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∗-TOPOLOGY AND #-TOPOLOGY

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Abstract. A new topology $\tau^{\#}$ on X, via an ideal $\mathcal{I}$, is introduced and investigated. $\tau^{\#}$ lies between $\tau^{\#} \cap \tau$ and τ^{*} properly, in general. Decompositions of $*$-continuity and $\#_r$-continuity are obtained – in particular, continuity and δ-continuity.

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1. INTRODUCTION AND PRELIMINARIES

Let (X, τ) be a topological space on which no separation axioms are imposed. If $S \subset X$ then the *interior* and the *closure* of S in (X, τ) are denoted by $\mathrm{int}(S)$ (or $\mathrm{int}_\tau(S)$) and $\mathrm{cl}(S)$ (or $\mathrm{cl}_\tau(S)$), respectively. A set S is called *regular open* (resp., *regular closed*) in (X, τ) if $S = \mathrm{int}(\mathrm{cl}(S))$ (resp., $S = \mathrm{cl}(\mathrm{int}(S))$). An $S \subset X$ is said to be α-open [19] (resp,. *semi-open* [14], preopen [16], b-open [2], β-*open* [1]) in (X, τ), if $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S)) \cup \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$).

The collection of all regular open (resp., α-open, semi-open, preopen, b-open, β-open) subsets of (X, τ) is denoted by $\mathrm{RO}(X, \tau)$ (resp., $\alpha(X, \tau)$, $\mathrm{SO}(X, \tau)$, $\mathrm{PO}(X, \tau)$, $\mathrm{BO}(X, \tau)$, $\beta(X, \tau)$).

A subset S of (X, τ) is said to be *g-closed* [15] (resp., *rg-closed* [23], αg^*-*closed* [17], *sg**-*closed* [25, 28, 22], *pg**-*closed*, *bg**-*closed*, βg^*-*closed* [17]) if $\mathrm{cl}(S) \subset U$ whenever $S \subset U$ and $U \in \tau$ (resp., $U \in \mathrm{RO}(X, \tau)$, $U \in \alpha(X, \tau)$, $U \in \mathrm{SO}(X, \tau)$, $U \in \mathrm{PO}(X, \tau)$, $U \in \mathrm{BO}(X, \tau)$, $U \in \beta(X, \tau)$). We remark that an sg*-closed set is called ω-*closed* in [25] and $\widehat{g}$-*closed* in [28].

The collection of all g-closed (resp., rg-closed, αg*-closed, sg*-closed, pg*-closed, bg*-closed, βg*-closed) subsets of (X, τ) is denoted by $\mathrm{g}(X, \tau)$ (resp., $\mathrm{rg}(X, \tau)$, $\alpha\mathrm{g}(X, \tau)$, $\mathrm{sg}(X, \tau)$, $\mathrm{pg}(X, \tau)$, $\mathrm{bg}(X, \tau)$, $\beta\mathrm{g}(X, \tau)$). The family of all closed subsets of (X, τ) we denote by $\mathrm{c}(\tau)$, and the family of all *semi-closed* subsets S of (X, τ) ($\mathrm{int}(\mathrm{cl}(S)) \subset S$) [4] will be denoted by $\mathrm{SC}(X, \tau)$.

Recall that the δ-closure of a subset S in (X, τ) is defined by $\{x \in X : S \cap \mathrm{int}(\mathrm{cl}(U)) \neq \emptyset$ for all $U \in \tau$ such that $x \in U\} =: \mathrm{cl}_\delta(S)$ [29]. If $S = \mathrm{cl}_\delta(S)$, then S is called δ-closed. The complement of a δ-closed set (to X) is called δ-open. For each space (X, τ) the family of all δ-open sets form a topology τ_δ on X [29]. Recall the following basic fact: $\tau_\delta = \tau_{\mathrm{s}}$ (on (X, τ)) [29], where τ_{s} is the so-called *semi-regularization of* τ, that is τ_{s} is a smaller topology on X with the family $\mathrm{RO}(X, \tau)$ as a base for it.

An ideal $\mathcal{I}$ on a topological space (X,τ) is a non-empty collection of subsets of X, which satisfies two conditions: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$, (2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$. If in a topological space (X,τ) we set an ideal $\mathcal{I}$, then the structure $(X,\tau,\mathcal{I})$ is called *ideal topological space* (or an ideal space). One type of topology via ideals has been defined by three independent authors: Vaidyanathaswamy (1945) [26, 27], Hashimoto (1976) [9], Hamlett, Rose and Janković (1990) [8, 11]. We recall this type of topology in the following fashion. First, for a subset S of $(X,\tau,\mathcal{I})$ we define the set $S^*(\mathcal{I},\tau) = \{x \in X : S \cap U \notin \mathcal{I}$ for every $U \in \tau$ with $x \in U\}$ which is called the *local function* of S with respect to $\mathcal{I}$ and τ [13]. When there is no ambiguity we simply write S^*. The following properties for local function are known.

Lemma 1.1 ([11, 27]). *Let $(X,\tau,\mathcal{I})$ be an ideal space and $S, S_1, S_2 \subset X$. Then the following hold:*

(1) $S_1 \subset S_2$ *implies* $S_1^* \subset S_2^*$,
(2) $S^* = \mathrm{cl}(S^*) \subset \mathrm{cl}(S)$,
(3) $(S^*)^* \subset S^*$,
(4) $(S_1 \cup S_2)^* = S_1^* \cup S_2^*$.

We now define an operator $\mathrm{cl}^*(S)$ for $S \subset X$: $\mathrm{cl}^*(S) := S \cup S^*$ [26, 27]. Using some properties from Lemma 1.1, one can prove that $\mathrm{cl}^*(\cdot)$ fulfils the Kuratowski closure operator axioms. Thus it determines a topology on X called ∗*-topology*, which is finer than τ. For ∗-topology we use denotation τ^*. The collection of all ∗-closed subsets of X ($\mathrm{cl}^*(S) = S$) is denoted by $\mathrm{c}(\tau^*)$.

