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THE INDECOMPOSABLE PREPROJECTIVE AND PREINJECTIVE REPRESENTATIONS OF THE QUIVER $\widetilde{\mathbb{D}}_5$

ÁBEL LŐRINCZI and CSABA SZÁNTÓ

Abstract. Consider the quiver $\widetilde{\mathbb{D}}_5$ and its finite dimensional representations over the field k. We know due to Ringel in [7] that indecomposable representations without self extensions (called exceptional representations) can be exhibited using matrices involving as coefficients 0 and 1, such that the number of nonzero coefficients is precisely $d-1$, where d is the global dimension of the representation. This means that the corresponding "coefficient quiver" is a tree, so we will call such a presentation a "tree presentation". In this paper we describe explicit tree presentations for the indecomposable preprojective and preinjective representations of the quiver $\widetilde{\mathbb{D}}_5$. In this way we extend some results obtained by Mróz in [5] involving the $\widetilde{\mathbb{D}}_4$ case.

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1. PRELIMINARIES

Let $Q=(Q_0,Q_1)$ be a tame quiver without oriented cycles (i.e. of type $\widetilde{\mathbb{A}}_n,\widetilde{\mathbb{D}}_n,\widetilde{\mathbb{E}}_6,\widetilde{\mathbb{E}}_7,\widetilde{\mathbb{E}}_8$). Suppose that the vertex set Q_0 has n elements and for an arrow $\alpha\in Q_1$ we denote by $t(\alpha),h(\alpha)\in Q_0$ the tail and head of α. The Euler form of Q is a bilinear form on $\mathbb{Z}Q_0\cong\mathbb{Z}^n$ given by $\langle x,y\rangle=\sum_{i\in Q_0}x_iy_i-\sum_{\alpha\in Q_1}x_{t(\alpha)}y_{h(\alpha)}$. Its quadratic form q_Q (called Tits form) is independent from the orientation of Q and in the tame case it is positive semidefinite with radical $\mathbb{Z}\delta$, where δ is a minimal positive imaginary root of the corresponding Kac-Moody root system. The defect of $x\in\mathbb{Z}Q_0$ is then $\partial x=\langle\delta,x\rangle$.

Let k be a field and consider the path algebra kQ. The category mod-kQ of finite dimensional right modules over kQ will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver Q. Recall that a k-representation of Q is defined as a set of finite dimensional k-spaces $\{M_i|i\in Q_0\}$ corresponding to the vertices together with k-linear maps $M_\alpha: M_{t(\alpha)}\to M_{h(\alpha)}$ corresponding to the arrows. Given two representations $M=(M_i,M_\alpha)$ and $N=(N_i,N_\alpha)$ of the quiver Q, a morphism $f:M\to N$ between them consists of a family of k-linear maps (corresponding to the vertices) $f_i:M_i\to N_i$, such that $N_\alpha f_{t(\alpha)}=f_{h(\alpha)}M_\alpha$ for all $\alpha\in Q_1$.

The dimension vector of a representation $M=(M_i,M_\alpha)$ is

$$\underline{\dim}M=(d_i)_{i\in Q_0}\in\mathbb{Z}^n,\ \text{where } d_i=\dim_k M_i.$$

The global dimension of M is $d=\sum_{i\in Q_0}d_i$. We will denote by $\partial M=\partial(\underline{\dim}M)$ the defect of M.

Following Ringel in [7] a basis $B = (B_i)$ of a representation $M = (M_i, M_\alpha)$ consists of a fixed basis B_i for each space M_i, where $i \in Q_0$. Let us assume that such a basis B of M is given. For any arrow α, we may replace the linear application M_α by the corresponding $d_{h(\alpha)} \times d_{t(\alpha)}$ matrix $M_{\alpha,B}$. Given $b \in B_{t(\alpha)}$ and $b' \in B_{h(\alpha)}$ we denote by $M_{\alpha,B}(b, b')$ the corresponding matrix coefficient, so $M_\alpha(b) = \sum_{b' \in B_{h(\alpha)}} M_{\alpha,B}(b, b')b'$. By definition, the coefficient quiver of M with respect to B has the set of vertices the disjoint union of all the bases B_i, and there is an arrow (α, b, b') if $M_{\alpha,B}(b, b') \neq 0$. We will call an indecomposable representation M of Q a tree module, provided there exists a basis B of M such that the corresponding coefficient quiver is a tree. Note that for a tree module M of global dimension d, there is a basis B of M such that precisely $d-1$ matrix coefficients are non-zero, and one may assume that all these coefficients are equal to 1 (see [7] for details). Thus, any tree module can be exhibited by (0,1)-matrices such that the number of 1-s is precisely $d-1$. Such a presentation is called a tree presentation.

