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OSCILLATION CRITERIA FOR NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH SEVERAL DELAYS

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Abstract. In this work, oscillatory behaviour of the solutions of a class of non-linear first-order neutral differential equations with several delays of the form

$$(E_1) \qquad (x(t)+p(t)x(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) = f(t)$$

and

$$(E_2) \qquad (x(t)+p(t)x(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) = 0$$

are studied under various ranges of $p(t)$. Sufficient conditions are obtained for existence of bounded positive solutions of (E_1).

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Key words. Oscillation, nonoscillation, non-linear, delay, neutral differential equations, existence of positive solution.

1. INTRODUCTION

An increasing interest in oscillation of solutions to functional differential equations during the last few decades has been stimulated by applications arising in engineering and natural sciences. The new classes of such equations provide challenges in these application areas. Equations involving delay, and those involving advance and a combination of both arise in the models on loss-less transmission lines in high speed computers which are used to interconnect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. There has been many investigations into the oscillation and nonoscillation of first order nonlinear neutral delay differential equations (See for example [1] - [3], [5], [7]-[16]). However, the study of oscillatory behaviour of solutions of (E_1) has received much less attention, which is due to mainly to the technical difficulties arising in its analysis.

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In [2], Ahmed et al. have studied the oscillation properties of a linear differential equations of the form

$$(E_3) \qquad (r(t)(x(t)+p(t)x(t-\tau))' + q(t)x(t-\sigma) = 0,$$

for the cases $p(t) \leq -1$, $-1 \leq p(t) < 0$ and $p(t) \equiv p \neq \pm 1$ and established sufficient conditions so that every solution of (E_3) is oscillatory. Their method has made the proof unnecessarily complicated and applicable to only homogeneous equations.

In [3], Das and Misra have made an attempt to study the oscillation properties of a nonlinear differential equations of type

$$(E_4) \qquad (x(t) - px(t-\tau))' + q(t)H(x(t-\sigma)) = f(t),$$

where $0 \leq p < 1$, $f(t) > 0$, and H satisfies the generalized sublinear condition

$$\int_0^{\pm k} \frac{\mathrm{d}t}{H(t)} < \infty,$$

for every positive constant k, and established necessary and sufficient conditions so that every solution of (E_4) either oscillates or tends to zero. Their method does not allow for $f \equiv 0$ and H to be superlinear.

Hence in this work, the author have made an attempt to establish the sufficient condition for oscillation of a class of nonlinear neutral delay differential equation

$$(1) \qquad (x(t) + p(t)x(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) = f(t),$$

where $\tau, \sigma_i \in \mathbb{R}_+ = (0, +\infty)$, $i = 1, 2..., m$, $p \in C([0,\infty), \mathbb{R})$, $q \in (\mathbb{R}_+, \mathbb{R}_+)$, $f \in C(\mathbb{R}, \mathbb{R})$, and $H \in C(\mathbb{R}, \mathbb{R})$ with $uH(u) > 0$ for $u \neq 0$. The objective of this work to establish the sufficient conditions for oscillation of solutions of (1) under various ranges of $p(t)$. Its associated homogenous equation

$$(2) \qquad (x(t) + p(t)x(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) = 0,$$

is also considered. Clearly, equations (E_3) (for $r(t) = 1$), and (E_4) are particular cases of equations (2) and (1) respectively. Therefore, it is interesting to study the more general equations (1) and (2). Unlike the work in [2] and [3] an attempt is made here to establish sufficient conditions under which every solution or every bounded solution of (1) and (2) oscillates. Of course, the impact of forcing term is considered. keeping in view of the influence of forcing function, this work is separated for forced and unforced equations.

Definition 1.1. By a solution of (1)/(2) we understand a function $x \in C([-\rho, \infty), \mathbb{R})$ such that $x(t) + p(t)x(t-\tau)$ is once continuously differentiable and (1) or (2) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \sigma_i\}$ for $i = 1, ..., m$, and $\sup\{|x(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (1)/(2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. NON-HOMOGENEOUS OSCILLATION

In this section, sufficient conditions are obtained for oscillation of solutions of the equation (1). We need the following conditions for this work in the sequel.

(A_1) There exists $\lambda > 0$ such that $H(u) + H(v) \geq \lambda H(u + v)$, for $u, v > 0$;
(A_2) $H(uv) = H(u)H(v)$, for $u, v \in \mathbb{R}$;
(A_3) $H(-u) = -H(u)$, for $u \in \mathbb{R}$;
(A_4) There exists $F \in C(\mathbb{R}, \mathbb{R})$ such that $F(t)$ changes sign with

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty \text{ and } F'(t) = f(t);$$

(A_5) $F^+(t) = \max\{F(t), 0\}$, $F^-(t) = \max\{-F(t), 0\}$;
(A_6) There exists $F \in C(\mathbb{R}, \mathbb{R})$ such that $F(t)$ changes sign with

$$\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = +\infty \text{ and } F'(t) = f(t).$$

Remark 2.1. Assumption (A_2) implies (A_3). Indeed, $H(1)H(1) = H(1)$ and $H(1) > 0$ imply that $H(1) = 1$. Further, $H(-1)H(-1) = H(1) = 1$ implies that $(H(1))^2 = 1$. Since $H(-1) < 0$, we conclude that $H(-1) = -1$. Hence,

$$H(-u) = H(-1)H(-u) = -H(u).$$

On the other hand, $H(uv) = H(u)H(v)$ for $u > 0$ and $v > 0$ and $H(-u) = -H(u)$ imply that $H(xy) = H(x)H(y)$ for every $x, y \in \mathbb{R}$.

