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A NOTE ON SOME SECOND-ORDER INTEGRO-DIFFERENTIAL INCLUSIONS WITH BOUNDARY CONDITIONS

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Abstract. We study the existence of solutions for two classes of second-order integro-differential inclusions with boundary conditions. We establish Filippov type existence results in the case of nonconvex set-valued maps.

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1. INTRODUCTION

In the last years we observe a remarkable amount of interest in the study of existence of solutions of several boundary value problems associated to problems of the form $\mathcal{D}x \in F(t,x)$, where $\mathcal{D}$ is a differential operator and F is a set-valued map. Most of these existence results are obtained using fixed point techniques and are based on an integral form of the right inverse to the operator $\mathcal{D}$. This means that for every f the unique solution y of the equation $\mathcal{D}y = f$ can be written in the form $y = \mathcal{R}f$, when the operator $\mathcal{R}$ has nonnegative Green's function.

This paper is concerned with the following second-order boundary value problems

$$(1.1)\qquad -x'' \in F(t,x,V(x)(t)) \ a.e.\ [0,1], \quad x(0)=0,\ x(1)=\int_0^1 h(t)x(t)\mathrm{d}t,$$

$$(1.2)\qquad -x'' - a(t)x' \in F(t,x,V(x)(t)) \quad a.e.\ [0,1], \quad x'(0)=0, \quad x(1)=0,$$

where $F : [0,1]\times \mathbf{R}\times\mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $h : [0,1] \to \mathbf{R}$ an integrable function, $a : (0,1] \to \mathbf{R}$ a nonnegative continuous function with $\int_0^1 a(t)\mathrm{d}t = \infty$ and $V : C([0,1],\mathbf{R}) \to C([0,1],\mathbf{R})$ a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))\mathrm{d}s$, with $k : [0,1]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ a given function.

In a relatively recent paper of Benchohra, Nieto and Ouahab ([2]), problem (1.1) is studied with F single valued and not depending on the last variable and it is obtained the existence of solutions using the Banach contraction principle and the nonlinear alternative of Leray Schauder type. In [4], Xiao, Cang and Liu obtained existence results for problem (1.2) with F not depending on the last variable. The existence results in [4] use fixed point techniques and are based on a nonlinear alternative of Leray-Schauder type due to Schafer, on the Bohnenblust-Karlin fixed point theorem and on the Covitz-Nadler contraction principle for set-valued maps.

The aim of this note is to show that Filippov's ideas ([3]) can be suitably adapted in order to obtain the existence of solutions for problems (1.1) and (1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([3]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In the case when F does not depend on the last variable our approach improves Theorem 3.3 in [2] and Theorem 3.1 in [4].

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. PRELIMINARIES

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets A, $B \subset X$ is defined by

$$\mathrm{d}_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},$$

where $d^*(A, B) = \sup\{d(a, B); a \in A\}$ and $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $I = [0, 1]$. We denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ endowed with the norm $||x||_C = \sup_{t \in I} |x(t)|$, and by $L^1(I, \mathbf{R})$ the Banach space of integrable functions $u : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^1 |u(t)|\mathrm{d}t$. With $AC^1(I, \mathbf{R})$ we denote the space of differentiable functions $x : (0, 1) \to \mathbf{R}$ whose first derivative x' is absolutely continuous.

In what follows we assume that $M_1 = \int_0^1 sh(s)\mathrm{d}s \neq 1$ and we need the following technical results.

Lemma 2.1. [2] *Let $\sigma : [0, 1] \to \mathbf{R}$ be an integrable function. Then the unique solution of the boundary value problem*

$$-x'' = \sigma(t) \quad a.e.\ (I), \quad x(0) = 0, \quad x(1) = \int_0^1 h(t)x(t)\mathrm{d}t$$

is given by

$$x(t) = \int_0^1 G(t, s)\sigma(s)\mathrm{d}s,$$

where

$$G(t, s) = H(t, s) + \frac{1}{1 - M_1}\int_0^1 H(r, s)h(r)\mathrm{d}r,$$

$$H(t, s) := \begin{cases} s(1 - t), & \text{if} \quad 0 \le s \le t \le 1, \\ t(1 - s), & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

