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DISSIPATIVE STURM-LIOUVILLE OPERATORS ON BOUNDED TIME SCALES

HÜSEYIN TUNA

Abstract. In this paper we consider a second-order Sturm–Liouville operator of the form

$$l(y) := -\left[p(t)\, y^{\Delta}(t)\right]^{\nabla} + q(t)\, y(t)$$

on bounded time scales. In this study, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, selfadjoint and other extensions of the dissipative Sturm-Liouville operators in terms of boundary conditions. Using by methods of Pavlov [28–30], we proved a theorem on completeness of the system of eigenvectors and associated vectors of the dissipative Sturm-Liouville operators on bounded time scales.

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1. INTRODUCTION

The study of dynamic equations on time scales is a new area of theoretical exploration in mathematics. The first fundamental results in this area were obtained by Hilger [19]. Time scale calculus unites the study of differential and difference equations. The study of time scales has led to several important applications, e.g., in the study of neural networks, heat transfer, and insect population models, epidemic models [1]. We refer the reader to consult the reference [7, 15, 16, 18, 23, 27] for some basic definitions.

The spectral analysis of non-selfadjoint (dissipative) operators is based on the ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. Using a functional model of a non-selfadjoint operator as a principal tool, spectral properties of such operators are investigated. The functional model of non-selfadjoint dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other disciplines. The construction of functional models for dissipative operators, natural analogues of spectral decompositions for selfadjoint operators is based on Sz. Nagy-Foias dilation theory [25] and Lax-Phillips scattering theory [24]. Pavlov's approach, [28-30], to the model construction of dissipative extensions of symmetric operators was followed by B. Allahverdiev in his works [2-6] and some others, and by the group of authors [12-14], where the theory of the dissipative Schrodinger operator on a finite interval was applied to the problems arising in the semiconductor physics. In

[8-11], Pavlov's functional model was extended to (general) dissipative operators which are finite dimensional extensions of a symmetric operator, and the corresponding dissipative and Lax-Phillips scattering problems were investigated in some detail.

The organization of this document is as follows: In Section 2, some time scale essentials are included for the convenience of the reader. In Section 3, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, selfadjoint and other extensions of the dissipative Sturm-Liouville operators in terms of boundary conditions. Later, we construct a selfadjoint dilation of this operator. We present its incoming and outcoming spectral representations which makes it possible to determine the scattering matrix of the dilation according to the Lax and Phillips scheme [24]. A functional model of this operator is constructed by methods of Pavlov [28-30] and its characteristic functions are defined. Finally, we prove a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators under consideration. While proving our results, we will use the machinery of [2-6].

2. PRELIMINARIES

In this section, first, we recall the essentials of time scales, and we refer to [7, 15, 16, 18, 23, 27] for more details.

Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, t \in \mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma : \mathbb{T} \to [0,\infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$. We introduce the sets $\mathbb{T}^k,\ \mathbb{T}_k,\ \mathbb{T}^*$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show (see [18])

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

Let $f:\mathbb{T}\rightarrow\mathbb{R}$ be a function, and $a,b\in\mathbb{T}$. If there exists a function $F:\mathbb{T}\rightarrow\mathbb{R}$, such that $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{k}$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_{a}^{b}f(t)\,\Delta t=F(b)-F(a)\ \text{ for } a,b\in\mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.

Let $L_{\Delta}^{2}(\mathbb{T}^{*})$ be the space of all functions defined on $\mathbb{T}^{*}$ such that

$$\|f\|:=\left(\int_{a}^{b}|f(t)|^{2}\,\Delta t\right)^{1/2}<\infty.$$

The space $L_{\Delta}^{2}(\mathbb{T}^{*})$ is a Hilbert space with the inner product (see [31])

$$(f,g):=\int_{a}^{b}f(t)\,\overline{g(t)}\Delta t,\quad f,g\in L_{\Delta}^{2}(\mathbb{T}^{*})\ .$$