An indecomposable module M is called exceptional if it has no self extensions (i.e. $\mathrm{Ext}^1(M, M) = 0$).

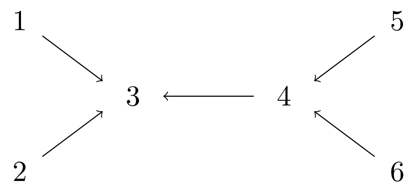
It is well-known that in the tame cases the indecomposable modules in mod-kQ are of three types: preprojectives (having negative defect), preinjectives (having positive defect) and regulars (having zero defect). For all the details we refer to [2],[3],[1],[4]. It is important to notice that indecomposable preprojectives and preinjectives are exceptional.

Having in mind all the notions above we are able now to formulate the main problem on which this article focuses.

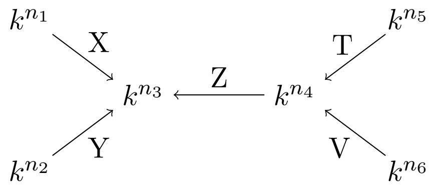
In [7] Ringel proves that any exceptional representation of Q over a field k is a tree module, so it has a tree presentation. However in many cases these presentations are not known explicitly.

The aim of this article is to describe explicitly these tree presentations in case of preprojective and preinjective indecomposable representations over the particular tame quiver $\widetilde{\mathbb{D}}_5$. In this way we extend similar results by A. Mróz obtained in the case $\widetilde{\mathbb{D}}_4$, with the four subspace orientation (see [5] and [6]).

We remark that it is enough to consider a specific orientation for $\widetilde{\mathbb{D}}_5$, since using reflection functors (see [3]) one can generalize the results for all the orientations. So we will consider the following oriented quiver of $\widetilde{\mathbb{D}}_5$ type (the numbers refer to the vertex numbering):



Using the quiver above, a representation M (up to isomorphism) will take the form



Here the matrices X, Y, Z, T, V correspond to the linear applications associated to the arrows (relative to the canonical bases).

Note that in this way our representation can be identifyed with the quintuple of matrices

$$(X, Y, Z, T, V) \in \mathbb{M}_{n_3\times n_1} \times \mathbb{M}_{n_3\times n_2} \times \mathbb{M}_{n_3\times n_4} \times \mathbb{M}_{n_4\times n_5} \times \mathbb{M}_{n_4\times n_6}.$$

2. TREE PRESENTATIONS FOR THE INDECOMPOSABLE PREPROJECTIVES AND PREINJECTIVES

For $n, m \geq 0$ we denote by $0_{n\times m} \in \mathbb{M}_{n\times m}$ and $0_n \in \mathbb{M}_{n\times n}$ the zero matrix and by $I_n \in \mathbb{M}_{n\times n}$ the identity matrix. In addition we introduce the following notations (taken from [6]):

$$^{\circ}\Pi_{n,m} = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{M}_{n\times m}, \quad \Pi^{\circ}_{n,m} = \begin{bmatrix} 0 & \cdots & 0 & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \in \mathbb{M}_{n\times m},$$

$$\overline{I}_n = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{M}_{n\times n}, \quad \Sigma_{n,n+1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 \; 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \in \mathbb{M}_{n\times(n+1)}$$

It is well known that the indecomposable preprojective (preinjective) representations of our quiver are up to isomorphism $P(n,i) := \tau^{-n}P(i)$ (respectively $I(n,i) := \tau^{n}I(i)$), where $P(i)$ is the projective (respectively $I(i)$ is the injective) indecomposable representation corresponding to the vertex i and τ is the Auslander-Reiten translation.