Remark 2.2. We may note that if $x(t)$ is a solution of (1), then $y(t) = -x(t)$ is also a solution of (1) provided that H satisfies (A_2) or (A_3).

Theorem 2.1. *Let $p(t) \geq 0$, $t \in \mathbb{R}_+$. If (A_2) and (A_6) hold, then every solution of the equation* (1) *is oscillatory.*

Proof. Suppose for contrary that $x(t)$ is a nonoscillatory solution of equation (1). Then there exists $t_0 \geq \rho$ such that $x(t) > 0$ or $x(t) < 0$, for $t \geq t_0$. Assume that $x(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = x(t) + p(t)x(t - \tau), \tag{3}$$

and

$$w(t) = z(t) - F(t), \tag{4}$$

it follows from (1) that

$$w'(t) = -\sum_{i=1}^{m} q_i(t)H(x(t - \sigma_i)) \leq 0 \tag{5}$$

for $t \geq t_1 > t_0$. Consequently, $w(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. Hence we have $w(t) < 0$ or $w(t) > 0$ for $t \geq t_2 > t_1$. Since $z(t) > 0$, then $w(t) < 0$, for $t \geq t_2$ implies that $\liminf_{t\to\infty} F(t) \geq 0$, for $t \geq t_2$, a contradiction to (A_6). Hence, $w(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} w(t)$ exists. Writing

$$z(t) = w(t) + F(t),$$

we notice that

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} z(t) = \liminf_{t\to\infty}(w(t) + F(t)) \\ &\leq \limsup_{t\to\infty} w(t) + \liminf_{t\to\infty} F(t) \\ &= \lim_{t\to\infty} w(t) + \liminf_{t\to\infty} F(t) \\ &= -\infty, \end{aligned}$$

a contradiction due to (A_6).

If $x(t) < 0$, for $t \geq t_0$, then we set $y(t) = -x(t)$, for $t \geq t_0$ in (1) and we find

$$(y(t) + p(t)y(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(y(t-\sigma_i)) = \widetilde{f}(t), \tag{6}$$

where $\widetilde{f}(t) = -f(t)$ due to (A_2). Let $\widetilde{F}(t) = -F(t)$. Then

$$-\infty < \liminf_{t\to\infty} \widetilde{F}(t) < 0 < \limsup_{t\to\infty} \widetilde{F}(t) < \infty$$

and $\widetilde{F}'(t) = \widetilde{f}(t)$ hold. Hence proceeding as above, we find a contradiction to (A_6). This completes the proof of the theorem. □

THEOREM 2.2. *Let* $0 \leq p(t) \leq p < \infty$, $t \in \mathbb{R}_+$. *Assume that* (A_1), (A_2), (A_4) *and* (A_5) *hold. Furthermore, assume that*

(A_7) $\int_\rho^\infty \sum_{i=1}^m Q_i(t)H(F^+(t-\sigma_i))\mathrm{d}t = \infty = \int_\rho^\infty \sum_{i=1}^m Q_i(t)H(F^-(t-\sigma_i))\mathrm{d}t$

hold, then conclusion of Theorem 2.1 *is true, where for* $t > \tau$, $Q_i(t) = min\{q_i(t), q_i(t-\tau)\}$; $i = 1, ..., m$.

Proof. On the contrary, we proceed as in the proof of the Theorem 2.1 to obtain that $w(t)$ is monotonic on $[t_2, \infty), t_2 > t_1$. Since $z(t) > 0$, then $w(t) < 0$, for $t \geq t_2$ implies that $F(t) > 0$, for $t \geq t_2$, a contradiction to (A_4). Hence, $w(t) > 0$ for $t \geq t_2$. Ultimately, $z(t) > F(t)$ and hence $z(t) > \max\{0, F(t)\} = F^+(t)$, for $t \geq t_2$. Note that $\lim_{t\to\infty} w(t)$ exists. Due to (4), (1) becomes

$$\begin{aligned} 0 = w'(t) &+ \sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) \\ &+ H(p)\left[w'(t-\tau) + \sum_{i=1}^{m} q_i(t-\tau)H(x(t-\tau-\sigma_i))\right] \end{aligned}$$

