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STARLIKENESS OF AN INTEGRAL TRANSFORM

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Abstract. The main objective of this paper is to present a differential inequality implying starlikeness of order β and as a consequence, to obtain conditions on the kernel function g such that the function defined by

$$f(z) = \int_0^1 \int_0^1 g(r, s, z)\mathrm{d}r\mathrm{d}s$$

is a starlike function of the same order.
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1. INTRODUCTION

Let $\mathcal{H}$ denotes the class of all analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$. For a positive integer n and $a \in \mathcal{C}$ define the classes of functions:

$$\begin{aligned} \mathcal{H}[a, n] &= \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}, \text{ and} \\ \mathcal{A}_n &= \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}, \end{aligned}$$

with $\mathcal{A}_1 = \mathcal{A}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions in E. A function f in $\mathcal{A}$ is said to be starlike of order β if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in E,$$

for some β $(0 \leq \beta < 1)$. We denote by $S^*(\beta)$, the subclass of S consisting of functions which are starlike of order β in E. Set $S^*(0) = S^*$. Also, a function f in $\mathcal{A}$ is said to be convex if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in E.$$

Let $f, g \in \mathcal{H}$ and let g be univalent in E. The function f is said to be subordinate to g (written $f(z) \prec g(z)$ or $f \prec g$) in E if $f(0) = g(0)$ and $f(E) \subset g(E)$.

In 2003, Fournier and Mocanu [1], investigated some differential inequalities in the unit disc E which imply starlikeness. In a recent paper, Miller and Mocanu [3] extended some of those results and also investigated starlikeness properties of functions f defined by double integral operators of the form

$$f(z) = \int_0^1 \int_0^1 W(r, s, z)\mathrm{d}r\mathrm{d}s.$$

In this paper, we propose a differential inequality which imply starlikeness of order β. As an application of this inequality, we construct a new starlike functions of order β which can be expressed in terms of double integrals of some functions in the class $\mathcal{H}$.

2. PRELIMINARY RESULTS

We shall need the following lemmas to prove our results.

LEMMA 2.1. ([2], p.71) *Let h be a convex function with $h(0) = a$ and let* $\mathrm{Re}(\gamma) > 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}\mathrm{d}t.$$

This result is sharp.

LEMMA 2.2. ([2], p.383) *Let n be a positive a integer and α real, with $0 \le \alpha < n$. Let $q \in \mathcal{H}$, with $q(0) = 0$, $q'(0) \neq 0$ and*

$$\mathrm{Re}\frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}. \tag{1}$$

If $p \in \mathcal{H}[0, n]$ satisfies

$$zp'(z) - \alpha p(z) \prec znq'(z) - \alpha q(z),$$

then $p(z) \prec q(z)$ and this result is sharp.

LEMMA 2.3. ([2], p.76) *Let h be a starlike function with $h(0) = 0$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = a + \frac{1}{n}\int_0^z \frac{h(t)}{t}\mathrm{d}t$$

and this result is sharp.

3. MAIN RESULT

THEOREM 3.1. *Let α and β be real numbers such that $0 \le \alpha < n+1$ and $0 \le \beta < 1$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)}, \quad z \in E, \tag{2}$$

then f is starlike of order β in E.

Proof. Rewriting inequality (2) in terms of subordination, we get

$$zf''(z)-\alpha\left(f'(z)-\frac{f(z)}{z}\right)\prec\frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)}z. \tag{3}$$

If we set

$$P(z)=f'(z)-\frac{f(z)}{z}=na_{n+1}z^n+(n+1)a_{n+2}z^{n+1}+\cdots,$$

then $P\in\mathcal{H}[0,n]$ and the subordination (3) becomes

$$(1-\alpha)P(z)+zP'(z)\prec\frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)}z. \tag{4}$$

In order to prove our result, we need to consider the following two cases:

Case I. When $0\le\alpha<1$, i.e. $0<1-\alpha\le 1$. Then, the differential subordination (4) can be written as

$$P(z)+\frac{zP'(z)}{1-\alpha}\prec\frac{n(n+1-\alpha)(1-\beta)}{(1-\alpha)(n+1-\beta)}z=h(z)\ (say).$$

It can be easily seen that h is convex and $h(0)=P(0)$. So, applying Lemma 2.1 (with $\gamma=1-\alpha$), we obtain

$$f'(z)-\frac{f(z)}{z}\prec\frac{n(1-\beta)}{(n+1-\beta)}z,\quad z\in E. \tag{5}$$

Case II. When $1\le\alpha<n+1$. In this case, differential subordination (4) can be written as

$$zP'(z)-(\alpha-1)P(z)\prec nzQ'(z)-(\alpha-1)Q(z), \tag{6}$$

where $Q(z)=\dfrac{n(1-\beta)}{(n+1-\beta)}z$, $Q(0)=0, Q'(0)\neq 0$ and satisfies in E

$$\operatorname{Re}\left(1+\frac{zQ''(z)}{Q'(z)}\right)>\frac{\alpha-1}{n},$$

since $\alpha<n+1$. So, in view of Lemma 2.2, the subordination (6) gives $P\prec Q$ in E or

$$f'(z)-\frac{f(z)}{z}\prec\frac{n(1-\beta)}{(n+1-\beta)}z,\quad z\in E. \tag{7}$$