In what follows we exhibit tree presentations for the indecomposables $P(n,i)$ and $I(n,i)$:

$P(6n,1)$,	$[2n+1, 2n, 4n+1, 4n, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ ^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} ^{\circ}\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^{\circ}_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$
$P(6n,2)$,	$[2n, 2n+1, 4n+1, 4n, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ ^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} ^{\circ}\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^{\circ}_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$

$P(6n,5)$,	$[2n, 2n, 4n+1, 4n+1, 2n+1, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix} \right)$
$P(6n,6)$,	$[2n, 2n, 4n+1, 4n+1, 2n, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix} \right)$
$P(6n+1,1)$,	$[2n, 2n+1, 4n+1, 4n+1, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix} \right)$
$P(6n+1,2)$,	$[2n+1, 2n, 4n+1, 4n+1, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \end{bmatrix} \right)$
$P(6n+1,5)$,	$[2n+1, 2n+1, 4n+2, 4n+1, 2n, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+1} \\ \Pi^{\circ}_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix} \right)$
$P(6n+1,6)$,	$[2n+1, 2n+1, 4n+2, 4n+1, 2n+1, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+1} \\ \Pi^{\circ}_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix} \right)$
$P(6n+2,1)$,	$[2n+1, 2n, 4n+2, 4n+2, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$P(6n+2,2)$,	$[2n, 2n+1, 4n+2, 4n+2, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$P(6n+2,5)$,	$[2n+1, 2n+1, 4n+2, 4n+2, 2n+1, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix} \right)$
$P(6n+2,6)$,	$[2n+1, 2n+1, 4n+2, 4n+2, 2n, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+2}], \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ I_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$

$P(6n+3,1)$, $n\geq 0$	$[2n+1,2n+2,4n+3,4n+2,2n+1,2n+1]$, $\left(\begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ {}^{\circ}\Pi_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+2}\\ 0_{1,4n+2}\\ \Pi^{\circ}_{2n+1,4n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{2n+1}\\ I_{2n+1}\end{bmatrix}\right)$
$P(6n+3,2)$, $n\geq 0$	$[2n+2,2n+1,4n+3,4n+2,2n+1,2n+1]$, $\left(\begin{bmatrix} I_{2n+2}\\ {}^{\circ}\Pi_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+2}\\ 0_{1,4n+2}\\ \Pi^{\circ}_{2n+1,4n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{2n+1}\\ I_{2n+1}\end{bmatrix}\right)$
$P(6n+3,5)$, $n\geq 0$	$[2n+1,2n+1,4n+3,4n+3,2n+1,2n+2]$, $\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}\right)$
$P(6n+3,6)$, $n\geq 0$	$[2n+1,2n+1,4n+3,4n+3,2n+2,2n+1]$, $\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$
$P(6n+4,1)$, $n\geq 0$	$[2n+2,2n+1,4n+3,4n+3,2n+1,2n+1]$, $\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$
$P(6n+4,2)$, $n\geq 0$	$[2n+1,2n+2,4n+3,4n+3,2n+1,2n+1]$, $\left(\begin{bmatrix} 0_{2n+2,2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+1,2n+2}\end{bmatrix}, [I_{4n+3}], \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ 0_{1,2n+1}\\ I_{2n+1}\end{bmatrix}\right)$
$P(6n+4,5)$, $n\geq 0$	$[2n+2,2n+2,4n+4,4n+3,2n+2,2n+1]$, $\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+2,4n+3}\\ \Pi^{\circ}_{2n+2,4n+3}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ {}^{\circ}\Pi_{2n+1,2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}\right)$
$P(6n+4,6)$, $n\geq 0$	$[2n+2,2n+2,4n+4,4n+3,2n+1,2n+2]$, $\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+2,4n+3}\\ \Pi^{\circ}_{2n+2,4n+3}\end{bmatrix}, \begin{bmatrix} I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix} \begin{bmatrix} I_{2n+2}\\ {}^{\circ}\Pi_{2n+1,2n+2}\end{bmatrix}\right)$
$P(6n+5,1)$, $n\geq 0$	$[2n+1,2n+2,4n+4,4n+4,2n+2,2n+2]$, $\left(\begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, [I_{4n+4}], \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}\right)$
$P(6n+5,2)$, $n\geq 0$	$[2n+2,2n+1,4n+4,4n+4,2n+2,2n+2]$, $\left(\begin{bmatrix} I_{2n+2}\\ 0_{2n+2}\end{bmatrix}, \begin{bmatrix} 0_{1,2n+1}\\ I_{2n+1}\\ I_{2n+1}\\ 0_{1,2n+1}\end{bmatrix}, [I_{4n+4}], \begin{bmatrix} 0_{2n+2}\\ I_{2n+2}\end{bmatrix}, \begin{bmatrix} I_{2n+2}\\ I_{2n+2}\end{bmatrix}\right)$