for $t \geq t_2$, and because of (A_1) and (A_2), we find that

$$
\begin{aligned}
0 &\geq w'(t) + H(p)w'(t-\tau)] \\
&\quad + \sum_{i=1}^{m} Q_i(t)\left[H(x(t-\sigma_i)) + H(p\,x(t-\tau-\sigma_i))\right] \\
&\geq w'(t) + H(p)w'(t-\tau) + \lambda \sum_{i=1}^{m} Q_i(t)H(z(t-\sigma_i)) \\
&\geq w'(t) + H(p)w'(t-\tau) + \lambda \sum_{i=1}^{m} Q_i(t)H(F^{+}(t-\sigma_i)),
\end{aligned} \tag{7}
$$

for $t \geq t_3 > t_2$. Integrating (7) from t_3 to $t(> t_3)$, we obtain

$$
\lambda \int_{t_3}^{t} \sum_{i=1}^{m} Q_i(s)H(F^{+}(t-\sigma_i))\mathrm{d}s \leq -[w(s) + H(p)w(s-\tau)]_{t_3}^{t} < \infty,
$$

as $t \to \infty$, a contradiction to (A_7).

If $x(t) < 0$, for $t \geq t_0$, then we set $y(t) = -x(t)$ to obtain $y(t) > 0$ for $t \geq t_0$ and hence using equation (6), we obtain a contradiction due to (A_7). This completes the proof of the theorem. □

Theorem 2.3. *Let* $-1 \leq p(t) \leq 0,\ t \in \mathbb{R}_+$. *Suppose that* (A_2), (A_4) *and* (A_5) *hold. If any one of the following conditions:*

(A_8) $\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)H(F^{+}(t-\sigma_i))\mathrm{d}t = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)H(F^{-}(t+\tau-\sigma_i))\mathrm{d}t,$

(A_9) $\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)H(F^{-}(t-\sigma_i))\mathrm{d}t = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)H(F^{+}(t+\tau-\sigma_i))\mathrm{d}t$

holds, then the conclusion of Theorem 2.1 *is true.*

Proof. On the contrary, we proceed as in the proof of Theorem 2.1 to obtain that $w(t)$ is monotonic on $[t_2, \infty), t_2 > t_1$. If $w(t) < 0$ for $t \geq t_2$, then $z(t) < F(t)$ is a contradiction due to (A_4) when $z(t) > 0$. Ultimately, $z(t) < 0$ and $z(t) < F(t)$ for $t \geq t_3 > t_2$. Using the fact that $z(t) < 0$ for $t \geq t_3$, it follows that

$$
x(t) \leq -p(t)x(t-\tau) \leq x(t-\tau) \leq x(t-2\tau) \leq ... \leq x(t_3),
$$

implies that $x(t)$ is bounded on $[t_3, \infty)$. Consequently, $\lim_{t\to\infty} w(t)$ exists. Clearly, $-z(t) > -F(t)$ implies that $-z(t) > \max\{0, -F(t)\} = F^{-}(t)$. Therefore, for $t \geq t_3 > t_2$

$$
-x(t-\tau) \leq p(t)x(t-\tau) \leq z(t) < -F^{-}(t)
$$

gives rise to $x(t-\sigma_i) > F^{-}(t+\tau-\sigma_i),\ \ t \geq t_4 > t_3$, for $i = 1, 2, ..., m$ and hence (5) reduced to

$$
w'(t) + \sum_{i=1}^{m} q_i(t)H(F^{-}(t+\tau-\sigma_i)) \leq 0,
$$

for $t \geq t_4 > t_3$. Integrating the last inequality from t_4 to ∞, we obtain

$$\int_{t_4}^{\infty} \sum_{i=1}^{m} q_i(s) H(F^-(s+\tau-\sigma_i)) \mathrm{d}s < \infty,$$

which contradicts (A_8). Hence $w(t) > 0$, for $t \geq t_2 > t_1$. We note that $z(t) > F(t)$ and $z(t) < 0$ is not possible due to (A_4). Therefore $z(t) > 0$ and $z(t) \leq x(t)$, for $t \geq t_3 > t_2$. In this case, $\lim_{t\to\infty} w(t)$ exists. Because, it happens that $z(t) > F^+(t)$ for $t \geq t_3 > t_2$, then (5) can be viewed as

$$w'(t) + \sum_{i=1}^{m} q_i(t) H(F^+(t-\sigma_i)) \leq 0.$$

Integrating the last inequality from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} \sum_{i=1}^{m} q_i(s) H(F^+(s-\sigma_i)) \mathrm{d}s < \infty,$$

a contradiction to (A_8). The case $x(t) < 0$, for $t \geq t_0$ is similar. Hence, the theorem is proved. □

THEOREM 2.4. *Let $-\infty < -p \leq p(t) \leq -1$, $t \in \mathbb{R}_+$ and $p > 0$. If all conditions of Theorem* 2.3 *are satisfied, then every bounded solution of (1) oscillates.*

Proof. The proof of the theorem can be followed from the proof of Theorem 2.3. Hence the details are omitted. □

REMARK 2.3. In Theorems 2.2–2.4, H could be linear, sublinear or superlinear.