Note that $|H(t, s)| \le \frac{1}{4}\ \forall\, t, s \in I$ and therefore

$$|G(t, s)| \le \frac{1}{4} + \frac{1}{|1 - M_1|}\frac{1}{4}\int_0^1 h(r)\mathrm{d}r =: G_0.$$

Lemma 2.2. [4] *Let $a : (0, 1] \to \mathbf{R}$ be a nonnegative continuous function with $\int_0^1 a(t)\mathrm{d}t = \infty$. For a given $f \in L^1(I, \mathbf{R}^n)$ the unique solution of the boundary value problem*

$$\begin{array}{l} -x''(t) - a(t)x'(t) = f(t) \quad a.e.\ ([0,1]), \\ x'(0) = 0, \quad x(1) = 0, \end{array}$$

is given by

$$x(t) = \int_0^1 G_1(t,s)f(s)\mathrm{d}s,$$

where

$$G_1(t,s) = \begin{cases} \int_s^1 e^{-\int_s^r a(\tau)\mathrm{d}\tau}\mathrm{d}r & \text{if} \quad 0 \le t \le s \le 1 \\ \int_t^1 e^{-\int_s^r a(\tau)\mathrm{d}\tau}\mathrm{d}r & \text{if} \quad 0 \le s \le t \le 1. \end{cases}$$

Remark 2.3. If $a(t) \geq \delta\ \forall t \in [0,1]$, where $\delta > 0$ is a given constant, then

$$0 < K := \max_{t,s \in I} G_1(t,s) = \max_{s \in I} \int_s^1 e^{-\int_s^r a(\tau)\mathrm{d}\tau}\mathrm{d}r < \frac{1 - e^{-\delta}}{\delta}.$$

In particular, since $1 - e^{-\delta} < \delta$, it follows that

$$\frac{2\delta}{1 + \delta - e^{-\delta}} K < 1.$$

3. THE MAIN RESULTS

In order to prove our results we need the following hypotheses.

Hypothesis 3.1. i) *$F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L \in L^1(I, (0, \infty))$ such that, for almost all $t \in I, F(t, \cdot, \cdot)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \le L(t)(|x_1 - x_2| + |y_1 - y_2|), \forall\ x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) *$k : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.*

iv) $|k(t,s,x) - k(t,s,y)| \le L(t)|x - y|$ *a.e.* $(t,s) \in I \times I, \quad \forall\, x, y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t)\left(1 + \int_0^t L(u)\mathrm{d}u\right),\ t \in I,\ M_0 = \int_0^1 M(t)\mathrm{d}t,\ L_0 := \int_0^1 L(t)\mathrm{d}t.$$

Theorem 3.2. *Assume that Hypothesis* 3.1 *is satisfied and $G_0M_0 < 1$. Let $y \in AC^1(I, \mathbf{R})$ be such that $y(0) = 0$, $y(1) = \int_0^1 h(t)y(t)\mathrm{d}t$ and such that there exists $p \in L^1(I, \mathbf{R}_+)$ with $d(-y''(t), F(t, y(t), V(y)(t))) \le p(t)$ a.e. on I.*

Then there exists $x \in C(I, \mathbf{R})$ *a solution of problem* (1.1) *satisfying for all* $t \in I$ the inequality

$$|x(t) - y(t)| \leq \frac{G_0}{1 - G_0 M_0} \int_0^1 p(t)\mathrm{d}t. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{-y''(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows (e.g., Theorem 1.14.1 in [1]) that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* on I such that

$$|f_1(t) + y''(t)| \leq p(t) \quad a.e.\ I \tag{3.2}$$

Defining $x_1(t) = \int_0^1 G_1(t, s) f_1(s)\mathrm{d}s$, one has

$$|x_1(t) - y(t)| \leq G_0 \int_0^1 p(t)\mathrm{d}t.$$

We claim that it is enough to construct the sequences (x_n) in $C(I, \mathbf{R})$ and (f_n) in $L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G_1(t, s) f_n(s)\mathrm{d}s, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\, I, \tag{3.4}$$

$$\begin{aligned} |f_{n+1}(t) - f_n(t)| \leq & L(t) \cdot |x_n(t) - x_{n-1}(t)| \\ & + L(t) \cdot \int_0^t L(s)|x_n(s) - x_{n-1}(s)|\mathrm{d}s\ a.e.\, I \end{aligned} \tag{3.5}$$