A subfamily $\mathcal{F}_X$ of the power set $\mathcal{P}(X)$, $X \neq \emptyset$, is called a *minimal structure* (briefly, m-structure) [24] on X if $\emptyset, X \in \mathcal{F}_X$. In the sequel m-structures will be denoted by $\mathcal{F}$. It is known that for some m-structures the following inclusion relationships hold:

$$\mathrm{RO}(X,\tau) \subset \tau \subset \alpha(X,\tau) \subset \mathrm{SO}(X,\tau) \subset \mathrm{BO}(X,\tau) \subset \beta(X,\tau)$$

and

$$\alpha(X,\tau) \subset \mathrm{PO}(X,\tau) \subset \mathrm{BO}(X,\tau).$$

All above inclusions are proper, in general.

2. ∗-CLOSED SETS

Let $(X,\tau,\mathcal{I})$ be an ideal space and let $\mathcal{F}$ be an arbitrarily chosen m-structure on X. Definitions 2.1 and 2.2 generalize the well-known notions of *locally closed* [7] (briefly, LC) subsets in (X,τ) and *g-closed* subsets in it.

Definition 2.1. A subset A of $(X,\tau,\mathcal{I})$ is said to be ∗-$\mathcal{I}$-$\mathcal{F}$-*locally closed* (briefly, ∗-$\mathcal{I}$-$\mathcal{F}$-LC) if $A = U \cap V$, where $U \in \mathcal{F}$ and $V \in \mathrm{c}(\tau^*)$.

Obviously, this definition also generalizes the known notions of *strongly-locally closed* [10] or *weakly-$\mathcal{I}$-LC* [12] sets.

Definition 2.2. A subset A of $(X,\tau,\mathcal{I})$ is said to be *$*$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed* if $\mathrm{cl}^*(A)\subset U$, whenever $A\subset U\in\mathcal{F}$.

We remark that equivalently: A is $*$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed if $A^*\subset U$ whenever $A\subset U\in\mathcal{F}$. Definition 2.2 generalizes such notions as *$\mathcal{I}_g$-closed* [5] or $\mathcal{I}_{rg}$-closed [18] sets. The family of all $*$-$\mathcal{I}$-$\mathcal{F}$-LC (resp. $*$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed subsets of $(X,\tau,\mathcal{I})$ we denote by $*$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)$ (resp. $*$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$).

Lemma 2.1. *If A is $*$-closed, then $A\in *$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$.*

Proof. Obvious. □

Lemma 2.2. *If $A\in *$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$, then A is $*$-closed.*

Proof. Since $A\in *$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)$, there exist $U\in\mathcal{F}$ and $V\in \mathrm{c}(\tau^*)$ such that $A=U\cap V$. Then $A\subset V$ and hence $\mathrm{cl}^*(A)\subset\mathrm{cl}^*(V)=V$. So, $A^*\subset V$. But $A\in *$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$. Thus $A^*\subset U$ for $U\in\mathcal{F}$ with $A\subset U$. Consequently, $A^*\subset U\cap V=A$ and $\mathrm{cl}^*(A)=A$. Therefore $A\in\mathrm{c}(\tau^*)$. □

Theorem 2.1. *Let $\mathcal{F}_1,\mathcal{F}_2\subset\mathcal{P}(X)$ be arbitrary m-structures on X such that $\mathcal{F}_1\supset\mathcal{F}_2$. Then the following statements are equivalent for any $A\subset X$:*

(1) *A is $*$-closed;*
(2) $A\in *$-$\mathcal{I}$-$\mathcal{F}_1$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}_1}(X,\tau)$;
(3) $A\in *$-$\mathcal{I}$-$\mathcal{F}_1$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}_2}(X,\tau)$;
(4) $A\in *$-$\mathcal{I}$-$\mathcal{F}_2$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}_1}(X,\tau)$;
(5) $A\in *$-$\mathcal{I}$-$\mathcal{F}_2$-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathcal{F}_2}(X,\tau)$.

Proof. (1)⇔(2) follows from Lemmas 2.1 and 2.2. Implications (2)⇒(3)⇒(5) and (2)⇒(4)⇒(5) are obvious. (1)⇔(5) follows from Lemmas 2.1 and 2.2. □

Using Theorem 2.1 and, respectively, m-structures from Section 1, one can obtain various equivalent conditions for a set being $*$-closed. We will put as $\mathcal{F}$: $\beta(X,\tau)$, $\mathrm{BO}(X,\tau)$, $\mathrm{SO}(X,\tau)$, $\mathrm{PO}(X,\tau)$, $\alpha(X,\tau)$, τ, $\mathrm{RO}(X,\tau)$ in $*$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)$, and in $*$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$ we will use, respectively, the following symbols: $(\mathcal{F}=)$ $\beta,\mathrm{b},\mathrm{s},\mathrm{p},\alpha,\tau,\mathrm{r}\tau$.

Theorem 2.2. *For $A\subset X$ the following statements are equivalent:*

(1) *A is $*$-closed;*
(2_1) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\beta}(X,\tau)$;
(2_2) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{b}}(X,\tau)$;
(2_3) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{s}}(X,\tau)$;
(2_4) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{p}}(X,\tau)$;
(2_5) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\alpha}(X,\tau)$;
(2_6) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\tau}(X,\tau)$;
(2_7) $A\in *$-$\mathcal{I}$-β-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{r}\tau}(X,\tau)$;
(3_1) $A\in *$-$\mathcal{I}$-b-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{b}}(X,\tau)$;
(3_2) $A\in *$-$\mathcal{I}$-b-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{s}}(X,\tau)$;
(3_3) $A\in *$-$\mathcal{I}$-b-$\mathrm{LC}(X,\tau)\cap *$-$\mathcal{I}$-$g_{\mathrm{p}}(X,\tau)$;

