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A SCHECHTER-TYPE CRITICAL POINT RESULT FOR LOCALLY LIPSCHITZ FUNCTIONS

ORSOLYA VAS

Abstract. Based on the variational principle of Ekeland, we prove a Schechter-type critical point existence theorem for locally Lipschitz functions defined on a ball of a Hilbert space. As application we give an existence result for a differential inclusion problem.

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1. INTRODUCTION

Concerning the critical points of a C^1-functional on a ball, Schechter proved [9] an existence and localization result. In this case, he also presents [10] a systematic way of finding critical points and shows that how this method can be used for solving partial differential equations. Schechter's original statements for extrema in a ball of a Hilbert space can be found in [10, Theorems 5.3.3 and 5.5.5].

In the articles [8, 6], Precup deals with the critical point theory [10] developed by Schechter. Based on the variational principle of Bishop-Phelps, he also gives in [8] a new proof to Schechter's theorem for these extrema.

The objective of the present paper is to extend the Schechter-type result of Precup [8] for locally Lipschitz functions. Confirming the applicability of this result, we present a differential inclusion problem.

The paper is structured as follows. In Section 2, we recall some definitions and properties of locally Lipschitz functions and generalized gradients. Section 3 describes the abstract framework in which we work, the formulation of our main theorem and its proof. Concerning the applicability of our abstract result, Section 4 presents a concrete application of the theorem.

2. PRELIMINARY RESULTS

In this section we recall some basic definitions and properties of locally Lipschitz functions from the theory developed by Clarke [2].

Let X be a Banach space, $X^\star$ be its topological dual space, U be an open subset of X and $f : U \to R$ be a function.

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DEFINITION 2.1. *The $f : U \to \mathbb{R}$ function is called locally Lipschitz if for each point $u \in U$ there exists a neighborhood $N_u \subset U$ such that*

$$|f(u_1) - f(u_2)| \leq K \|u_1 - u_2\|, \quad \forall u_1, u_2 \in N_u,$$

for a constant $K > 0$ depending on N_u.

DEFINITION 2.2. *The generalized directional derivative of the locally Lipschitz function $f : U \to \mathbb{R}$ at the point $u \in U$ in the direction $v \in X$ is defined by*

$$f^{\circ}(u; v) := \limsup_{\substack{w \to u \\ t \downarrow 0}} \frac{1}{t} \left[f(w + tv) - f(w) \right].$$

PROPOSITION 2.1. *The generalized directional derivative of the locally Lipschitz function $f : U \to \mathbb{R}$ has the following properties*

a) *For every $u \in U$ the function $f^{\circ}(u; \cdot) : X \to R$ is positively homogenous and subadditive and satisfies*

$$|f^{\circ}(u; v)| \leq K \|u\|, \quad \forall v \in X.$$

Moreover, it is Lipschitz continuous on X with the Lipschitz constant K, where $K > 0$ is a Lipschitz constant of f near u.

b) *$f^{\circ}(\cdot; \cdot) : U \times X \to R$ is upper semicontinuous.*

c) *$f^{\circ}(u; -v) := (-f)^{\circ}(u; v), \quad \forall u \in U, \forall v \in X.$*

DEFINITION 2.3. *The generalized gradient of the locally Lipschitz function $f : U \to \mathbb{R}$ at the point $u \in U$ is a subset of $X^{\star}$, defined by*

$$\partial f(u) = \{z \in X^{\star} : \langle z, v \rangle \leq f^{\circ}(u; v), \forall v \in X\}.$$

PROPOSITION 2.2. *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then the following assertions hold:*

a) *For every $u \in U$, $\partial f(u)$ is a nonempty, convex and weakly-compact subset of $X^{\star}$ which is bounded by the Lipschitz constant $K > 0$ of f near u.*

b) *For every $u \in U, f^{\circ}(u; \cdot)$ is the support function of $\partial f(u)$,*

$$f^{\circ}(u; v) = \max\{\langle z, v \rangle : z \in \partial f(u)\}, \forall v \in X.$$

c) *The set valued map $\partial f : U \to X^{\star}$ is weakly-closed, that is, if $\{u_n\} \subset U$ and $\{z_n\} \subset X^{\star}$ are sequences such that $u_n \to u$ strongly in $X, z_n \in \partial f(u_n)$ and $z_n \to z$ weakly in $X^{\star}$ for $u \in U, z \in X^{\star}$, then $z \in \partial f(u)$.*