THEOREM 2.5. *Let $-\infty < -p \leq p(t) \leq -1$, $t \in \mathbb{R}_+$, $p > 0$ and $\tau \geq \sigma_i$, $i = 1, ..., m$. Assume that (A_2), (A_4), (A_5) and (A_8) hold. Furthermore, assume that*

(A_{10}) $\frac{H(u)}{u^\beta} \geq \frac{H(v)}{v^\beta}$, $u \geq v > 0$, $\beta > 1$

and

(A_{11}) $\int_\rho^\infty \sum_{i=1}^{m} \frac{q_i(t) H(F^+(t+\tau-\sigma_i))}{[F^+(t+\tau-\sigma_i)]^\beta} \mathrm{d}t = \infty = \int_\rho^\infty \sum_{i=1}^{m} \frac{q_i(t) H(F^-(t+\tau-\sigma_i))}{[F^-(t+\tau-\sigma_i)]^\beta} \mathrm{d}t$

hold. Then conclusion of Theorem 2.1 *is true.*

Proof. The proof of the theorem follows from the proof of Theorem 2.3 except for the case when $w(t) < 0$, $z(t) < 0$, for $t \geq t_3 > t_2$. Since $z(t) \geq p(t)x(t-\tau)$, then

$$w(t) = z(t) - F(t) \geq p(t)x(t-\tau) - F(t), \quad t \geq t_3$$

implies that $w(t) - p(t)x(t-\tau) \geq -F(t)$, for $t \geq t_3$. Clearly, $w(t) - p(t)x(t-\tau) < 0$ is not possible due to (A_4) and the fact that $w(t) - p(t)x(t-\tau) = x(t) - F(t) \geq$

$-F(t)$ if and only if $x(t) > 0$, for $t \geq t_3$. Ultimately, $w(t) - p(t)x(t-\tau) > 0$ and hence

$$w(t) - p(t)x(t-\tau) \geq \max\{0, -F(t)\} = F^-(t),$$

that is,

$$w(t) \geq p(t)x(t-\tau) + F^-(t) \geq -px(t-\tau) + F^-(t) > -px(t-\tau) \tag{8}$$

for $t \geq t_4 > t_3$. Since $w(t)$ is decreasing and $\tau \geq \sigma_i$, for $i = 1, 2, ..., m$, then it follows that

$$-w(t) \leq -w(t+\tau-\sigma_i) < px(t-\sigma_i), \ \ t \geq t_4, \ i = 1, 2, ..., m.$$

Therefore,

$$\frac{H(x(t-\sigma_i))}{[-w(t)]^\beta} \geq \frac{H(x(t-\sigma_i))}{p^\beta x^\beta(t-\sigma_i)}, \ \ t \geq t_4, i = 1, 2, ..., m. \tag{9}$$

Consequently,

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}[-w(t)]^{1-\beta} &= -(1-\beta)[-w(t)]^{-\beta}\left[-w('t)\right] \\ &= (\beta-1)[-w(t)]^{-\beta}\sum_{i=1}^{m} q_i(t)H(x(t-\sigma_i)) \\ &\geq (\beta-1)\sum_{i=1}^{m} q_i(t)\frac{H(x(t-\sigma_i))}{p^\beta x^\beta(t-\sigma_i)}, \ \ t \geq t_4 \end{aligned}$$

due to (5) and (9). We may note from (8) that $0 > w(t) > -px(t-\tau) + F^-(t)$ implies that $x(t-\sigma_i) > p^{-1}F^-(t+\tau-\sigma_i)$, for $i = 1, 2, ..., m$ and hence

$$-\frac{\mathrm{d}}{\mathrm{d}t}[-w(t)]^{1-\beta} \geq (\beta-1)\ \sum_{i=1}^{m} q_i(t)\ \frac{H(p^{-1}F^-(t+\tau-\sigma_i))}{p^\beta[p^{-1}F^-(t+\tau-\sigma_i)]^\beta}, \tag{10}$$

for $t \geq t_4$ due to (A_{13}). Integrating (10) from t_4 to t, we get

$$\begin{aligned} (\beta-1)H(p^{-1})\int_{t_4}^{t}\sum_{i=1}^{m} q_i(s)\ \frac{H(F^-(s+\tau-\sigma_i))}{[F^-(s+\tau-\sigma_i)]^\beta}\mathrm{d}s &\leq -\left[-w(s)^{1-\beta}\right]_{t_4}^{t} \\ &< \infty, \text{ as } t \to \infty, \end{aligned}$$

due to (A_2), a contradiction to (A_{11}).