(3$_4$) $A \in *\text{-}\mathcal{I}\text{-b-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
(3$_5$) $A \in *\text{-}\mathcal{I}\text{-b-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$;
(3$_6$) $A \in *\text{-}\mathcal{I}\text{-b-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$;
(4$_1$) $A \in *\text{-}\mathcal{I}\text{-s-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{s}}(X,\tau)$;
(4$_2$) $A \in *\text{-}\mathcal{I}\text{-s-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
(4$_3$) $A \in *\text{-}\mathcal{I}\text{-s-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$;
(4$_4$) $A \in *\text{-}\mathcal{I}\text{-s-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$;
(5$_1$) $A \in *\text{-}\mathcal{I}\text{-p-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{p}}(X,\tau)$;
(5$_2$) $A \in *\text{-}\mathcal{I}\text{-p-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
(5$_3$) $A \in *\text{-}\mathcal{I}\text{-p-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$;
(5$_4$) $A \in *\text{-}\mathcal{I}\text{-p-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$;
(6$_1$) $A \in *\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
(6$_2$) $A \in *\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$;
(6$_3$) $A \in *\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$;
(7$_1$) $A \in *\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$ [10, Theorem 2.5];
(7$_2$) $A \in *\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$;
(8) $A \in *\text{-}\mathcal{I}\text{-r}\tau\text{-}\mathrm{LC}(X,\tau) \cap *\text{-}\mathcal{I}\text{-}g_{\mathrm{r}\tau}(X,\tau)$ [10, Theorem 2.6(3)].

The remaining cases are left to the reader.

3. $\#$-TOPOLOGY

In this section we introduce a new topology on (X,τ) via ideals. For a subset $S \subset X$ we define a set $S^{\#}(\mathcal{I},\tau)$ as follows:

$$S^{\#}(\mathcal{I},\tau) = \big\{x \in X : S \cap \mathrm{int}(\mathrm{cl}(U)) \notin \mathcal{I} \text{ for every } U \in \tau \text{ with } x \in U\big\}.$$

For $S^{\#}(\mathcal{I},\tau)$ we use the notation $S^{\#}$ (if there is no risk of confusion). In Lemma 3.2 below we list some properties of the operator $(\cdot)^{\#}$.

Recall a generalization (due to Noiri and the author) of some well-known fact (for the "open" case):

LEMMA 3.1. [6, 21] *Let (X,τ) be an arbitrary space. If either $S_1 \in \mathrm{SO}(X,\tau) \cup \mathrm{SC}(X,\tau)$ or $S_2 \in \mathrm{SO}(X,\tau) \cup \mathrm{SC}(X,\tau)$, then*

$$\mathrm{int}(\mathrm{cl}(S_1 \cap S_2)) = \mathrm{int}(\mathrm{cl}(S_1)) \cap \mathrm{int}(\mathrm{cl}(S_2)). \tag{1}$$

LEMMA 3.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $S, S_1, S_2 \subset X$ be arbitrary. Then the following hold:*

(1) $S_1 \subset S_2$ *implies* $S_1^{\#} \subset S_2^{\#}$;
(2) $(S^{\#})^{\#} \subset S^{\#}$;
(3) $(S_1 \cup S_2)^{\#} = S_1^{\#} \cup S_2^{\#}$;
(4) $S^{\#} = \mathrm{cl}(S^{\#})$;
(5) $S^{\#} \subset (\mathrm{cl}(S))^{\#} \subset \mathrm{cl}_\delta(S)$;
(6) $\mathrm{cl}(S^{\#}) \subset \mathrm{cl}(S)$ *for every* $S \in \mathrm{PO}(X,\tau)$;
(7) $(\mathrm{cl}(S))^{\#} \subset \mathrm{cl}(S)$ *for every* $S \in \mathrm{PO}(X,\tau)$.

Proof. (1) Obvious.

(2) Suppose there exists a point x such that $x \in (S^{\#})^{\#}$ and $x \notin S^{\#}$. Then, by definition of operation $(\cdot)^{\#}$, we have $S^{\#} \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$ for any $U \in \tau$ with $x \in U$. On the other hand, there exists a $V \in \tau$ with $x \in V$ such that $S \cap \operatorname{int}(\operatorname{cl}(V) \in \mathcal{I}$. Hence, for $U = V$ we get $\emptyset \neq S^{\#} \cap \operatorname{int}(\operatorname{cl}(V)) \in \mathcal{I}$. Let $y \in S^{\#} \cap \operatorname{int}(\operatorname{cl}(V))$. Then $y \in S^{\#}$ and $y \in \operatorname{int}(\operatorname{cl}(V)) = W$. So, we have $S \cap \operatorname{int}(\operatorname{cl}(W)) \notin \mathcal{I}$, that is $S \cap \operatorname{int}(\operatorname{cl}(V)) \notin \mathcal{I}$. This contradiction establishes our inclusion.

(3) Let $x \in (S_1 \cup S_2)^{\#}$ be an arbitrarily chosen point. Then $S_1 \cup S_2) \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$ for each $U \in \tau$ with $x \in U$. So, we have $[S_1 \cap \operatorname{int}(\operatorname{cl}(U))] \cup [S_2 \cap \operatorname{int}(\operatorname{cl}(U))] \notin \mathcal{I}$ which implies $S_1 \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$ or $S_2 \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$—in the opposite case one gets $(S_1 \cup S_2) \cap \operatorname{int}(\operatorname{cl}(U)) \in \mathcal{I}$, a contradiction. Therefore we obtain that $x \in S_1^{\#} \cup S_2^{\#}$.

Let now $x \in S_1^{\#} \cup S_2^{\#}$ be arbitrary. So, $x \in S_1^{\#}$ or $x \in S_2^{\#}$. Thus for any $U \in \tau$ with $x \in U$, $S_1 \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$or, for any $V \in \tau$ with $x \in V$, $S_2 \cap \operatorname{int}(\operatorname{cl}(V)) \notin \mathcal{I}$. Therefore $[S_1 \cap \operatorname{int}(\operatorname{cl}(U))] \cup [S_2 \cap \operatorname{int}(\operatorname{cl}(V))] \notin \mathcal{I}$, and for $U = V$ we get $(S_1 \cup S_2) \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$, which shows that $S_1^{\#} \cup S_2^{\#} \subset (S_1 \cup S_2)^{\#}$.