DEFINITION 2.4 (Palais–Smale condition). *The locally Lipschitz function $f : U \to \mathbb{R}$ satisfies the non-smooth Palais-Smale condition at level $c \in \mathbb{R}$ if any sequence $\{u_n\} \subset U$ which satisfies*

a) *$f(u_n) \to c$;*

b) *there exists $\{\varepsilon_n\} \subset \mathbb{R}, \varepsilon_n \downarrow 0$ such that $f^{\circ}(u_n; v - u_n) + \varepsilon_n \|v - u_n\| \geq 0$, for all $v \in U$ and all $n \in \mathbb{N}$*

admits a convergent subsequence. If this holds for every $c \in \mathbb{R}$ we say that f satisfies the non-smooth Palais-Smale condition.

THEOREM 2.1 (Ekeland's variational principle). *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a lower semicontinuous, proper and bounded from below function. For any $\varepsilon > 0$, there exists some point $x_\varepsilon \in X$ such that*

$$f(x_\varepsilon) \leq \inf_{x\in X} f(x) + \varepsilon;$$
$$f(y) > f(x_\varepsilon) - \varepsilon d(x_\varepsilon, y), \forall y \in X.$$

3. MAIN RESULT

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, consider the origin centered closed ball $\overline{B}_R = \{x \in X : \|x\| \leq R\}$ of X with radius $R > 0$ and denote by $B_R = \{x \in X : \|x\| < R\}$ the origin centered open ball of X having radius $R > 0$.

DEFINITION 3.1. *Let $F : X \to \mathbb{R}$ be a locally Lipschitz function. In the space X we consider the sphere S_R of center 0 and radius $R > 0$, i.e., $S_R = \{x \in X : \|x\| = R\}$. The generalized gradient $\partial (F\mid_{S_R})(u)$ at $u \in S_R$ is defined by*

$$\partial (F\mid_{S_R})(u) = \left\{ w^\star - \frac{1}{R^2} \langle w^\star, u \rangle \Lambda u; w^\star \in \partial F(u) \right\},$$

where $\Lambda : X \to X^\star$ denotes the duality mapping.

Using the aforementioned notations we can state the main result of the paper.

THEOREM 3.1. *Let $F : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz function, which is bounded from below. There exist a sequence $(x_n) \subset \overline{B}_R$, such that $F(x_n) \longrightarrow \inf F(\overline{B}_R)$ and one of the following two situations holds:*

a) $\lambda_F(x_n) \longrightarrow 0$;
b) $\|x_n\| = R$ *and* $\langle w^\star_{n,} x_n \rangle \leq 0$, *for all n and $w^\star_n \in \partial F(x_n)$, then* $\lambda_{F,S_R}(x_n) \longrightarrow 0$,

where $\partial F(x_n)$ is the generalized gradient of the locally Lipschitz function F and

$$\lambda_{F,S_R}(x_n) = \inf\{w^\star - \frac{1}{R^2} \langle w^\star_, x_n \rangle \Lambda x_n, w^\star \in \partial F(x_n)\}.$$

If in addition $\langle x^\star_, x \rangle \geq -a > -\infty$ for all $x \in \partial\overline{B}_R$, $x^\star \in \partial F(x)$, F satisfies a Palais-Smale type compactness condition and the boundary condition

$$x^\star + \mu\Lambda x \neq 0 \text{ for all } x \in \partial\overline{B}_R \text{ and } \mu > 0, \quad (3.1)$$

holds, then there exists $x \in \partial\overline{B}_R$ with

$$F(x) = \inf F(\overline{B}_R) .$$

Proof. We apply the variational principle of Ekeland stated in Theorem 2.1, to the closed set $X = \overline{B}_R$, to the continuous and bounded from below function $f = F$, to distance $d(x,y) = \|x-y\|$, $\varepsilon = \frac{1}{n}(n \in N \setminus \{0\})$ and to $x_\varepsilon \in \overline{B}_R$ with

$$F(x_\varepsilon) \leq \inf_{x \in \overline{B}_R} F(\overline{B}_R) + \frac{1}{n}.$$