Let $a\leq b$ be fixed points in $\mathbb{T}$ and $a\in\mathbb{T}_{k},b\in\mathbb{T}^{k}$. We will consider the Sturm-Liouville equation

$$l(y):=-\left[p(t)\,y^{\Delta}(t)\right]^{\nabla}+q(t)\,y(t)\,,\ t\in[a,b], \tag{1}$$

where $q:\mathbb{T}\rightarrow\mathbb{C}$ is continuous function, $p:\mathbb{T}\rightarrow\mathbb{C}$ is ∇-differentiable on $\mathbb{T}^{k}$, $p(t)\neq0$ for all $t\in\mathbb{T}$, and $p^{\nabla}:\mathbb{T}_{k}\rightarrow\mathbb{C}$ is continuous. The Wronskian of y,z is defined by (see [18])

$$W(y,z)(t):=p(t)\left[y(t)\,z^{\Delta}(t)-y^{\Delta}(t)\,z(t)\right],\quad t\in\mathbb{T}^{*}.$$

Let L_{0} denote the closure of the minimal operator generated by (1) and by D_{0} its domain. Besides, we denote by D the set of all functions $y(t)$ from $L_{\Delta}^{2}(\mathbb{T}^{*})$ such that $l(y)\in L_{\Delta}^{2}(\mathbb{T}^{*})$; D is the domain of the maximal operator L. Furthermore $L=L_{0}^{*}$ (see [26]). Suppose that the operator L_{0} has defect index $(2,2)$.

For every $y,z\in D$ we have Lagrange's identity (see [18])

$$(Ly,z)-(y,Lz)=[y,\overline{z}](b)-[y,\overline{z}](a)\,,$$

where $[y,z]:=p(t)\left[y(t)\,z^{\Delta}(t)-y^{\Delta}(t)\,z(t)\right]$.

Let us define by Γ_{1},Γ_{2} the linear maps from D to $\mathbb{C}^{2}$ by the formula

$$\Gamma_{1}y=\begin{pmatrix}-y(a)\\ y(b)\end{pmatrix},\ \Gamma_{2}y=\begin{pmatrix}p(t)\,y^{\Delta}(a)\\ p(t)\,y^{\Delta}(b)\end{pmatrix},\ y\in D. \tag{2}$$

For any $y,z\in D$, we have

$$\begin{aligned}(Ly,z)-(y,Lz)&=[y,\overline{z}](b)-[y,\overline{z}](a)\\ &=(\Gamma_{1}y,\Gamma_{2}z)_{\mathbb{C}^{2}}-(\Gamma_{2}y,\Gamma_{1}z)_{\mathbb{C}^{2}}\,.\end{aligned} \tag{3}$$

3. MAIN RESULTS

THEOREM 1. *The triple* $\left(\mathbb{C}^2, \Gamma_1, \Gamma_2\right)$ *defined by* (2) *is a boundary spaces of the operator* L_0.

Proof. The proof is obtained from the definition of boundary value space and (3). □

Recall that a linear operator T (with dense domain $D(T)$) acting on some Hilbert space H is called dissipative (accretive) if $\mathrm{Im}\,(Tf, f) \geq 0$ ($\mathrm{Im}\,(Tf, f) \leq 0$) for all $f \in D(T)$ and maximal dissipative (maximal accretive) if it does not have a proper dissipative (accretive) extension.

From [17-21], the following theorem is obtained.

THEOREM 2. *For any contraction* K *in* $\mathbb{C}^2$ *the restriction of the operator* L *to the set of functions* $y \in D$ *satisfying either*

$$(K - \mathrm{i})\,\Gamma_1 y + i\,(K + I)\,\Gamma_2 y = 0 \tag{4}$$

or

$$(K - \mathrm{i})\,\Gamma_1 y - \mathrm{i}\,(K + I)\,\Gamma_2 y = 0 \tag{5}$$

is respectively the maximal dissipative and accretive extension of the operator L_0. *Conversely, every maximal dissipative (accretive) extension of the operator* L_0 *is the restriction of* L *to the set of functions* $y \in D$ *satisfying* (4) *(resp.* (5)*), and the extension uniquely determines the contraction* K. *Conditions* (4) *(resp.* (5)*), in which* K *is an isometry describe the maximal symmetric extensions of* L_0 *in* $L^2_\Delta(\mathbb{T}^*)$. *If* K *is unitary, these conditions define selfadjoint extensions. In particular, the boundary conditions*

$$y(a) - h_1 p(t)\, y^\Delta(a) = 0 \tag{6}$$

$$y(b) - h_2\; p(t)\, y^\Delta(b) = 0 \tag{7}$$

with $\mathrm{Im}h_1 \geq 0$ *or* $h_1 = \infty$, $\mathrm{Im}h_2 \geq 0$ *or* $h_2 = \infty$, *describe the maximal dissipative (resp. selfadjoint) extensions of* L_0 *with separated boundary conditions.*

Throughout the rest of the paper, while proving our results, we will use the machinery and methods of [2-6].