$P(6n+5,5)$,	$[2n+2, 2n+2, 4n+4, 4n+4, 2n+1, 2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}, [I_{4n+4}], \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ I_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ I_{2n+2} \end{bmatrix} \right)$
$P(6n+5,6)$,	$[2n+2, 2n+2, 4n+4, 4n+4, 2n+2, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}, [I_{4n+4}], \begin{bmatrix} I_{2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ I_{2n+1} \\ 0_{1,2n+1} \end{bmatrix} \right)$
$P(3n,3)$,	$[2n, 2n, 4n+1, 4n, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,4n} \\ 0_{1,4n} \\ \Pi^{\circ}_{2n,4n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$
$P(3n,4)$,	$[2n, 2n, 4n+1, 4n+1, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n} \\ 0_{2n+1,2n} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, [I_{4n+1}], \begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix} \right)$
$P(3n+1,3)$,	$[2n+1, 2n+1, 4n+2, 4n+1, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+1} \\ \Pi^{\circ}_{2n+1,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ \overline{I}_{2n} \\ 0_{1,2n} \end{bmatrix}, \begin{bmatrix} 0_{1,2n} \\ I_{2n} \\ \overline{I}_{2n} \end{bmatrix} \right)$
$P(3n+1,4)$,	$[2n+1, 2n+1, 4n+3, 4n+2, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,4n+2} \\ 0_{1,4n+2} \\ \Pi^{\circ}_{2n+1,4n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$P(3n+2,3)$,	$[2n+1, 2n+1, 4n+3, 4n+3, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+2,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, [I_{4n+3}], \begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix} \right)$
$P(3n+2,4)$,	$[2n+2, 2n+2, 4n+4, 4n+3, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+2,4n+3} \\ \Pi^{\circ}_{2n+2,4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \overline{I}_{2n+1} \\ 0_{1,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,2n+1} \\ I_{2n+1} \\ \overline{I}_{2n+1} \end{bmatrix} \right)$
$I(6n,1)$,	$[2n+1, 2n, 4n, 4n, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, [I_{4n}], \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix} \right)$
$I(6n,2)$,	$[2n, 2n+1, 4n, 4n, 2n, 2n]$,
$n \geq 0$	$\left(\begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, [I_{4n}], \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix} \right)$

$I(6n,5)$, $n \geq 0$	$[2n, 2n, 4n, 4n, 2n+1, 2n]$, $\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix} \right)$
$I(6n,6)$, $n \geq 0$	$[2n, 2n, 4n, 4n, 2n, 2n+1]$, $\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix} \right)$
$I(6n+1,1)$, $n \geq 0$	$[2n, 2n+1, 4n+1, 4n+1, 2n+1, 2n+1]$, $\left(\begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix} \right)$
$I(6n+1,2)$, $n \geq 0$	$[2n+1, 2n, 4n+1, 4n+1, 2n+1, 2n+1]$, $\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix} \right)$
$I(6n+1,5)$, $n \geq 0$	$[2n, 2n, 4n, 4n+1, 2n, 2n+1]$, $\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix} \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,4n+1} \\ \Pi^{\circ}_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$I(6n+1,6)$, $n \geq 0$	$[2n, 2n, 4n, 4n+1, 2n+1, 2n]$, $\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix} \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,4n+1} \\ \Pi^{\circ}_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix} \right)$
$I(6n+2,1)$, $n \geq 0$	$[2n+1, 2n, 4n+1, 4n+2, 2n+1, 2n+1]$, $\left(\begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+1,4n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$I(6n+2,2)$, $n \geq 0$	$[2n, 2n+1, 4n+1, 4n+2, 2n+1, 2n+1]$, $\left(\begin{bmatrix} I_{2n} \\ 0_{1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+1,4n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$I(6n+2,5)$, $n \geq 0$	$[2n+1, 2n+1, 4n+1, 4n+1, 2n+1, 2n]$, $\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix} \right)$
$I(6n+2,6)$, $n \geq 0$	$[2n+1, 2n+1, 4n+1, 4n+1, 2n, 2n+1]$, $\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ \Pi^{\circ}_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ {}^{\circ}\Pi_{2n,2n+1} \end{bmatrix} \right)$
$I(6n+3,1)$, $n \geq 0$	$[2n+1, 2n+2, 4n+2, 4n+2, 2n+1, 2n+1]$, $\left(\begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^{\circ}\Pi_{2n+1,2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \right)$
$I(6n+3,2)$, $n \geq 0$	$[2n+2, 2n+1, 4n+2, 4n+2, 2n+1, 2n+1]$, $\left(\begin{bmatrix} {}^{\circ}\Pi_{2n+1,2n+2} \\ \Pi^{\circ}_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \right)$