If this construction is realized, then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq G_0 (G_0 M_0)^n \int_0^1 p(t)\mathrm{d}t \quad \forall n \in \mathbf{N}.$$

Indeed, assuming that the last inequality is true for $n-1$, we prove it for n. We get that

$$\begin{aligned} &|x_{n+1}(t) - x_n(t)| \leq \int_0^1 |G_1(t, t_1)| \cdot |f_{n+1}(t_1) - f_n(t_1)|\mathrm{d}t_1 \\ &\leq G_0 \int_0^1 L(t_1) \left(|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} L(s)|x_n(s) - x_{n-1}(s)|\mathrm{d}s \right) \mathrm{d}t_1 \\ &\leq G_0 \int_0^1 L(t_1) \left(1 + \int_0^{t_1} L(s)\mathrm{d}s \right) \mathrm{d}t_1 \cdot G_0^n M_0^{n-1} \int_0^1 p(t)\mathrm{d}t \\ &= G_0 (G_0 M_0)^n \int_0^1 p(t)\mathrm{d}t. \end{aligned}$$

Therefore (x_n) is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence it converges uniformly to some $x \in C(I,\mathbf{R})$. Therefore, by (3.5), for almost all $t \in I$, the sequence $(f_n(t))$ is Cauchy in $\mathbf{R}$. Let f be the pointwise limit of (f_n).

Moreover, one has that

$$\begin{aligned}
|x_n(t)-y(t)| &\le |x_1(t)-y(t)| + \sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \\
&\le G_0\int_0^1 p(t)\mathrm{d}t + \sum_{i=1}^{n-1} G_0\int_0^1 p(t)\mathrm{d}t(G_0M_0)^i \qquad (3.6)\\
&= \frac{G_0\int_0^1 p(t)\mathrm{d}t}{1-G_0M_0}.
\end{aligned}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$

$$|f_n(t)+y''(t)| \le \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)| + |f_1(t)+y''(t)| \le L(t)\frac{G_0\int_0^1 p(t)\mathrm{d}t}{1-G_0M_0} + p(t).$$

Hence the sequence (f_n) is integrable bounded and thus $f \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3) and (3.4), we deduce that x is a solution of (1.1). Finally, passing to the limit in (3.6), we obtained the desired estimate for x.

It remains to construct the sequences (x_n) and (f_n) with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \ge 1$ we have already constructed $x_n \in C(I,\mathbf{R})$ and $f_n \in L^1(I,\mathbf{R})$, $n = 1,2,\dots N$, satisfying (3.3), (3.5) for $n = 1,2,\dots N$, and (3.4) for $n = 1,2,\dots N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t)-x_{N-1}(t)| + \int_0^t L(s)|x_N(s)-x_{N-1}(s)|\mathrm{d}s)$ is measurable. Since $F(t,\cdot)$ is a Lipschitz function, we have, for almost all $t \in I$, that the intersection of $F(t, x_N(t))$ with

$$\left\{ f_N(t) + L(t)\left(|x_N(t)-x_{N-1}(t)| + \int_0^t L(s)|x_N(s)-x_{N-1}(s)|\mathrm{d}s\right)[-1,1]\right\}$$

is not empty. Theorem 1.14.1 in [1] yields that there exists a measurable selection f_{N+1} of $F(\cdot, x_N(\cdot), V(x_N)(\cdot))$ such that for almost $t \in I$

$$|f_{N+1}(t)-f_N(t)| \le L(t)(|x_N(t)-x_{N-1}(t)| + \int_0^t L(s)|x_N(s)-x_{N-1}(s)|\mathrm{d}s).$$

We define x_{N+1} as in (3.3) with $n = N+1$. Thus f_{N+1} satisfies (3.4) and (3.5) and the proof is finished. □

If in Theorem 3.2, we have that $y(\cdot) = 0$, we obtain the following consequence of Theorem 3.2.