Thus, in both the cases, we arrive at the same conclusion. Now, if we write

$$p(z)=\frac{f(z)}{z}=1+a_{n+1}z^n+a_{n+2}z^{n+1}+\cdots,$$

then, $p\in\mathcal{H}[1,n]$ and the subordination (7) becomes

$$zp'(z)\prec\frac{n(1-\beta)}{(n+1-\beta)}z=h_1(z)\ (say).$$

The function h_1 satisfies the conditions of Lemma 2.3. Thus, we obtain

$$p(z) \prec 1 + \frac{1}{n}\int_0^z \frac{n(1-\beta)}{(n+1-\beta)}\mathrm{d}t$$

or

$$\frac{f(z)}{z} \prec 1 + \frac{(1-\beta)}{(n+1-\beta)}z. \tag{8}$$

It follows from the subordination (7) that

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{n(1-\beta)}{(n+1-\beta)}, \quad z \in E,$$

while from the subordination (8), we have

$$\left|\frac{f(z)}{z}\right| > \frac{n}{n+1-\beta}, \quad z \in E.$$

Combining the above two inequalities, we get

$$\frac{n}{n+1-\beta}\left|\frac{zf'(z)}{f(z)} - 1\right| < \left|\frac{f(z)}{z}\right|\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|f'(z) - \frac{f(z)}{z}\right| < \frac{n(1-\beta)}{(n+1-\beta)},$$

which implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-\beta).$$

This proves that f is starlike of order β in E. □

Letting $\beta = 0$ in Theorem 3.1, we obtain the following result of Miller and Mocanu [3].

COROLLARY 3.1. *Let $f \in \mathcal{A}_n$ and $0 \le \alpha < n+1$. If*

$$\left|zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(n+1-\alpha)}{(n+1)},$$

then $f \in S^*$.

For $\alpha = \beta = 0$ and $n = 1$, Theorem 3.1 reduces to the following result of Obradovic [4]:

COROLLARY 3.2. *Let $f \in \mathcal{A}$ be such that $|zf''(z)| < 1$ in E. Then $f \in S^*$.*

4. APPLICATION

As an application of Theorem 3.1, we prove the starlikeness of an integral operator in the following result.

THEOREM 4.1. *Let $g \in \mathcal{H}$ satisfy $|g(z)| \le \frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)}$ in E for some $0 \le \alpha < n+1$ and $0 \le \beta < 1$. Then, the function f given by*

$$f(z) = z + z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-\alpha}s^{n-1}\mathrm{d}r\mathrm{d}s \tag{9}$$

is starlike of order β in E.

Proof. Let $f \in \mathcal{A}_n$ satisfy the differential equation

$$zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) = z^n g(z). \tag{10}$$

Clearly,

$$\left|zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)}, \quad z \in E.$$

Thus, from the Theorem 3.1, we see that the solution f of the differential equation (10) must be starlike of order β.

Setting $\phi(z) = f'(z) - \frac{f(z)}{z} \in \mathcal{H}[0, n]$ in the differential equation (10), we obtain

$$z\phi'(z) + (1-\alpha)\phi(z) = z^n g(z).$$

Solving this equation, we get

$$\phi(z) = z^{-1+\alpha}\int_0^z \zeta^{n-\alpha} g(\zeta)\mathrm{d}\zeta = z^n \int_0^1 r^{n-\alpha} g(rz)\mathrm{d}r.$$

Since $\phi(z) = f'(z) - \frac{f(z)}{z}$, we have

$$f'(z) - \frac{f(z)}{z} = z^n \int_0^1 r^{n-\alpha} g(rz)\mathrm{d}r$$

or

$$\left(\frac{f(z)}{z}\right)' = z^{n-1}\int_0^1 r^{n-\alpha} g(rz)\mathrm{d}r.$$

Integrating, we get

$$\frac{f(z)}{z} = 1 + \int_0^z \zeta^{n-1}\int_0^1 g(r\zeta) r^{n-\alpha}\mathrm{d}r\mathrm{d}\zeta.$$

Thus, putting $\zeta = sz$, we have

$$f(z) = z + z^{n+1}\int_0^1\int_0^1 g(rsz) r^{n-\alpha} s^{n-1}\mathrm{d}r\mathrm{d}s.$$

This completes the proof of the theorem. □

Taking various permissible values of α and n, we obtain several special cases of above result. However, we mention only one such result by taking $\alpha = 0$ and $n = 1$.

COROLLARY 4.1. *If $g \in \mathcal{H}$ and $|g(z)| < \frac{2(1-\beta)}{2-\beta}$ for $z \in E$, then for some β $(0 \le \beta < 1)$,*

$$f(z) = z + z^2\int_0^1\int_0^1 g(rsz) r\mathrm{d}r\mathrm{d}s \in S^*(\beta).$$

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