$I(6n+3,5)$,	$[2n+1,2n+1,4n+2,4n+2,2n+1,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}\right)$
$I(6n+3,6)$,	$[2n+1,2n+1,4n+2,4n+2,2n+2,2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+2} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}\right)$
$I(6n+4,1)$,	$[2n+2,2n+1,4n+3,4n+3,2n+2,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}\right)$
$I(6n+4,2)$,	$[2n+1,2n+2,4n+3,4n+3,2n+2,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}\right)$
$I(6n+4,5)$,	$[2n+1,2n+1,4n+2,4n+3,2n+2,2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+3} \\ \Pi^\circ_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}\right)$
$I(6n+4,6)$,	$[2n+1,2n+1,4n+2,4n+3,2n+1,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix} \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+3} \\ \Pi^\circ_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}\right)$
$I(6n+5,1)$,	$[2n+1,2n+2,4n+3,4n+4,2n+2,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+3,4n+4} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}\right)$
$I(6n+5,2)$,	$[2n+2,2n+1,4n+3,4n+4,2n+2,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{1,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+3,4n+4} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix}\right)$
$I(6n+5,5)$,	$[2n+2,2n+2,4n+3,4n+3,2n+1,2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}\right)$
$I(6n+5,6)$,	$[2n+2,2n+2,4n+3,4n+3,2n+2,2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ \Pi^\circ_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2,2n+1} \\ I_{2n+1} \end{bmatrix}\right)$
$I(3n,3)$,	$[2n+1,2n+1,4n+1,4n+1,2n+1,2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n,2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} I_{4n+1} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}\right)$
$I(3n,4)$,	$[2n,2n,4n,4n+1,2n+1,2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}, \begin{bmatrix} 0_{2n} \\ I_{2n} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n,4n+1} \\ \Pi^\circ_{2n,4n+1} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}\right)$

$I(3n+1,3)$,	$[2n+1, 2n+1, 4n+2, 4n+3, 2n+2, 2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix}, \begin{bmatrix} {}^\circ\Pi_{2n+1,4n+3} \\ \Pi^\circ_{2n+1,4n+3} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix} \right)$
$I(3n+1,4)$,	$[2n+1, 2n+1, 4n+1, 4n+2, 2n+1, 2n+1]$,
$n \geq 0$	$\left(\begin{bmatrix} \overline{I}_{2n+1} \\ {}^\circ\Pi_{2n,2n+1} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n,2n+1} \\ \overline{I}_{2n+1} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+1,4n+2} \end{bmatrix}, \begin{bmatrix} I_{2n+1} \\ 0_{2n+1} \end{bmatrix}, \begin{bmatrix} 0_{2n+1} \\ I_{2n+1} \end{bmatrix} \right)$
$I(3n+2,3)$,	$[2n+2, 2n+2, 4n+3, 4n+4, 2n+2, 2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} \overline{I}_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix}, \begin{bmatrix} \Sigma_{4n+3,4n+4} \end{bmatrix}, \begin{bmatrix} I_{2n+2} \\ 0_{2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+2} \\ I_{2n+2} \end{bmatrix} \right)$
$I(3n+2,4)$,	$[2n+2, 2n+2, 4n+3, 4n+3, 2n+2, 2n+2]$,
$n \geq 0$	$\left(\begin{bmatrix} I_{2n+2} \\ 0_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} 0_{2n+1,2n+2} \\ I_{2n+2} \end{bmatrix}, \begin{bmatrix} I_{4n+3} \end{bmatrix}, \begin{bmatrix} \overline{I}_{2n+2} \\ {}^\circ\Pi_{2n+1,2n+2} \end{bmatrix}, \begin{bmatrix} \Pi^\circ_{2n+1,2n+2} \\ \overline{I}_{2n+2} \end{bmatrix} \right)$