COROLLARY 3.3. *Assume that Hypothesis* 3.1 *is satisfied and that there exists* $p \in L^1(I, \mathbf{R}_+)$ *such that* $d(0, F(t, 0, V(0)(t))) \le p(t)$ *a.e. on* I *and* $G_0 M_0 < 1$. *Then there exists a solution* x *of problem* (1.1) *satisfying for all* $t \in I$

$$|x(t)| \le \frac{G_0}{1 - G_0 M_0} \int_0^1 p(t) \mathrm{d}t.$$

If F does not depend on the last variable, problem (1.1) reduces to

$$-x'' \in F(t, x) \quad a.e.\ [0,1], \quad x(0) = 0, \quad x(1) = \int_0^1 h(t) x(t) \mathrm{d}t. \tag{3.7}$$

HYPOTHESIS 3.4. i) $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and, for every* $x \in \mathbf{R}$, $F(\cdot, x)$ *is measurable.*

ii) *There exists* $L \in L^1(I, \mathbf{R}_+)$ *such that, for almost all* $t \in I, F(t, \cdot)$ *is* $L(t)$*-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t, x), F(t, y)) \le L(t)|x - y|, \forall \ x, y \in \mathbf{R}.$$

iii) $d(0, F(t, 0)) \le L(t)$ *a.e.* I.

In this case, Corollary 3.3 has the following statement.

PROPOSITION 3.5. *Assume that Hypothesis* 3.4 *is satisfied and* $G_0 L_0 < 1$. *Then there exists a solution* x *of problem* (3.7) *satisfying for all* $t \in I$

$$|x(t)| \le \frac{G_0 L_0}{1 - G_0 L_0}. \tag{3.8}$$

REMARK 3.6. In particular, if F is single-valued, the above Proposition 3.5 yields Theorem 3.3 in [2]. We note that the approach in [2] does not provide a priori bounds as in (3.8).

We are concerned next with problem (1.2).

HYPOTHESIS 3.7. *Hypothesis* 3.1 *is satisfied, there exists* $\delta > 0$ *such that* $a(t) \ge \delta$ *for any* $t \in (0, 1]$, *and* $M_0 = \int_0^1 M(t) \mathrm{d}t < \frac{2\delta}{1 + \delta - e^{-\delta}}$.

The proof of the next theorem is similar to the proof of Theorem 3.2.

THEOREM 3.8. *Assume that Hypothesis* 3.7 *is satisfied and that* $K M_0 < 1$. *Let* $y \in AC^1(I, \mathbf{R})$ *be such that* $y'(0) = 0$, $y(1) = 0$ *and such that there exists* $q \in L^1(I, \mathbf{R}_+)$ *with* $d(-y''(t) - a(t) y'(t), F(t, y(t), V(y)(t))) \le q(t)$ *a.e. on* I.

Then there exists a solution $x \in C(I, \mathbf{R})$ *of problem* (1.2) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{K}{1 - K M_0} \int_0^1 q(t) \mathrm{d}t.$$

If in Theorem 3.2 the function $y(\cdot) = 0$, we obtain the following consequence of Theorem 3.2.

Corollary 3.9. *Assume that Hypothesis* 3.7 *is satisfied, that there exists* $q \in L^1(I, \mathbf{R}_+)$ *with* $d(0, F(t, 0, V(0)(t))) \leq q(t)$ *a.e. on* I*, and that* $KM_0 < 1$. *Then there exists a solution* x *of problem* (1.1) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{K}{1 - KM_0} \int_0^1 q(t) \mathrm{d}t.$$

If F does not depend on the last variable, problem (1.2) reduces to

$$-x'' - a(t)x' \in F(t, x) \quad a.e.\ [0,1], \quad x'(0) = 0, \quad x(1) = 0. \tag{3.9}$$

Hypothesis 3.10. *Hypothesis* 3.4 *is satisfied, there exists* $\delta > 0$ *such that* $a(t) \geq \delta$ *for any* $t \in (0, 1]$*, and* $L_0 = \int_0^1 L(t) \mathrm{d}t < \frac{2\delta}{1+\delta-e^{-\delta}}$.

In this case, Corollary 3.9 becomes the next proposition.

Proposition 3.11. *Assume that Hypothesis* 3.10 *is satisfied and* $KL_0 < 1$. *Then there exists a solution* x *of problem* (3.9) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{KL_0}{1 - KL_0}. \tag{3.10}$$

Remark 3.12. A similar result to Proposition 3.11 is obtained in Theorem 3.1 of [4] using the Covitz-Nadler contraction principle for set-valued maps. We note that the approach in [4] does not provides a priori bounds as in (3.10).

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