(4) We shall show that $\operatorname{cl}(S^{\#}) \subset S^{\#}$. Suppose there exists a point $x \in X$ with $x \in \operatorname{cl}(S^{\#})$ and $x \notin S^{\#}$. Hence, for any $V \in \tau$ with $x \in V$, $S^{\#} \cap V \neq \emptyset$ and for a certain $V_1 \in \tau$ with $x \in V_1$, $S \cap \operatorname{int}(\operatorname{cl}(V_1)) \in \mathcal{I}$. Let $y \in S^{\#} \cap V_1$. Thus $y \in V_1$ and $S \cap \operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{I}$ for each $U \in \tau$ with $y \in U$. For $U = V_1$ one has $S \cap \operatorname{int}(\operatorname{cl}(V_1)) \notin \mathcal{I}$, a contradiction.

(5) Inclusion $S^{\#} \subset (\operatorname{cl}(S))^{\#}$ is obvious by (1). Let $x \in (\operatorname{cl}(S))^{\#}$ be arbitrary. Then $\operatorname{int}(\operatorname{cl}(U)) \cap \operatorname{cl}(S) \notin \mathcal{I}$ for any $U \in \tau$ with $x \in U$. We get $\operatorname{cl}\big[S \cap \operatorname{int}(\operatorname{cl}(U))\big] \notin \mathcal{I}$. It cannot be $S \cap \operatorname{int}(\operatorname{cl}(U)) = \emptyset$, since $\operatorname{cl}(\emptyset) = \emptyset \in \mathcal{I}$. Therefore $S \cap \operatorname{int}(\operatorname{cl}(U)) \neq \emptyset$, which shows that $x \in \operatorname{cl}_\delta(S)$.

(6) Suppose $x \in \operatorname{cl}(S^{\#})$ and $x \notin \operatorname{cl}(S)$. Hence, there exists a $V_1 \in \tau$ with $x \in V_1$ such that $S \cap V_1 = \emptyset$. Using Lemma 3.1 we obtain $\operatorname{int}(\operatorname{cl}(S)) \cap \operatorname{int}(\operatorname{cl}(V_1)) = \emptyset$ and since $S \in \operatorname{PO}(X, \tau)$, $S \cap \operatorname{int}(\operatorname{cl}(V_1)) = \emptyset \in \mathcal{I}$. On the other hand, $S^{\#} \cap U \neq \emptyset$ for any $U \in \tau$ with $x \in U$. Let $U = V_1$ and $y \in S^{\#} \cap V_1$, that is $S \cap \operatorname{int}(\operatorname{cl}(V_1)) \notin \mathcal{I}$. Consequently our inclusion holds.

(7) Let $x \in (\operatorname{cl}(S))^{\#}$ and $x \notin \operatorname{cl}(S)$. Then for a certain $V_1 \in \tau$ with $x \in V_1$ we have $V_1 \cap S = \emptyset$. Obviously, by Lemma 3.1, $\operatorname{int}(\operatorname{cl}(V_1)) \cap \operatorname{int}(\operatorname{cl}(S)) = \emptyset$ and since $S \in \operatorname{PO}(X, \tau)$, $\operatorname{int}(\operatorname{cl}(V_1)) \cap S = \emptyset \in \mathcal{I}$. On the other hand, $\operatorname{int}(\operatorname{cl}(U)) \cap \operatorname{cl}(S) \notin \mathcal{I}$ for each $U \in \tau$ with $x \in U$. So, we get for $U = V_1$, $\operatorname{cl}\big[S \cap \operatorname{int}(\operatorname{cl}(V_1))\big] \notin \mathcal{I}$. Simultaneously, by $\operatorname{int}(\operatorname{cl}(V_1)) \cap S = \emptyset$, we have $\operatorname{cl}\big[\operatorname{int}(\operatorname{cl}(V_1)) \cap S\big] = \emptyset \in \mathcal{I}$. This contradicts our inclusion. □

Corollary 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If*
(a) $S \in \operatorname{PO}(X, \tau)$ *and*
(b) $\operatorname{int}(\operatorname{cl}(S)) \subset \operatorname{int}(\operatorname{cl}(S^{\#})) = \operatorname{int}(S^{\#})$, *then*

(1) $\operatorname{cl}(S) = \operatorname{cl}(S^{\#})$,
(2) $\operatorname{cl}(S) = (\operatorname{cl}(S))^{\#}$.

Proof. (1) In virtue of Lemma 3.2 (6), it is enough to prove $\mathrm{cl}(S) \subset \mathrm{cl}(S^{\#})$. Suppose $x \in \mathrm{cl}(S)$ and $x \notin \mathrm{cl}(S^{\#})$. From $x \notin \mathrm{cl}(S^{\#})$ we infer that there exists a $V_1 \in \tau$ with $x \in V_1$ such that $V_1 \cap S^{\#} = \emptyset$. Making use of Lemma 3.1 we get $\mathrm{int}(\mathrm{cl}(V_1)) \cap \mathrm{int}(\mathrm{cl}(S^{\#})) = \emptyset$. By assumptions, $V_1 \cap S = \emptyset$. But $x \in \mathrm{cl}(S)$ and so $V_1 \cap S \neq \emptyset$. A contradiction.

