in \mathcal{I}$, by the assumption that $m_X \sim \mathcal{I}$.

Sufficiency. Let $A \subseteq X$ and assume that, for each $x \in A$, there exists $U_x \in m_X(x)$ such that $m - cl(U_x) \cap A \in \mathcal{I}$. Observe that $\overline{\Psi}(X-A) - (X-A) = A - A^*_m = \{x :$ there exists $U_x \in m_X(x)$ such that $x \in m - cl(U_x) \cap A \in \mathcal{I}\}$. Thus we have $A \subseteq \overline{\Psi}(X-A) - (X-A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$, by the heredity of $\mathcal{I}$. □

PROPOSITION 4.8. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \widetilde{\sim} \mathcal{I}$, $A \subseteq X$. If N is a nonempty m-open subset of $A^*_m \cap \overline{\Psi}(A)$, then $N - A \in \mathcal{I}$ and $m - cl(N) \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq A^*_m \cap \overline{\Psi}(A)$, then $N - A \subseteq \overline{\Psi}(A) - A \in \mathcal{I}$ by Theorem 4.7 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in m_X - \{\phi\}$ and $N \subseteq A^*_m$, we have $m - cl(N) \cap A \notin \mathcal{I}$ by the definition of A^*_m. □

In [4], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 4.3(5), we have that if $A = B$ [mod $\mathcal{I}$], then $\overline{\Psi}(A) = \overline{\Psi}(B)$.

DEFINITION 4.9. Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. A subset A of X is called an *m-Baire set* with respect to m_X and $\mathcal{I}$ (we denote $A \in \mathcal{B}_r(X, m_X, \mathcal{I})$), if there exists an m_θ-open set U such that $A = U$ [mod $\mathcal{I}$].

LEMMA 4.10. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \widetilde{\sim} \mathcal{I}$. If U and V are m_θ-open sets and $\overline{\Psi}(U) = \overline{\Psi}(V)$, then $U = V$ [mod $\mathcal{I}$].*

Proof. Since U is m_θ-open, by Corollary 4.4, we have $U \subseteq \overline{\Psi}(U)$ and hence $U - V \subseteq \overline{\Psi}(U) - V = \overline{\Psi}(V) - V \in \mathcal{I}$, by Theorem 4.7. Therefore, $U - V \in \mathcal{I}$. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$, by additivity. Hence $U = V$ [mod $\mathcal{I}$]. □

THEOREM 4.11. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \widetilde{\sim} \mathcal{I}$. If m_X has property [I], A, $B \in \mathcal{B}_r(X, m_X, \mathcal{I})$ and $\overline{\Psi}(A) = \overline{\Psi}(B)$, then $A = B$ [mod $\mathcal{I}$].*

Proof. Let U and V be m_θ-open sets such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now $\overline{\Psi}(A) = \overline{\Psi}(U)$ and $\overline{\Psi}(B) = \overline{\Psi}(V)$, by Theorem 4.3 (5). Since $\overline{\Psi}(A) = \overline{\Psi}(B)$, $\overline{\Psi}(U) = \overline{\Psi}(V)$ and hence $U = V$ [mod $\mathcal{I}$], by Lemma 4.10. Hence $A = B$ [mod $\mathcal{I}$], by transitivity. □

PROPOSITION 4.12. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space.*

1. *If $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$, then there exists nonempty m_θ-open set A such that $B = A$ [mod $\mathcal{I}$].*
2. *Let $m - cl(m_X) \cap \mathcal{I} = \phi$. Then $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$ if and only if there exists a nonempty m_θ-open set A such that $B = A$ [mod $\mathcal{I}$].*

Proof. (1) Assume that $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$. Then $B \in \mathcal{B}_r(X, m_X, \mathcal{I})$. Hence there exists m_θ-open set A such that $B = A$ [mod $\mathcal{I}$]. If $A = \phi$, then we have $B = \phi$ [mod $\mathcal{I}$]. This implies that $B \in \mathcal{I}$, which is a contradiction.

(2) Assume there exists a nonempty m_θ-open set A such that $B = A$ [mod $\mathcal{I}$]. Hence, by Definition 4.9, $B \in \mathcal{B}_r(X, m_X, \mathcal{I})$. Then $A = (B - J) \cup I$, where $J = B - A$, $I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$, by heredity and additivity. Since $A \in \mathcal{M}_\theta - \{\phi\}$, $A \neq \phi$ and there exists $U \in m_X$ such that $\phi \neq U \subseteq m - cl(U) \subseteq A$. Since $A \in \mathcal{I}$, $m - cl(U) \in \mathcal{I}$ and thus $m - cl(U) \in m - cl(m_X) \cap \mathcal{I}$. This contradicts $m - cl(m_X) \cap \mathcal{I} = \phi$. □