The case $x(t) < 0$ for $t \geq t_0$ can similarly be dealt with. Hence the theorem is proved. □

Example 2.1. Consider

$$(x(t) + x(t-\pi))' + x(t-2\pi) + x(t-4\pi) = 2\sin t, \tag{11}$$

where $p(t) = 1$, $q_1(t) = q_2(t) = 1$, $\tau = \pi$, $m = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi$, $H(x) = x$ and $f(t) = 2\sin t$. Indeed, if we choose $F(t) = -2\ \cos t$, then $F'(t) = f(t)$.

Since

$$F^+(t) = \begin{cases} -2\cos t, & 2n\pi + \frac{\pi}{2} \le t \le 2n\pi + \frac{3\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-(t) = \begin{cases} 2\cos t, & 2n\pi + \frac{3\pi}{2} \le t \le 2n\pi + \frac{5\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$F^+(t-2\pi) = \begin{cases} -2\cos t, & 2n\pi + \frac{5\pi}{2} \le t \le 2n\pi + \frac{7\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-(t-2\pi) = \begin{cases} 2\cos t, & 2n\pi + \frac{7\pi}{2} \le t \le 2n\pi + \frac{9\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$F^+(t-4\pi) = \begin{cases} -2\cos t, & 2n\pi + \frac{9\pi}{2} \le t \le 2n\pi + \frac{11\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-(t-4\pi) = \begin{cases} 2\cos t, & 2n\pi + \frac{11\pi}{2} \le t \le 2n\pi + \frac{13\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\int_{4\pi}^{\infty} [Q_1(t)F^+(t-2\pi) + Q_2(t)F^+(t-4\pi)]\mathrm{d}t = I_1 + I_2,$$

where for $n = 0, 1, 2...$, we get

$$\begin{aligned} I_1 = \int_{4\pi}^{\infty} F^+(t-2\pi)\mathrm{d}t &= \sum_{n=0}^{\infty} \int_{2n\pi+\frac{5\pi}{2}}^{2n\pi+\frac{7\pi}{2}} [-2\cos t]\mathrm{d}t \\ &= -2\sum_{n=0}^{\infty} [\sin t]_{2n\pi+\frac{5\pi}{2}}^{2n\pi+\frac{7\pi}{2}} = +\infty, \end{aligned}$$

$$\begin{aligned} I_2 = \int_{4\pi}^{\infty} F^+(t-4\pi)\mathrm{d}t &= \sum_{n=0}^{\infty} \int_{2n\pi+\frac{9\pi}{2}}^{2n\pi+\frac{11\pi}{2}} [-2\cos t]\mathrm{d}t \\ &= -2\sum_{n=0}^{\infty} [\sin t]_{2n\pi+\frac{9\pi}{2}}^{2n\pi+\frac{11\pi}{2}} = +\infty. \end{aligned}$$

Clearly, (A_1), (A_2), (A_4), (A_5) and (A_7) are satisfied. Hence, by Theorem 2.2, every solution of (11) is oscillatory. Thus, in particular, $x(t) = \sin t$ is an oscillatory solution of the equation (11).

EXAMPLE 2.2. Consider

$$(x(t) - x(t-2\pi))' + x(t-2\pi) + x(t-4\pi) = 2\sin t, \tag{12}$$

where $p(t) = -1$, $q_1(t) = q_2(t) = 1$, $\tau = 2\pi$, $m = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi$, $H(x) = x$ and $f(t) = 2\sin t$. Indeed, if we choose $F(t) = -2\ \cos t$, then $F'(t) = f(t)$. Clearly, (A_2), (A_4), (A_5), (A_8) and (A_9) are hold true. Hence, Theorem 2.3 can be applied to (12), that is, every solution of (12) oscillates. Indeed, $x(t) = \sin t$ is such a solution of (12).

3. HOMOGENEOUS OSCILLATION

This section deals with the oscillatory behaviour of solutions of equation (2). Here H could be linear, sublinear or superlinear.

THEOREM 3.1. *Let* $0 \le p(t) \le p < \infty$, $t \in \mathbb{R}_+$ *and* $\tau \le \sigma_i$, $i = 1, ..., m$. *Assume that* (A_1) *and* (A_2) *hold. Furthermore, assume that*

(A_{12}) $\int\limits_{c_1}^{\pm c_2} \frac{\mathrm{d}t}{H(t)} < \infty, \quad c_1, c_2 > 0$

and

(A_{13}) $\int\limits_{\tau}^{\infty} \sum\limits_{i=1}^{m} Q_i(t)\mathrm{d}t = \infty$

hold, where $\sum\limits_{i=1}^{m} Q_i(t)$ *is defined as in Theorem* 2.2. *Then every solution of the equation* (2) *is oscillatory.*