(2) By Lemma 3.2, items (4) and (5), we have

$$\mathrm{cl}(S^{\#}) \subset (\mathrm{cl}(S))^{\#} \subset \mathrm{cl}_\delta(S).$$

Using (1) of Lemma 3.2 one gets

$$\mathrm{cl}(S) \subset (\mathrm{cl}(S))^{\#} \subset \mathrm{cl}_\delta(S).$$

Then

$$\mathrm{cl}(S) \subset \mathrm{cl}(S) \cap (\mathrm{cl}(S))^{\#} \subset \mathrm{cl}(S) \cap \mathrm{cl}_\delta(S) = \mathrm{cl}(S),$$

and consequently $\mathrm{cl}(S) = \mathrm{cl}(S) \cap (\mathrm{cl}(S))^{\#}$. Thus $\mathrm{cl}(S) \subset (\mathrm{cl}(S))^{\#}$ and, by Lemma 3.2 (7), $\mathrm{cl}(S) = (\mathrm{cl}(S))^{\#}$. □

We define now operator $\mathrm{cl}^{\#}(\,\cdot\,)$ on $(X, \tau, \mathcal{I})$ as follows:

$$\mathrm{cl}^{\#}(S) = S \cup S^{\#} \qquad \text{for each} \quad S \subset X.$$

COROLLARY 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Under conditions* (a) *and* (b) *of Corollary* 3.1 *we have* $\mathrm{cl}^{\#}(S) = \mathrm{cl}(S) = \mathrm{cl}^{\#}(\mathrm{cl}(S))$.

Proof. Using Lemma 2.2(4) and Corollary 3.1 (1) we get what follows: $\mathrm{cl}^{\#}(S) = S \cup S^{\#} = S \cup \mathrm{cl}(S^{\#}) = S \cup \mathrm{cl}(S) = \mathrm{cl}(S)$.

By Corollary 3.1 (2) we have $\mathrm{cl}^{\#}(\mathrm{cl}(S)) = \mathrm{cl}(S) \cup (\mathrm{cl}(S))^{\#} = \mathrm{cl}(S)$. □

THEOREM 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the operator $\mathrm{cl}^{\#}(\,\cdot\,)$ fulfills the Kuratowski closure axioms.*

Proof. **[K.1]** $\mathrm{cl}^{\#}(\emptyset) = \emptyset$**.** Obvious.

[K.2] $S \subset \mathrm{cl}^{\#}(S)$ for each $S \subset X$. Obvious.

[K.3] $\mathrm{cl}^{\#}(\mathrm{cl}^{\#}(S)) = \mathrm{cl}^{\#}(S)$ for each $S \subset X$. We calculate as follows (using Lemma 3.2, (2) & (3)): $\mathrm{cl}^{\#}(\mathrm{cl}^{\#}(S)) = \mathrm{cl}^{\#}(S \cup S^{\#}) = (S \cup S^{\#}) \cup (S \cup S^{\#})^{\#} = (S \cup S^{\#}) \cup (S^{\#} \cup (S^{\#})^{\#}) = S \cup S^{\#} \cup (S^{\#})^{\#} = S \cup S^{\#} = \mathrm{cl}^{\#}(S)$.

[K.4] $\mathrm{cl}^{\#}(S_1 \cup S_2) = \mathrm{cl}^{\#}(S_1) \cup \mathrm{cl}^{\#}(S_2)$**.** By Lemma 3.2 (3), $\mathrm{cl}^{\#}(S_1 \cup S_2) = (S_1 \cup S_2) \cup (S_1 \cup S_2)^{\#} = (S_1 \cup S_2) \cup (S_1^{\#} \cup S_2^{\#}) = \mathrm{cl}^{\#}(S_1) \cup \mathrm{cl}^{\#}(S_2)$. □

Thus $\mathrm{cl}^{\#}(\,\cdot\,)$ is a Kuratowski closure operator on X and thus it determines a topology $\tau^{\#}(\mathcal{I}, \tau)$ on X, which we call *$\#$-topology*. For the sake of brevity we denote this topology as $\tau^{\#}$ (if there is no risk of confusion). By Lemma 3.2 (5) we have $S^{\#} \subset \mathrm{cl}_\delta(S)$. Thus $\mathrm{cl}^{\#}(S) = S \cup S^{\#} \subset \mathrm{cl}_\delta(S)$ for every $S \subset X$. Also, directly from respective definitions we have $S^* \subset S^{\#}$. So, $\mathrm{cl}^*(S) \subset \mathrm{cl}^{\#}(S)$ for each $S \subset X$. Using a known fact: $\tau_1 \subset \tau_2$ iff $\mathrm{cl}_{\tau_2}(S) \subset \mathrm{cl}_{\tau_1}(S)$ for $S \subset X$, we infer that for any ideal space $(X, \tau, \mathcal{I})$ the following inclusions hold:

$$\tau_\delta \subset \tau^{\#} \subset \tau^*.$$

There exists an ideal space for which the above inclusions are proper, as the following example shows.

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$, $\tau = \big\{\emptyset, X, \{b\}, \{b, c, d\}\big\}$ and $\mathcal{I} = \big\{\emptyset, \{c\}, \{d\}, \{c, d\}\big\}$.

(a) We have $\{a, d\}^* = \{a\}$ and hence $\mathrm{cl}^*(\{a, d\}) = \{a, d\}$. Consequently, $\mathrm{int}^*(\{b, c\}) = X \setminus \mathrm{cl}^*(\{a, d\}) = \{b, c\}$. Thus $\{b, c\} \in \tau^*$.

(b) $\{a, b\}^{\#} = X$ since for every $U \in \tau$, $\mathrm{int}_\tau(\mathrm{cl}_\tau(U)) = X$. It shows that $\mathrm{int}^{\#}(\{b, c\}) = X \setminus \mathrm{cl}^{\#}(\{a, d\}) = \emptyset$ and consequently $\{b, c\} \notin \tau^{\#}$.

(c) We have $\{c, d\}^{\#} = \emptyset$, because $\{c, d\} \in \mathcal{I}$. Hence, $\mathrm{int}^{\#}(\{a, b\}) = X \setminus \mathrm{cl}^{\#}(\{c, d\}) = \{a, b\}$. Therefore $\{a, b\} \in \tau^{\#} \setminus \tau$ $(\{\emptyset, X\} = \tau_\delta \subsetneq \tau)$.