In this case there exists a sequence $(x_n) \in \overline{B}_R$ such that

$$F(x_n) \leq F(x_\varepsilon) \leq \inf_{x \in \overline{B}_R} F(\overline{B}_R) + \frac{1}{n} \tag{3.2}$$

and

$$F(x_n) < F(y) + \frac{1}{n}\|x_n - y\| \text{ for every } y \in \overline{B}_R \setminus \{x_n\}. \tag{3.3}$$

From (3.2) follows that $F(x_n) \to \inf F(\overline{B}_R)$.

The sequence (x_n) belongs to $\overline{B}_R$, hence we distinguish two possible cases:

(1) there exists a subsequence of (x_n), also denoted by (x_n), such that $\|x_n\| < R$ for all $n \in \mathbb{N}$;

(2) there exists a subsequence of (x_n), also denoted by (x_n), such that $\|x_n\| = R$ for all $n \in \mathbb{N}$.

In case (1) we suppose that $\|x_n\| < R$ for all $n \in \mathbb{N}$. For a fixed n and any $z \in X$ with $\|z\| = 1$ let $y := x_n - tz$, where $t > 0$. For t small enough $\|y\| \leq R$, thus y still belongs to $\overline{B}_R$. By consequence (3.3) of the variational principle of Ekeland we have

$$F(x_n) < F(x_n - tz) + \frac{1}{n}\|x_n + tz - x_n\|,$$

therefore

$$F(x_n) - F(x_n - tz) < \frac{t}{n}.$$

Dividing by $t > 0$ and taking $t \to 0$, we obtain

$$-F^\circ(x_n, -z) < \frac{1}{n}.$$

The property of the generalized directional derivative stated in Proposition 2.2 gives us

$$-\max\{\langle w_n^\star, -z\rangle : w_n^\star \in \partial F(x_n)\} < \frac{1}{n},$$

thus

$$\min\{\langle w_n^\star, z\rangle : w_n^\star \in \partial F(x_n)\} < \frac{1}{n} \text{ for any } z \in X, \text{ where } \|z\| = 1.$$

Then there exists $w_n^\star \in \partial F(x_n)$ such that $z = \frac{w_n^\star}{\|w_n^\star\|}$, and

$$\begin{aligned}\min\left\{\left\langle w_n^\star, \frac{w_n^\star}{\|w_n^\star\|}\right\rangle : w_n^\star \in \partial F(x_n)\right\} &< \frac{1}{n}\\ \min\left\{\frac{\|w_n^\star\|^2}{\|w_n^\star\|} : w_n^\star \in \partial F(x_n)\right\} &< \frac{1}{n}\\ \min\left\{\|w_n^\star\| : w_n^\star \in \partial F(x_n)\right\} &< \frac{1}{n},\end{aligned}$$

whence it follows that $\lambda_F(x_n) \longrightarrow 0$. Therefore, in this case, the property (a) of the theorem holds.

In case (2) we suppose that $\|x_n\| = R$, for all $n \in \mathbb{N}$. For a fixed n and any $z \in X$ with $\|z\| = 1$ let $y := x_n - tz$, where $t > 0$. For such a y we have

$$\begin{aligned}\|y\|^2 &= \|x_n - tz\|^2 = \langle x_n - tz, x_n - tz\rangle = \|x_n\|^2 - 2t\langle x_n, z\rangle + t^2\|z\|^2\\ &= R^2 - 2t\langle x_n, z\rangle + t^2.\end{aligned}$$