Proposition 4.13. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \tilde{\sim} \mathcal{I}$. If $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$ and m_X has property $[I]$, then $\overline{\Psi}(B) \cap m - Int_\theta(B^*_m) \neq \phi$.*

Proof. Assume that $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$. Then, by Proposition 4.12(1), there exists $A \in \mathcal{M}_\theta - \{\phi\}$ such that $B = A$ [mod $\mathcal{I}$]. By Theorem 3.5 and Lemma 2.7, $A = A \cap X = A \cap X^*_m \subseteq (A \cap X)^*_m = A^*_m$. This implies that $\phi \neq A \subseteq A^*_m = ((B - J) \cup I)^*_m = B^*_m$, where $J = B - A, I = A - B \in \mathcal{I}$ by Corollary 2.10. Since A is m_θ-open set, $A \subseteq m - Int_\theta(B^*_m)$. Also, $\phi \neq A \subseteq \overline{\Psi}(A) = \overline{\Psi}(B)$, by Corollary 4.4 and Theorem 4.3(5). Consequently, we obtain $A \subseteq \overline{\Psi}(B) \cap m - Int_\theta(B^*_m)$. □

Given an ideal minimal space $(X, m_X, \mathcal{I})$, let $\mathcal{U}(X, m_X, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 4.14. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \tilde{\sim} \mathcal{I}$. If every m-open set is m_θ-open, then the following statements are equivalent:*

(1) $A \in \mathcal{U}(X, m_X, \mathcal{I})$;
(2) $\overline{\Psi}(A) \cap m - Int_\theta(A^*_m) \neq \phi$;
(3) $\overline{\Psi}(A) \cap A^*_m \neq \phi$;
(4) *There exists $N \in m_X - \{\phi\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. (1) ⇒ (2): Let $A \in \mathcal{U}(X, m_X, \mathcal{I})$. Then there exists $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $m - Int_\theta(B^*_m) \subseteq m - Int_\theta(A^*_m)$ and $\overline{\Psi}(B) \subseteq \overline{\Psi}(A)$ and hence $m - Int_\theta(B^*_m) \cap \overline{\Psi}(B) \subseteq m - Int_\theta(A^*_m) \cap \overline{\Psi}(A)$. By Proposition 4.13, we have $\overline{\Psi}(A) \cap m - Int_\theta(A^*_m) \neq \phi$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Suppose that $\overline{\Psi}(A) \cap A^*_m \neq \phi$. Then there exists a point $x \in X$ such that $x \in \overline{\Psi}(A)$ and $x \in A^*_m$. Since $x \in \overline{\Psi}(A)$, there exists $U \in m_X(x)$ such that $m - Cl(U) - A \in \mathcal{I}$. Furthermore, since $x \in A^*_m$, $m - Cl(V) \cap A \notin \mathcal{I}$, for every $V \in m_X(x)$. By our assumption, we deduce that $U \in m_X(x)$ and $m_X = \mathcal{M}_\theta$ and there exists $N \in m_X$ such that $x \in N \subset m - Cl(N) \subset U$. Hence $U \cap A \notin \mathcal{I}$. On the other hand, $U - A \subset m - Cl(U) - A \in \mathcal{I}$ and hence $U - A \in \mathcal{I}$. Therefore, (4) holds.

(4) ⇒ (1): Let $B = N \cap A \notin \mathcal{I}$ with N nonempty m_θ-open set and $N - A \in \mathcal{I}$. Then $B \in \mathcal{B}_r(X, m_X, \mathcal{I}) - \mathcal{I}$, since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$. □

Theorem 4.15. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space with $m_X \tilde{\sim} \mathcal{I}$, if m_X has property $[I]$, where $m - cl(m_X) \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\overline{\Psi}(A) \subseteq A^*_m$.*

Proof. Suppose $x \in \overline{\Psi}(A)$ and $x \notin A^*_m$. Then there exists a nonempty neighborhood $U_x \in m_X(x)$ such that $m - cl(U_x) \cap A \in \mathcal{I}$. Since $x \in \overline{\Psi}(A)$, by Theorem 4.5 we deduce that $x \in \cup\{U \in m_X : m - cl(U) - A \in \mathcal{I}\}$ and that there exists $V \in m_X(x)$ such that $m - cl(V) - A \in \mathcal{I}$. Now we have $U_x \cap V \in m_X(x)$, $m - cl(U_x \cap V) \cap A \in \mathcal{I}$ and $m - cl(U_x \cap V) - A \in \mathcal{I}$, by heredity.

Hence, by finite additivity, we have $(m-cl(U_x \cap V) \cap A) \cup (m-cl(U_x \cap V) - A) = m - cl(U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in m_X(x)$, this is in contradiction with $m - cl(m_X) \cap \mathcal{I} = \phi$. Therefore, $x \in A^*_m$. This implies that $\overline{\Psi}(A) \subseteq A^*_m$. $\square$

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