Proof. Let $x(t)$ be nonoscillatory solution of equation (2) such that $x(t) > 0$ for $t \ge t_0$. Setting as in (3), (2) can be written as

$$z'(t) = -\sum_{i=1}^{m} q_i(t) H(x(t-\sigma_i)) \le 0 \tag{13}$$

for $t \ge t_1 > t_0$. Consequently, $z(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. Since $z(t) > 0$ for $t_2 > t_1$. Due to (13), (2) becomes

$$\begin{aligned} 0 =& z'(t) + \sum_{i=1}^{m} q_i(t) H(x(t-\sigma_i)) \\ &+ H(p) \left[z'(t-\tau) + \sum_{i=1}^{m} q_i(t-\tau) H(x(t-\tau-\sigma_i)) \right] \end{aligned}$$

for $t \ge t_2$ and because of (A_1) and (A_2), we find that

$$\begin{aligned} 0 &\ge z'(t) + H(p) z'(t-\tau) + \sum_{i=1}^{m} Q_i(t) \left[H(x(t-\sigma_i)) + H(p\, x(t-\tau-\sigma_i)) \right] \\ &\ge z'(t) + H(p) z'(t-\tau) + \lambda \sum_{i=1}^{m} Q_i(t) H(z(t-\sigma_i)). \end{aligned}$$

Consequently, for $i = 1, 2, ..., m$ there exists $t_3 > t_2$ such that

$$\frac{z'(t)}{H(z(t-\sigma_i))} + H(p)\frac{z'(t-\tau)}{H(z(t-\sigma_i))} + \lambda\sum_{i=1}^{m} Q_i(t) < 0. \tag{14}$$

Because $z(t)$ is decreasing on $[t_3, \infty)$ and $\tau \le \sigma_i$, for $i = 1, 2, ..., m$ the inequalities in (14) become

$$\frac{z'(t)}{H(z(t))} + H(p)\frac{z'(t-\tau)}{H(z(t-\tau))} + \lambda\sum_{i=1}^{m} Q_i(t) < 0.$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_3 to t, we get

$$\int_{t_3}^{t} \frac{z'(s)}{H(z(s))}\mathrm{d}s + H(p)\int_{t_3}^{t} \frac{z'(s-\tau)}{H(z(s-\tau))}\mathrm{d}s + \lambda\int_{t_3}^{t}\sum_{i=1}^{m} Q_i(s)\mathrm{d}s < 0,$$

that is,

$$\lambda\int_{t_3}^{t}\sum_{i=1}^{m} Q_i(s)\mathrm{d}s < -\left[\int_{z(t_3)}^{z(t)} \frac{\mathrm{d}y}{H(y)} + H(p)\int_{z(t_3-\tau)}^{z(t-\tau)} \frac{\mathrm{d}y}{H(y)}\right]$$
$$< \infty, \text{ as } t \to \infty,$$

due to (A_{12}), a contradiction to (A_{13}).

If $x(t) < 0$, for $t \ge t_0$, then we set $y(t) = -x(t)$, for $t \ge t_0$ in (1) and we find

$$(y(t) + p(t)y(t-\tau))' + \sum_{i=1}^{m} q_i(t)H(y(t-\sigma_i)) = 0.$$

Then proceeding as above, we find the same contradiction. This completes the proof of the theorem. □

THEOREM 3.2. *Let* $-\infty < -p \le p(t) \le -1,\ t \in \mathbb{R}_+,\ p > 0$ *and* $\tau > \sigma_i,\ i = 1, ..., m$. *Assume that* (A_2) *holds. If*

(A_{14}) $\int_0^{\pm\infty} \frac{\mathrm{d}t}{H(t)} < \infty$

and

(A_{15}) $\int_0^{\infty} \sum_{i=1}^{m} q_i(t)\mathrm{d}t = \infty$

hold, then also the conclusion of Theorem 3.1 *is true.*

Proof. On the contrary, we proceed as in the proof of the Theorem 3.1 to obtain $z(t)$ is monotonic on $[t_2, \infty), t_2 > t_1$. We claim that $z(t) < 0$, for $t \ge t_2$. If not, let $z(t) \ge 0$, for $t \ge t_2 > t_1$. Consequently,

$$x(t) \ge -p(t)x(t-\tau) \ge x(t-\tau) \ge x(t-2\tau) \ge x(t-3\tau) \ge ... \ge x(t_2)$$

implies that x is bounded from below by $m > 0$. Integrating (13) from t_2 to $t(> t_2)$, we obtain

$$z(t) - z(t_3) + \int_{t_2}^{t} \sum_{i=1}^{m} q_i(s) H(x(s - \sigma_i)) \mathrm{d}s = 0,$$

that is,

$$z(t) - z(t_3) + H(m) \int_{t_2}^{t} \sum_{i=1}^{m} q_i(s) \mathrm{d}s < 0.$$

Therefore,

$$z(t) < z(t_3) - H(m) \int_{t_2}^{t} \sum_{i=1}^{m} q_i(s) \mathrm{d}s \to -\infty \ \text{ as } \ t \to \infty,$$