(a), (b) and (c) show that for this ideal space, $\tau_\delta \subsetneq \tau^{\#} \subsetneq \tau^*$. Observe, moreover, that $\{a\}^{\#} = X$, hence $\mathrm{int}^{\#}(\{b, c, d\}) = \emptyset$. Therefore, $\{b, c, d\} \in \tau \setminus \tau^{\#}$. It shows that in general, there is no inclusion relationship between τ and $\tau^{\#}$ (cf. (c)).

Example 3.1 leads to the following theorem.

THEOREM 3.2. *There exists an ideal space $(X, \tau, \mathcal{I})$ for which the following statements hold:*

1) $\tau_\delta \subsetneq \tau^{\#} \cap \tau$;
2) *there exist sets $S_1, S_2 \subset X$ with $S_1 \in \tau^{\#} \setminus \tau$ and $S_2 \in \tau \setminus \tau^{\#}$;*
3) $\tau^{\#} \cap \tau \subsetneq \tau \subsetneq \tau^*$;
4) $\tau^{\#} \cap \tau \subsetneq \tau^{\#} \subsetneq \tau^*$.

Proof. For 1), observe that $\{b, c\} \in \tau^{\#} \cap \tau \setminus \tau_\delta$. For 3), check that $\{b, c, d\} \in \tau \setminus (\tau^{\#} \cap \tau)$ and $\{a, b\} \in \tau^* \setminus \tau$. For 2) and 4) we refer straight to Example 3.1. □

A natural question, in the context, is: for what types of open subsets S of $(X, \tau, \mathcal{I})$ there is always $S \in \tau^{\#}$?

THEOREM 3.3. *Let $(X, \tau, \rangle)$ be an ideal space. If S is clopen (in (X, τ)) and $X \setminus S \subset \mathrm{int}_\tau((X \setminus S)^{\#})$, then $S \in \tau^{\#}$.*

Proof. Since S is clopen, $X \setminus S = \mathrm{int}(\mathrm{cl}(X \setminus S))$. Also, $\mathrm{int}(\mathrm{cl}(X \setminus S)) \subset \mathrm{int}(\mathrm{cl}((X \setminus S)^{\#}))$ by our second assumption and Lemma 3.2 (4). Hence, by the use of Corollary 3.2 we calculate as follows:

$$S = \mathrm{int}(S) = X \setminus \mathrm{cl}(X \setminus S) = X \setminus \mathrm{cl}^{\#}(X \setminus S) = \mathrm{int}^{\#}(S).$$

So, $S \in \tau^{\#}$. □

We finish this section by the observation that two ideal spaces $(X, \tau_1, \mathcal{I})$, $(X, \tau_2, \mathcal{I})$, where $\tau_1 \neq \tau_2$, may generate the same $\#$-topology, i.e. $\tau_1^{\#}(\mathcal{I}, \tau_1) = \tau_2^{\#}(\mathcal{I}, \tau_2)$–see the following example.

EXAMPLE 3.2. Let $X = \{a,b\}$, $\mathcal{I} = \{\emptyset, X\}$, $\tau_1 = \big\{\emptyset, X, \{a\}\big\}$ and $\tau_2 = \big\{\emptyset, X, \{b\}\big\}$. We get that $\tau_1^{\#} = \{\emptyset, X\} = \tau_2^{\#}$: calculations are left to the reader.

PROBLEM 1. *Are there two ideal spaces $(X,\tau,\mathcal{I}_1)$, $(X,\tau,\mathcal{I}_2)$, with $\mathcal{I}_1 \neq \mathcal{I}_2$, such that $\tau^{\#}(\mathcal{I}_1,\tau) = \tau^{\#}(\mathcal{I}_2,\tau)$?*

4. $\#$-CLOSED SETS

The content of this section is analogous to that of Section 2, with the difference that it concerns $\#$-topology. Let $(X,\tau,\mathcal{I})$ be an ideal space and $\mathcal{F}$ be an m-structure on X.

DEFINITION 4.1. A subset A of $(X,\tau,\mathcal{I})$ is said to be *$\#$-$\mathcal{I}$-$\mathcal{F}$-locally closed* (briefly: $\#$-$\mathcal{I}$-$\mathcal{F}$-LC) if $A = U \cap V$, where $U \in \mathcal{F}$ and $V \in \mathrm{c}(\tau^{\#})$.

DEFINITION 4.2. A subset A of $(X,\tau,\mathcal{I})$ is said to be *$\#$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed*, if $\mathrm{cl}^{\#}((A) \subset U$ whenever $A \subset U$ and $U \in \mathcal{F}$.

Equivalently, we have: A is $\#$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed if $A^{\#} \subset U$ whenever $A \subset U$ and $U \in \{$. The family of all $\#$-$\mathcal{I}$-$\mathcal{F}$-LC (resp. $\#$-$\mathcal{I}$-$g_{\mathcal{F}}$-closed) subsets of $(X,\tau,\mathcal{I})$ we denote by $\#$-$\mathcal{I}$-$\mathcal{F}$-$\mathrm{LC}(X,\tau)$ (resp. $\#$-$\mathcal{I}$-$g_{\mathcal{F}}(X,\tau)$).

LEMMA 4.1. *If A is $\#$-closed, then $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}}(X,\tau)$.*

Proof. $A = X \cap A$, where $X \in \mathcal{F}$. For $A \subset U$, $U \in \mathcal{F}$, $\mathrm{cl}^{\#}(A) \subset U$. □

LEMMA 4.2. *If $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}}(X,\tau)$, then A is $\#$-closed.*

Proof. Similar to that for Lemma 2.2. □

THEOREM 4.1. *Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{P}(X)$ be m-structures on X with $\mathcal{F}_1 \supset \mathcal{F}_2$. Then the following are equivalent for each $A \subset X$:*

(1′) *A is $\#$-closed;*
(2′) $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}_1\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}_1}(X,\tau)$;
(3′) $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}_1\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}_2}(X,\tau)$;
(4′) $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}_2\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}_1}(X,\tau)$;
(5′) $A \in \#\text{-}\mathcal{I}\text{-}\mathcal{F}_2\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_{\mathcal{F}_2}(X,\tau)$.