If $\langle x_{n,} z\rangle > 0$, then there exists $t \in (0, 2\langle x_n, z\rangle)$ for which y still belongs to $\overline{B}_R$, thus we also have

$$-F^\circ(x_n, -z) < \frac{1}{n}. \tag{3.4}$$

If $\langle x_{n,} z\rangle = 0$ for any fixed n we choose a subsequence (z_k) such that $z_k \longrightarrow z$ and $\langle x_n, z_k\rangle > 0$. Then $F^\circ(x_n, -z_k) \to F^\circ(x_n, -z)$ holds, and we obtain

$$-F^\circ(x_n, -z) < \frac{1}{n}. \tag{3.5}$$

Hence (3.4) and (3.5) gives us

$$-F^\circ(x_n, -z) < \frac{1}{n} \text{ for every } z \in X \text{ with } \|z\| = 1 \text{ and } \langle x_{n,} z\rangle \geq 0. \tag{3.6}$$

Henceforward, two subcases are possible:

(i) there exists a subsequence of (x_n), also denoted by (x_n), such that $\langle w_{n,}^\star x_n\rangle > 0$ for $w_n^\star \in \partial F(x_n)$ and $n \in \mathbb{N}$;
(ii) there exists a subsequence of (x_n), also denoted by (x_n), such that $\langle w_{n,}^\star x_n\rangle \leq 0$ for $w_n^\star \in \partial F(x_n)$ and $n \in \mathbb{N}$.

In case (i), by taking $z = \frac{w_n^\star}{\|w_n^\star\|}$ in (3.6), we also obtain

$$-F^\circ(x_n, -z) < \frac{1}{n},$$

whence it follows that $\lambda_F(x_n) \longrightarrow 0$, thus for the sequence (x_n) the property (a) of the theorem holds.

In case (ii) let $z \in X$ be such that $\|z\| = 1$. By taking $z = \frac{z_n^\star}{\|z_n^\star\|}$, where $z_n^\star = w_n^\star - \frac{\langle w_n^\star, x_n \rangle}{R^2} x_n$ in (3.6) we have

$$\begin{aligned}
\langle w_n^\star, z \rangle &= \left\langle w_n^\star, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle \\
&= \left\langle w_n^\star - \frac{\langle w_n^\star, x_n \rangle}{R^2} x_n, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle + \frac{\langle w_n^\star, x_n \rangle}{R^2} \left\langle x_n, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle \\
&= \left\langle w_n^\star - \frac{\langle w_n^\star, x_n \rangle}{R^2} x_n, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle + \frac{\langle w_n^\star, x_n \rangle}{R^2 \|z_n^\star\|} \langle x_n, z_n^\star \rangle .
\end{aligned}$$

But

$$\begin{aligned}
\langle z_n^\star, x_n \rangle &= \langle w_n^\star, x_n \rangle - \frac{\langle w_n^\star, x_n \rangle}{R^2} \langle x_n, x_n \rangle \\
&= \langle w_n^\star, x_n \rangle - \frac{\langle w_n^\star, x_n \rangle}{R^2} \|x_n\|^2 \\
&= \langle w_n^\star, x_n \rangle - \frac{\langle w_n^\star, x_n \rangle}{R^2} R^2 \\
&= 0,
\end{aligned}$$

and by the definition of $z_n^\star$ we get

$$\langle w_n^\star, z \rangle = \left\langle w_n^\star - \frac{\langle w_n^\star, x_n \rangle}{R^2} x_n, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle = \left\langle z_n^\star, \frac{z_n^\star}{\|z_n^\star\|} \right\rangle = \|z_n^\star\| ,$$

hence

$$\lambda_{F,S_R}(x_n) = \min\{\|z_n^\star\| : w_n^\star \in \partial F(x_n)\} \longrightarrow 0.$$

Thus for the sequence (x_n) the property (b) of the theorem holds.