a contradiction to the fact that $z(t) > 0$ on $[t_2, \infty)$. So our claim holds. From (3), it follows that $z(t + \tau - \sigma_i) > p(t + \tau - \sigma_i) x(t - \sigma_i)$ for $i = 1, 2, ..., m$. Hence, (13) becomes

$$z'(t) + \sum_{i=1}^{m} \frac{q_i(t)}{H(-p)} H(z(t + \tau - \sigma_i)) \leq 0, \tag{15}$$

due to (A_2). Because z is decreasing on $[t_2, \infty)$, then

$$z'(t) + \sum_{i=1}^{m} \frac{q_i(t)}{H(-p)} H(z(t)) \leq 0.$$

Integrating the last inequality from t_2 to $t(> t_2)$, we get

$$\int_{t_2}^{t} \frac{z'(s)}{H(z(s))} \mathrm{d}s + \frac{1}{H(-p)} \int_{t_2}^{t} \sum_{i=1}^{m} q_i(s) \mathrm{d}s \geq 0,$$

that is,

$$\int_{t_2}^{t} \sum_{i=1}^{m} q_i(s) \mathrm{d}s \leq -H(-p) \int_{z(t_2)}^{z(t)} \frac{\mathrm{d}y}{H(y)} < \infty, \ \text{ as } \ t \to \infty,$$

due to (A_{14}), a contradiction to (A_{15}). The case $x(t) < 0$ is similar. Hence the theorem is proved. □

THEOREM 3.3. *Let $-\infty < -p \leq p(t) \leq -1$, $t \in \mathbb{R}_+$ and $p > 0$. Assume that (A_2) and (A_{15}) hold. Then every bounded solution of (2) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.2, we have that $z(t) < 0$, for $t \geq t_2 > t_1$. Hence the inequality (15) holds. Because z is decreasing, there exist $t_3 > t_2$ and $k > 0$ such that $z(t) \leq -k$, for $t \geq t_3$. Therefore, the inequality (15) can be viewed as

$$z'(t) + \frac{H(-k)}{H(-p)} \sum_{i=1}^{m} q_i(t) < 0, \tag{16}$$

for $t \geq t_3$. Integrating (16) from t_3 to $t(> t_3)$, we obtain

$$\frac{H(-k)}{H(-p)} \int_{t_3}^{t} \sum_{i=1}^{m} q_i(s)\mathrm{d}s < - \left[z(s)\right]_{t_3}^{t}.$$

Since $x(t)$ is bounded, then $z(t)$ is bounded and hence for $t \to \infty$ the last inequality becomes

$$\frac{H(-k)}{H(-p)} \int_{t_3}^{\infty} \sum_{i=1}^{m} q_i(s)\mathrm{d}s < \infty,$$

a contradiction to (A_{15}). The case $x(t) < 0$ is similar dealt with. Hence the proof of the theorem is completed. □

THEOREM 3.4. *Let $-1 < -p \leq p(t) \leq 0,\ t \in \mathbb{R}_+,\ p > 0$ and $\tau > \sigma_i,\ i = 1, ..., m$. If (A_1), (A_{12}) and (A_{15}) hold, then also the conclusion of Theorem 3.1 is true.*

Proof. Proceeding as in Theorem 3.1, we may note that $z(t)$ is monotonic on $[t_2, \infty), t_2 > t_1$. Hence there exists $t_3 > t_2$ such that $z(t) > 0$ or $z(t) < 0$. Let $z(t) > 0$ for $t_3 > t_2$. From (3), it follows that $z(t) \leq x(t)$ on $[t_3, \infty)$. Consequently, (13) becomes

$$z'(t) + \sum_{i=1}^{m} q_i(t) H(z(t - \sigma_i)) < 0,$$

that is,

$$\frac{z'(t)}{H(z(t))} + \sum_{i=1}^{m} q_i(t) < 0.$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_3 to t, we get

$$\int_{t_3}^{t} \sum_{i=1}^{m} q_i(s)\mathrm{d}s < - \int_{z(t_3)}^{z(t)} \frac{\mathrm{d}y}{H(y)} < \infty, \quad \text{as } t \to \infty,$$

due to (A_{12}), a contradiction to (A_{15}). Hence $z(t) < 0$, for $t_3 > t_2$. Proceedings as above proof of Theorem 2.3, we obtain that $x(t)$ is bounded on $[t_3, \infty)$. The rest of the theorem follows from Theorem 3.3. This completes the proof. □

EXAMPLE 3.1. Consider

$$(x(t) + x(t - \pi))' + x(t - 2\pi) + x(t - 3\pi) = 0, \tag{17}$$

where $p(t) = 1,\ q_1(t) = q_2(t) = 1,\ \tau = \pi,\ m = 2,\ \sigma_1 = 2\pi,\ \sigma_2 = 3\pi,\ H(x) = x$. Clearly, (A_1), (A_2), (A_{12}) and

$$\int_{\pi}^{\infty} [Q_1(t) + Q_2(t)]\mathrm{d}t = \infty,$$

hold, where $Q_1(t) = Q_2(t) = 1$. Hence, Theorem 3.1 can be applied to (17), that is, every solution of (17) oscillates. Indeed, $x(t) = \sin t$ is such a solution of (17).