Proof. Compare the proof of Theorem 2.1. □

As in Theorem 2.2 of the second section, using the families $\mathcal{F}$, respectively, $\beta(X,\tau)$, $\mathrm{BO}(X,\tau)$, $\mathrm{SO}(X,\tau)$, $\mathrm{PO}(X,\tau)$, $\alpha(X,\tau)$, τ, $\mathrm{RO}(X,\tau)$, one may obtain Theorem 4.2 for the $\#$ case.

It is enough to replace in Theorem 2.2 the prefixes $*$ with $\#$. Since there are too many statements that are included in Theorem 4.2, we list only those we have dropped in Theorem 2.2 (for the $*$ case).

THEOREM 4.2. *Let $A \subset X$. The following statements are equivalent:*

(1′) *A is $\#$-closed;*

($2'_1$) $A \in \#\text{-}\mathcal{I}\text{-}\beta\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($2'_2$) $A \in \#\text{-}\mathcal{I}\text{-}\beta\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;

(*cases* ($2'_3$) *until* ($8'$) *are omitted*)

($9'_1$) $A \in \#\text{-}\mathcal{I}\text{-}b\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($9'_2$) $A \in \#\text{-}\mathcal{I}\text{-}s\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($9'_3$) $A \in \#\text{-}\mathcal{I}\text{-}p\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($9'_4$) $A \in \#\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($9'_5$) $A \in \#\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($9'_6$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\beta(X,\tau)$;
($10'_1$) $A \in \#\text{-}\mathcal{I}\text{-}s\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;
($10'_2$) $A \in \#\text{-}\mathcal{I}\text{-}p\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;
($10'_3$) $A \in \#\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;
($10'_4$) $A \in \#\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;
($10'_5$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_b(X,\tau)$;
($11'_1$) $A \in \#\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_s(X,\tau)$;
($11'_2$) $A \in \#\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_s(X,\tau)$;
($12'_1$) $A \in \#\text{-}\mathcal{I}\text{-}\alpha\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_p(X,\tau)$;
($12'_2$) $A \in \#\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_p(X,\tau)$;
($12'_3$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_p(X,\tau)$;
($13'_1$) $A \in \#\text{-}\mathcal{I}\text{-}\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
($13'_2$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\alpha(X,\tau)$;
($13'_3$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$;
($14'$) $A \in \#\text{-}\mathcal{I}\text{-}r\tau\text{-}\mathrm{LC}(X,\tau) \cap \#\text{-}\mathcal{I}\text{-}g_\tau(X,\tau)$.

5. DECOMPOSITIONS OF CONTINUITY

DEFINITION 5.1. A map $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be $*$-*continuous* [10, Definition 3.1] (resp. $\#$-*continuous*) on X if $f^{-1}(F) \in \mathrm{c}(\tau^*)$ (resp. $f^{-1}(F) \in \mathrm{c}(\tau^\#)$) for each $F \in \mathrm{c}(\sigma)$.

In order to consider applications of some particular cases, it will be convenient to use the following definition.

DEFINITION 5.2. A map $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be $\#_r$-*continuous* on X if $f^{-1}(F) \in \mathrm{c}(\tau^\#)$ for any $F \in \mathrm{RC}(Y,\sigma)$.

The following corollary is obvious.

COROLLARY 5.1. *A map* $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ *is* $*$-*continuous* (*resp.*, $\#$-*continuous*, $\#_r$-*continuous*) *on* X *if* $f^{-1}(V) \in \tau^*$ (*resp.*, $f^{-1}(V) \in \tau^\#$, $f^{-1}(V) \in \tau^\#$) *for each* $V \in \sigma$ (*resp.*, $V \in \sigma$, $V \in \mathrm{RO}(Y,\sigma)$).

The key theorems to obtain many decomposition results concerning continuities from Definitions 5.1 and 5.2 are Theorems 2.1 and 4.1.

First, we should define the respective notions of continuity, some of which are known from literature. Let $\mathcal{F}$ be an m-structure.

Definition 5.3. (a) A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ will be called *$*$-$\mathcal{I}$-$\mathfrak{F}$-LC* (resp., *$*$-$\mathcal{I}$-$g_{\mathfrak{F}}$)-continuous* on X, if $f^{-1}(F) \in *\text{-}\mathcal{I}\text{-}\mathfrak{F}\text{-}\mathrm{LC}(X,\tau)$ (resp., $f^{-1}(F) \in *\text{-}\mathcal{I}\text{-}g_{\mathfrak{F}}(X,\tau)$) for every $F \in \mathrm{c}(\sigma)$.

(b) A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ will be called *$\#$-$\mathcal{I}$-$\mathfrak{F}$-LC* (resp., *$\#$-$\mathcal{I}$-$g_{\mathfrak{F}}$)-continuous* on X, if, for each $F \in \mathrm{c}(\sigma)$, we have $f^{-1}(F) \in \#\text{-}\mathcal{I}\text{-}\mathfrak{F}\text{-}\mathrm{LC}(X,\tau)$ (resp., $f^{-1}(F) \in \#\text{-}\mathcal{I}\text{-}g_{\mathfrak{F}}(X,\tau)$).

(c) A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ will be called *$\#_r$-$\mathcal{I}$-$\mathfrak{F}$-LC* (resp., *$\#_r$-$\mathcal{I}$-$g_{\mathfrak{F}}$)-continuous* on X, if, for each $F \in \mathrm{RC}(Y,\sigma)$, the following relation $f^{-1}(F) \in \#\text{-}\mathcal{I}\text{-}\mathfrak{F}\text{-}\mathrm{LC}(X,\tau)$ (resp., $f^{-1}(F) \in \#\text{-}\mathcal{I}\text{-}g_{\mathfrak{F}}(X,\tau)$) holds true.