3. THE PROOF

The presentations above were obtained more or less in an intuitive way (using the known presentations due to Mróz in the $\widetilde{\mathbb{D}}_4$ four subspace orientation case).

To check their correctness we need to check three things: the indecomposability, the defect and the fact that these presentations are indeed tree presentations.

Since for a preprojective (or preinjective) indecomposable M we have that $\dim_k \operatorname{End}(M) = 1$ and conversely $\dim_k \operatorname{End}(M) = 1$ implies indecomposability, we need to check first for each representation if its endomorphism ring is one dimensional.

We take for example the representation $P(6n+1,5)$ and observe that an endomorphism of this representation consists of a sextuple of square matrices (A, B, C, D, E, F) corresponding to the vertices 1, 2, 3, 4, 5, 6 and satisfying the matrix equations $CX = XA, CY = YB, CZ = ZD, DT = TE, DV = VF$. Here (X, Y, Z, T, V) is the quintuple of matrices associated to $P(6n+1,5)$.

Expanding the matrix equations we get

$$(1) \qquad \begin{bmatrix} c_{1,1} & \cdots & c_{1,2n+1} \\ \vdots & \ddots & \vdots \\ c_{2n+1,1} & \cdots & c_{2n+1,2n+1} \\ c_{2n+2,1} & \cdots & c_{2n+2,2n+1} \\ \vdots & \ddots & \vdots \\ c_{4n+2,1} & \cdots & c_{4n+2,2n+1} \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,2n+1} \\ \vdots & \ddots & \vdots \\ a_{2n+1,1} & \cdots & a_{2n+1,2n+1} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_{1,2n+2} & \cdots & c_{1,4n+2} \\ \vdots & \ddots & \vdots \\ c_{2n+1,2n+2} & \cdots & c_{2n+1,4n+2} \\ c_{2n+2,2n+2} & \cdots & c_{2n+2,4n+2} \\ \vdots & \ddots & \vdots \\ c_{4n+2,2n+2} & \cdots & c_{4n+2,4n+2} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ b_{1,1} & \cdots & b_{1,2n+1} \\ \vdots & \ddots & \vdots \\ b_{2n+1,1} & \cdots & b_{2n+1,2n+1} \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} c_{1,1} & \cdots & c_{1,2n+1}+c_{1,2n+2} & \cdots & c_{1,4n+2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{2n+1,1} & \cdots & c_{2n+1,2n+1}+c_{2n+1,2n+2} & \cdots & c_{2n+1,4n+2} \\ c_{2n+2,1} & \cdots & c_{2n+2,2n+1}+c_{2n+2,2n+2} & \cdots & c_{2n+2,4n+2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{4n+2,1} & \cdots & c_{4n+2,2n+1}+c_{4n+2,2n+2} & \cdots & c_{4n+2,4n+2} \end{bmatrix} = \begin{bmatrix} d_{1,1} & \cdots & d_{1,2n+1} & \cdots & d_{1,4n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{2n+1,1} & \cdots & d_{2n+1,2n+1} & \cdots & d_{2n+1,4n+1} \\ d_{2n+1,1} & \cdots & d_{2n+1,2n+1} & \cdots & d_{2n+1,4n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{4n+1,1} & \cdots & d_{4n+1,2n+1} & \cdots & d_{4n+1,4n+1} \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} d_{1,1}+d_{1,2n+1} & \cdots & d_{1,2n}+d_{1,4n} \\ \vdots & \ddots & \vdots \\ d_{2n,1}+d_{2n,2n+1} & \cdots & d_{2n,2n}+d_{2n,4n} \\ d_{2n+1,1}+d_{2n+1,2n+1} & \cdots & d_{2n+1,2n}+d_{2n+1,4n} \\ \vdots & \ddots & \vdots \\ d_{4n,1}+d_{4n,2n+1} & \cdots & d_{4n,2n}+d_{4n,4n} \\ d_{4n+1,1}+d_{4n+1,2n+1} & \cdots & d_{4n+1,2n}+d_{4n+1,4n} \end{bmatrix} = \begin{bmatrix} e_{1,1} & \cdots & e_{1,2n} \\ \vdots & \ddots & \vdots \\ e_{2n,1} & \cdots & e_{2n,2n} \\ e_{1,1} & \cdots & e_{1,2n} \\ \vdots & \ddots & \vdots \\ e_{2n,1} & \cdots & e_{2n,2n} \\ 0 & \cdots & 0 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} d_{1,1}+d_{1,2n+2} & \cdots & d_{1,2n}+d_{1,4n+1} & d_{1,2n+1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{2n+1,1}+d_{2n+1,2n+2} & \cdots & d_{2n+1,2n}+d_{2n+1,4n+1} & d_{2n+1,2n+1} \\ d_{2n+2,1}+d_{2n+2,2n+2} & \cdots & d_{2n+2,2n}+d_{2n+2,4n+1} & d_{2n+2,2n+1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{4n+1,1}+d_{4n+1,2n+2} & \cdots & d_{4n+1,2n}+d_{4n+1,4n+1} & d_{4n+1,2n+1} \end{bmatrix} = \begin{bmatrix} f_{1,1} & \cdots & f_{1,2n+1} \\ \vdots & \ddots & \vdots \\ f_{2n+1,1} & \cdots & f_{2n+1,2n+1} \\ f_{1,1} & \cdots & f_{1,2n+1} \\ \vdots & \ddots & \vdots \\ f_{2n,1} & \cdots & f_{2n,2n+1} \end{bmatrix} \tag{5}$$