Finally, if $\langle x^\star, x \rangle \geq -a > -\infty$ holds for all $x \in \partial \overline{B}_R, x^\star \in \partial F(x)$ and the function F satisfies the Palais-Smale condition, we may assume that $\langle x_n^\star, x_n \rangle \to b$, where $b \leq 0$. Then, by the Palais-Smale condition, there exists a convergent subsequence (x_n) such that $x_n \to x$, $\|x_n\| = R$, where $x \in B_R$. Using Proposition 2.2, there exist sequences $(x_n) \subset B_R$ and $x_n^\star \in \partial F(x_n)$ such that $x_n \to x$ strongly in X and $x_n^\star \to x^\star$ weakly in $X^\star$ for $x \in B_R$ and $x^\star \in \partial F(x)$. Then

$$x_n^\star - \frac{\langle x_n^\star, x_n \rangle}{R^2} \Lambda x_n = x^\star - \frac{b}{R^2} \Lambda x = 0,$$

whence

$$x^\star + \mu \Lambda x = 0,$$

where $\mu = -\frac{\langle x^\star, x \rangle}{R^2} = -\frac{b}{R^2} \geq 0$. The boundary condition (3.1) excludes the case that $\mu > 0$, thus we obtain $x^\star = 0$, consequently

$$F(x_n) = F(x) = \inf F(\overline{B}_R).$$

□

4. APPLICATION

In this section we give a concrete application of our main result.

Let Ω denote a bounded domain in $\mathbb{R}^N$ with the C^1 regular boundary $\partial\Omega$. Consider the Sobolev space $W_0^{1,2}(\Omega)$ equipped with the norm

$$\|u\|_{W_0^{1,2}} = \left(\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \sqrt{\langle u, u\rangle}.$$

Let $W_0^{-1,2}(\Omega)$ denote the topological dual space $(W_0^{1,2}(\Omega))^\star$.

From the Sobolev embedding theorem, [1], we know that the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in (1, 2^\star = \frac{2N}{N-2})$, hence there exists a constant $C > 0$ such that

$$\|u\|_{L^q} \le C \|u\|_{W_0^{1,2}}, \quad \forall u \in W_0^{1,2}(\Omega).$$

Let the Carathéodory function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions:

a) $F(\cdot, u)$ is measurable for each $u \in \mathbb{R}$;
b) $F(x, \cdot)$ is locally Lipschitz for each $x \in \Omega$;
c) $F(\cdot, 0) \in L^1(\Omega)$.

We consider the following non-smooth Dirichlet problem

$$\text{(P)} \qquad \begin{cases} -\Delta u \in \partial_y F(x,u) & \text{a.e.} \quad x \in \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We assume that the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the growth condition

$$(4.1) \qquad |z| \le a(x) + b(x)\,|y|^{q-1}, \quad \text{for } \forall z \in \partial_y F(x,y),\ (x,y) \in (\Omega \times \mathbb{R}),$$

where $a \in L^{\frac{q}{q-1}}(\Omega)$, $b \in L^\infty(\Omega)$ are positive functions and $q \in (1, 2^\star)$, with $2^\star = \frac{2N}{N-2}$.

We introduce the notations

$$\overline{B}_R = \{u \in W_0^{1,2}(\Omega) : \|u\|_{W_0^{1,2}} \le R\}$$

and

$$S_R = \{u \in W_0^{1,2}(\Omega) : \|u\|_{W_0^{1,2}} = R\}.$$

Definition 4.1. *A function $u \in W_0^{1,2}(\Omega)$ is a weak solution of problem (P) if there exists $w_F(x) \in \partial_y F(x, u(x))$ for a.e. $x \in \Omega$ such that for all $v \in W_0^{1,2}(\Omega)$ we have*

$$\int_\Omega \nabla u(x)\nabla v(x)\mathrm{d}x = \int_\Omega w_F(x) \cdot v(x)\mathrm{d}x.$$

Let $I : W_0^{1,2}(\Omega) \to \mathbb{R}$

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \quad \forall u \in W_0^{1,2}(\Omega),$$

be the energy functional associated to the problem (P) .The critical points of the energy functional I are the weak solutions of (P).