Translating some of the above generalized notions via ideals (Definitions 5.1, 5.2, and 5.3) to the usual case ($\mathcal{I} = \{\emptyset\}$), one obtains that $*$-continuity is a continuity and $\#_r$-continuity is δ-continuity [20, Theorem 2.2]. Also, $*$-$\mathcal{I}$-$g_{\mathfrak{F}}$-continuity ($\mathcal{I}_g$-continuity in [10]) is a g-continuity [3]; $*$-$\mathcal{I}$-$g_{r\tau}$-continuity ($\mathcal{I}_{rg}$-continuity in [10]) is an rg-continuity [23]; $*$-$\mathcal{I}$-$r\tau$-LC-continuity (strong-$\mathcal{I}$-LC-continuity in [10]) is a strong-LC continuity [10]. For known examples of decompositions of $*$-continuity and continuity the reader is advised to see [10, Theorem 3.7 and Corollary 3.8], respectively.

Using Theorems 2.1 and **??** we establish a series of decompositions of $*$-continuity and $\#_r$-continuity, respectively.

Theorem 5.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and (Y,σ) be a topological space. Let $\mathfrak{F}_1 \subset \mathfrak{F}_2$ be arbitrary m-structures on X. Then the following are equivalent for any $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$*

(1^*) *f is $*$-continuous;*

(2^*) *f is $*$-$\mathcal{I}$-$\mathfrak{F}_1$-LC-continuous and $*$-$\mathcal{I}$-$g_{\mathfrak{F}_1}$-continuous;*

(3^*) *f is $*$-$\mathcal{I}$-$\mathfrak{F}_1$-LC-continuous and $*$-$\mathcal{I}$-$g_{\mathfrak{F}_2}$-continuous;*

(4^*) *f is $*$-$\mathcal{I}$-$\mathfrak{F}_2$-LC-continuous and $*$-$\mathcal{I}$-$g_{\mathfrak{F}_1}$-continuous;*

(5^*) *f is $*$-$\mathcal{I}$-$\mathfrak{F}_2$-LC-continuous and $*$-$\mathcal{I}$-$g_{\mathfrak{F}_2}$-continuous.*

Theorem 5.2. *Under the same assumptions as in Theorem 5.1, the following are equivalent:*

($1^{\#_r}$) *f is $\#_r$-continuous;*

($2^{\#_r}$) *f is $\#_r$-$\mathcal{I}$-$\mathfrak{F}_1$-LC-continuous and $\#_r$-$\mathcal{I}$-$g_{\mathfrak{F}_1}$-continuous;*

($3^{\#_r}$) *f is $\#_r$-$\mathcal{I}$-$\mathfrak{F}_1$-LC-continuous and $\#_r$-$\mathcal{I}$-$g_{\mathfrak{F}_2}$-continuous;*

($4^{\#_r}$) *f is $\#_r$-$\mathcal{I}$-$\mathfrak{F}_2$-LC-continuous and $\#_r$-$\mathcal{I}$-$g_{\mathfrak{F}_1}$-continuous;*

($5^{\#_r}$) *f is $\#_r$-$\mathcal{I}$-$\mathfrak{F}_2$-LC-continuous and $\#_r$-$\mathcal{I}$-$g_{\mathfrak{F}_2}$-continuous.*

For $\mathcal{I} = \{\emptyset\}$ we give below some examples of decompositions of continuity and δ-continuity. We set $r\tau, b, s, p$ for $\mathrm{RO}(X,\tau)$, $\mathrm{BO}(X,\tau)$, $\mathrm{SO}(X,\tau)$, and $\mathrm{PO}(X,\tau)$, respectively.

Corollary 5.2. *Let $(X,\tau,\mathcal{I} = \{\emptyset\})$ be an ideal space, (Y,σ) a topological space. For any function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ the following statements are equivalent:*

(1^*) *f is continuous;*

(2*) *f is* $*$-$\{\emptyset\}$-*p*-*LC*-*continuous and* $*$-$\{\emptyset\}$-$g_{r\tau}$-*continuous;*
(3*) *f is* $*$-$\{\emptyset\}$-*s*-*LC*-*continuous and* $*$-$\{\emptyset\}$-g_s-*continuous;*
(4*) *f is* $*$-$\{\emptyset\}$-*b*-*LC*-*continuous and* $*$-$\{\emptyset\}$-g_p-*continuous.*

We suggest the reader to compare items (2*), (3*), and (4*) with statements (5_4), (4_1), and (3_3) of Theorem 2.2, respectively.

COROLLARY 5.3. *Under the same assumptions as in Corollary* 5.2 *the following are equivalent:*

($1^{\#_r}$) *f is* δ-*continuous;*
($2^{\#_r}$) *f is* $*$-$\{\emptyset\}$-*s*-*LC*-*continuous and* $\#_r$-$\{\emptyset\}$-g_b-*continuous;*
($3^{\#_r}$) *f is* $*$-$\{\emptyset\}$-τ-*LC*-*continuous and* $\#_r$-$\{\emptyset\}$-g_s-*continuous;*
($4^{\#_r}$) *f is* $*$-$\{\emptyset\}$-τ-*LC*-*continuous and* $\#_r$-$\{\emptyset\}$-g_p-*continuous.*

Compare ($10'_1$), ($11'_2$) and ($12'_2$) of Theorem 4.2 with ($2^{\#_r}$), ($3^{\#_r}$) and ($4^{\#_r}$), respectively.

REMARK 5.1. In connection to known notions (see Section 1) one can observe that in Corollary 5.2, f is $*$-$\{\emptyset\}$-$g_{r\tau}$-continuous (resp., $*$-$\{\emptyset\}$-g_s-continuous; $*$-$\{\emptyset\}$-g_p-continuous) if for each $F \in \mathrm{c}(\sigma)$, the preimage $f^{-1}(F)$ is rg-closed (resp., sg^*-closed; pg^*-closed).

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