We introduce the following notations to make the deduction easier:

$D = (d_{i,j}) \in \mathbb{M}_{(4n+1)\times(4n+1)}$,

$D = \begin{bmatrix} D_1 \\ \vdots \\ D_{4n+1} \end{bmatrix}$, where $D_i \in \mathbb{M}_{1\times(4n+1)}$, for all $i \in \{1, \ldots, 4n+1\}$,

$d_i^{00} = (0, \ldots, 0, d_{i,i}, 0, \ldots, 0) \in \mathbb{M}_{1\times(4n+1)}$, for all $i \in \{1, \ldots, 4n+1\}$,

$d_i^{01} = \begin{cases} (0, \ldots, 0, d_{i,i}, d_{i,i+1}, \ldots, d_{i,2n}, 0, \ldots, 0) \in \mathbb{M}_{1\times(4n+1)} & \text{if } i \leq 2n \\ (0, \ldots, 0, d_{i,i}, \ldots, d_{i,4n+1}) \in \mathbb{M}_{1\times(4n+1)} & \text{if } i \geq 2n+1 \end{cases}.$

Using all the (labelled) equations above we can deduce the following:

$$\begin{cases} (1),(3) \Rightarrow \begin{bmatrix} d_{2n+1,1} & \dots & d_{2n+1,2n} \\ \vdots & \ddots & \vdots \\ d_{4n+1,1} & \dots & d_{4n+1,2n} \end{bmatrix} = 0_{2n+1,2n} \\ (2),(3) \Rightarrow \begin{bmatrix} d_{1,2n+2} & \dots & d_{1,4n+1} \\ \vdots & \ddots & \vdots \\ d_{2n+1,2n+2} & \dots & d_{2n+1,4n+1} \end{bmatrix} = 0_{2n+1,2n} \\ (1),(2),(3) \Rightarrow D_{2n+1} = d^{00}_{2n+1} \end{cases}$$

1.

$$\begin{cases} (4)(4n+1) \Rightarrow D_{4n+1} = d^{00}_{4n+1} \\ (5)(2n,4n+1) \Rightarrow D_{2n} = d^{00}_{2n} \end{cases}$$