4. EXISTENCE OF POSITIVE SOLUTION

In this section, sufficient conditions are obtained to show that equation (1) admits a positive bounded solution.

THEOREM 4.1. *Let* $0 \le p(t) \le p_1 < 1, t \in \mathbb{R}$ *and* H *be Lipschitzian on the interval of the form* $[a, b],\ \ 0 < a < b < \infty$. *Suppose that* $f(t)$ *satisfies* (A_4). *If*

(A_{16}) $\int_0^\infty \sum_{i=1}^m q_i(t)\mathrm{d}t < \infty,$

holds, then equation (1) *admits a positive bounded solution.*

Proof. Due to (A_{16}), it is possible to find $t_1 > 0$ such that

$$\int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(s)\mathrm{d}s < \frac{1 - p_1}{5K},$$

where $K = \max\{K_1, H(1)\},\ K_1$ is the Lipschitz constant on $[\frac{1-p_1}{10}, 1]$. Let F be such that $-\frac{1-p_1}{10} \le F(t) \le \frac{1-p_1}{10}$ for $t \ge t_2$. For $t_3 > \max\{t_1, t_2\}$, we set $Y = BC([t_3, \infty), \mathbb{R})$, the space of real valued bounded continuous functions on $[t_3, \infty)$. Clearly, Y is a Banach space with respect to supremum norm defined by

$$||y|| = \sup\{|y(t)| : t \ge t_3\}.$$

Let us define

$$S = \left\{ u \in Y : \ \frac{1 - p_1}{10} \le u(t) \le 1,\ t \ge t_3 \right\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T : S \to S$ be defined by

$$Tx(t) = \begin{cases} Tx(t_3 + \rho), \text{ for } t \in [t_3, t_3 + \rho] \\ \\ -p(t)x(t - \tau) + \frac{1+4p_1}{5} + F(t) \\ + \int_t^\infty \sum_{i=1}^m q_i(s)H(x(s - \sigma_i))\mathrm{d}s, \text{ for } t \ge t_3 + \rho. \end{cases}$$

For every $x \in S$,

$$Tx(t) \leq \frac{1-p_1}{10} + \frac{1+4p_1}{5} + H(1)\left[\int_t^\infty \sum_{i=1}^m q_i(s)\mathrm{d}s\right]$$
$$< \frac{1-p_1}{10} + \frac{1+4p_1}{5} + \frac{1-p_1}{5} = \frac{1+p_1}{2} < 1$$

and

$$Tx(t) \geq -p(t)x(t-\tau) + \frac{1+4p_1}{5} + F(t)$$
$$\geq -p_1 + \frac{1+4p_1}{5} - \frac{1-p_1}{10} = \frac{1-p_1}{10}$$

implies that $Tx \in S$. Now, for $y_1, y_2 \in S$

$$|Ty_1(t) - Ty_2(t)| \leq |p(t)||y_1(t-\tau) - y_2(t-\tau)|$$
$$+ K_1 \int_t^\infty \sum_{i=1}^m q_i(s)|y_1(s-\sigma_i) - y_2(s-\sigma_i)|\mathrm{d}s$$
$$\leq p_1||y_1 - y_2|| + K_1||y_1 - y_2|| \left[\int_t^\infty \sum_{i=1}^m q_i(s)\mathrm{d}s\right]$$
$$< \left(p_1 + \frac{1-p_1}{5}\right)||y_1 - y_2||,$$

that is, $||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||$, that is, T is a contraction mapping, where $\mu = \frac{1+4p_1}{5} < 1$. Since S is complete and T is a contraction on S, then by the Banach's fixed point theorem T has a unique fixed point on $\left[\frac{1-p_1}{10}, 1\right]$. Hence $Tx = x$ and

$$x(t) = \begin{cases} x(t_3+\rho), \text{ for } t \in [t_3, t_3+\rho] \\ \\ -p(t)x(t-\tau) + \frac{1+4p_1}{5} + F(t) \\ + \int_t^\infty \sum_{i=1}^m q_i(s)H(x(s-\sigma_i))\mathrm{d}s, \text{ for } t \geq t_3+\rho \end{cases}$$

is a bounded positive solution of the equation (1) on $\left[\frac{1-p_1}{10}, 1\right]$. This completes the proof of the theorem. □

Remark 4.1. Theorems similar to Theorem 4.1 can be proved in other ranges of $p(t)$.

5. SUMMARY

It is worth noticing that both unforced and forced equations (1) and (2) are studied keeping in view assumptions (A_1) – (A_{15}). The results concerning equations (1) and (2) are completely oscillatory due to the analysis corporated here. Of course, the forcing term can be considered to (1).

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