Proposition 4.1. *If $R>0$ is the solution of the inequality in $\mathbb{R}$*

$$R-\|b\|_{L^\infty}\cdot C_q^q\cdot R^{q-1}>\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q, \tag{4.2}$$

then

$$\langle u^\star,u\rangle+\mu\cdot\langle\Lambda u,u\rangle\neq 0,\quad \forall u^\star\in\partial I(u)$$

for any $\mu>0$, where $u\in S_R$.

Proof. We prove this proposition by contradiction. We assume that there exists $u\in S_R$ and $\mu>0$ such that $\langle u^\star,u\rangle+\mu\cdot\langle\Lambda u,u\rangle=0$ for any $u^\star\in\partial I(u)$. Then using the growth condition (4.1) we obtain

$$\begin{aligned}\int_\Omega w_F(x)\cdot u(x)\mathrm{d}x &\le \int_\Omega a(x)\cdot u(x)+b(x)\cdot|u(x)|^q\,\mathrm{d}x<R^2\\ &\le \int_\Omega a(x)\cdot u(x)\mathrm{d}x+\int_\Omega b(x)\cdot|u(x)|^q\,\mathrm{d}x\\ &\le \left(\int_\Omega |u(x)|^q\right)^{\frac{1}{q}}\cdot\|a\|_{L^{\frac{q}{q-1}}}+\left[\left(\int_\Omega|u(x)|^q\right)^{\frac{1}{q}}\right]^q\cdot\|b\|_{L^\infty}\\ &\le \|a\|_{L^{\frac{q}{q-1}}}\cdot\|u\|_{L^q}+\|b\|_{L^\infty}\cdot\|u\|_{L^q}^q.\end{aligned}$$

By the Sobolev embedding theorem

$$\int_\Omega w_F(x)\cdot u(x)\mathrm{d}x\le\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q\,\|u\|_{W_0^{1,2}}+\|b\|_{L^\infty}\cdot C_q^q\,\|u\|_{W_0^{1,2}}^q.$$

Since our assumption gives us

$$(1+\mu)\,\|u\|_{W_0^{1,2}}^2=\int_\Omega w_F(x)\cdot u(x)\mathrm{d}x,$$

we have

$$(1+\mu)\,\|u\|_{W_0^{1,2}}^2\le\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q\,\|u\|_{W_0^{1,2}}+\|b\|_{L^\infty}\cdot C_q^q\,\|u\|_{W_0^{1,2}}^q.$$

We know that $1+\mu>0$, therefore

$$\|u\|_{W_0^{1,2}}^2\le(1+\mu)\,\|u\|_{W_0^{1,2}}^2\le\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q\cdot\|u\|_{W_0^{1,2}}+\|b\|_{L^\infty}\cdot C_q^q\cdot\|u\|_{W_0^{1,2}}^q.$$

Using that $u\in S_R$, namely $\|u\|_{W_0^{1,2}}=R$, we get

$$R^2\le(1+\mu)R^2\le\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q\cdot R+\|b\|_{L^\infty}\cdot C_q^q\cdot R^q.$$

Dividing the inequality by $R>0$ and rearranging it, we obtain

$$R-\|b\|_{L^\infty}\cdot C_q^q\cdot R^{q-1}\le\|a\|_{L^{\frac{q}{q-1}}}\cdot C_q. \tag{4.3}$$

The condition (4.2) implies that (4.3) cannot be satisfied. □

Using the conditions of Proposition 4.1 and Theorem 3.1, we can state the next result.

THEOREM 4.1. *If we choose $R > 0$ to be the solution of the inequality*

$$R - \|b\|_{L^\infty} \cdot C_q^q \cdot R^{q-1} > \|a\|_{L^{\frac{q}{q-1}}} \cdot C_q,$$

in $\mathbb{R}$ then, the problem (P) admits a weak solution $u \in \overline{B}_R$, which minimizes I on $\overline{B}_R$.

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Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. M. Kogălniceanu 1
400084 Cluj Napoca, Romania
E-mail: vas.orsolya@math.ubbcluj.ro