$2n+1.$

$$\begin{cases} (4)(1,2n+1) \Rightarrow D_1 = d^{00}_1 \\ (5)(1,2n+2) \Rightarrow D_{2n+2} = d^{00}_{2n+2} \end{cases}$$

2.

$$\begin{cases} (4)(2n,4n) \Rightarrow D_{4n} = d^{01}_{4n} \\ (5)(2n-1,4n) \Rightarrow D_{2n-1} = d^{01}_{2n-1} \end{cases}$$

$2n+2.$

$$\begin{cases} (4)(2,2n+2) \Rightarrow D_2 = d^{00}_2 \\ (5)(2,2n+3) \Rightarrow D_{2n+3} = d^{00}_{2n+3} \end{cases}$$

3.

$$\begin{cases} (4)(2n-1,4n-1) \Rightarrow D_{4n-1} = d^{01}_{4n-1} \\ (5)(2n-2,4n-1) \Rightarrow D_{2n-2} = d^{01}_{2n-2} \end{cases}$$

$2n+3.$

$$\begin{cases} (4)(3,2n+3) \Rightarrow D_3 = d^{00}_3 \\ (5)(3,2n+4) \Rightarrow D_{2n+4} = d^{00}_{2n+4} \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

$2n.$

$$\begin{cases} (4)(2,2n+2) \Rightarrow D_{2n+2} = d^{01}_{2n+2} \\ (5)(1,2n+2) \Rightarrow D_1 = d^{01}_1 \end{cases}$$

$4n-1.$

$$\begin{cases} (4)(2n-1,4n-1) \Rightarrow D_{2n} = d^{00}_{2n-1} \\ (5)(2n-1,4n) \Rightarrow D_{4n} = d^{00}_{4n} \end{cases}$$

It follows that $D_i = d^{00}_i$ for all $i \in \{1,\dots,4n+1\}$, i.e. the D matrix is diagonal.

1.

$$\begin{cases} (4)(1,2n+1) \Rightarrow d_{2n+1,2n+1} = d_{1,1} = e_{1,1} \\ (5)(1,2n+2) \Rightarrow f_{1,1} = d_{1,1} = d_{2n+2,2n+2} \end{cases}$$

2.

$$\begin{cases}(4)(2,2n+2)\Rightarrow d_{2n+2,2n+2}=d_{2,2}=e_{2,2}\\(5)(2,2n+3)\Rightarrow f_{2,2}=d_{2,2}=d_{2n+3,2n+3}\end{cases}$$

3.

$$\begin{cases}(4)(3,2n+3)\Rightarrow d_{2n+3,2n+3}=d_{3,3}=e_{3,3}\\(5)(3,2n+4)\Rightarrow f_{3,3}=d_{3,3}=d_{2n+4,2n+4}\end{cases}$$

$$\vdots$$

$2n$.

$$\begin{cases}(4)(2n,4n)\Rightarrow d_{4n,4n}=d_{2n,2n}=e_{2n,2n}\\(5)(2n,4n+1)\Rightarrow f_{2n,2n}=d_{2n,2n}=d_{4n+1,4n+1}\end{cases}$$

From this, we get that $d_{i,i}=e_{i,i}=f_{i,i}=\lambda\in k$ for all $i\in\{1,\dots,4n+1\}$.

Now, from (1), (2) and (3) we conclude that the solutions are: $a_{i,j}=b_{i,j}=c_{i,j}=d_{i,j}=e_{i,j}=f_{i,j}=\lambda\delta_{ij}$, where δ_{ij} is the Kronecker delta, and $\lambda\in k$, which means that $\dim_k \operatorname{End} P(6n+1,5)=1$.

The indecomposability check for all the other representations can be performed in the same way as above.

The defect of the given representations can be easily calculated using the Euler form.

Finally, we need to check the fact that the given presentation is a tree presentation. We know due to Ringel in [7] that if M is indecomposable, then its coefficient tree associated to any basis is connected. Since we can easily check that the number of 1's in each presentation is the global dimension minus one, in our connected coefficient quiver the number of edges equals the number of vertices minus one. But this means that the coefficient quiver is a tree.

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Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: 92abel@gmail.com
E-mail: szanto.cs@gmail.com