We will study the maximal dissipative operators L_K generated by the conditions (6) - (7) and l. Let us add the "incoming" and "outgoing" subspaces $D_- = L^2(-\infty, 0)$ and $D_+ = L^2(0, \infty)$ to $H = L^2_\Delta(\mathbb{T}^*)$. The orthogonal sum $\mathcal{H}$=$D_- \oplus H \oplus D_+$ is called *main Hilbert space of the dilation*.

In the space $\mathcal{H}$, we consider the operator $\mathcal{L}$ on the set $D(\mathcal{L})$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathcal{L}\langle \varphi_-, y, \varphi_+ \rangle = \left\langle \mathrm{i}\frac{\mathrm{d}\varphi_-}{\mathrm{d}\xi}, l(y), \mathrm{i}\frac{\mathrm{d}\varphi_+}{\mathrm{d}\xi} \right\rangle \tag{8}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0)$, $\varphi_+ \in W_2^1(0, \infty)$, $y \in H$,

$$\begin{aligned} y(a) - h_1 p(t) y^\Delta(a) &= C\varphi_-(0), \ y(a) - \overline{h_1} p(t) y^\Delta(a) = C\varphi_+(0), \\ y(b) - h_2\, p(t) y^\Delta(b) &= 0, \end{aligned}$$

where W_2^1 are Sobolev spaces and $C^2 := 2\mathrm{Im}G$, $C > 0$.

THEOREM 3. *The operator $\mathcal{L}$ is selfadjoint in $\mathcal{H}$. Moreover, it is a selfadjoint dilation of the operator $\widetilde{L}\,(= L_K)$.*

Proof. We first prove that $\mathcal{L}$ is symmetric in $\mathcal{H}$. Namely

$$(\mathcal{L}f, g)_{\mathcal{H}} - (f, \mathcal{L}g)_{\mathcal{H}} = 0.$$

Let $f, g \in D(\mathcal{L})$, $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Then we have

$$\begin{aligned} (\mathcal{L}f, g)_{\mathcal{H}} \ &- \ (f, \mathcal{L}g)_{\mathcal{H}} \\ &= (\mathcal{L}\langle \varphi_-, y, \varphi_+ \rangle, \langle \psi_-, z, \psi_+ \rangle) - (\langle \varphi_-, y, \varphi_+ \rangle, \mathcal{L}\langle \psi_-, z, \psi_+ \rangle) \\ &= \int_{-\infty}^0 \mathrm{i}\varphi_-' \overline{\psi}_- \mathrm{d}\xi + (l(y), z)_H + \int_0^\infty \mathrm{i}\varphi_+' \overline{\psi}_+ \mathrm{d}\xi \\ &\quad - \int_{-\infty}^0 \mathrm{i}\psi_-' \overline{\varphi}_- \mathrm{d}\xi - (y, l(z))_H - \int_0^\infty \mathrm{i}\psi_+' \overline{\varphi}_+ \mathrm{d}\xi \\ &= \int_{-\infty}^0 \mathrm{i}\varphi_-' \overline{\psi}_- \mathrm{d}\xi + [y, z](b) + \int_0^\infty \mathrm{i}\varphi_+' \overline{\psi}_+ \mathrm{d}\xi \\ &\quad - \int_{-\infty}^0 \mathrm{i}\psi_-' \overline{\varphi}_- \mathrm{d}\xi - [y, z](a) - \int_0^\infty \mathrm{i}\psi_+' \overline{\varphi}_+ \mathrm{d}\xi \\ &= \mathrm{i}\psi_-(0)\, \overline{\varphi}_-(0) - \mathrm{i}\varphi_+(0)\, \overline{\psi}_+(0) + [y, z](b) - [y, z](a) \end{aligned}$$

We obtain by direct computation that

$$\mathrm{i}\psi_-(0)\, \overline{\varphi}_-(0) - \mathrm{i}\varphi_+(0)\, \overline{\psi}_+(0) + [y, z](b) - [y, z](a) = 0.$$

Thus, $\mathcal{L}$ is a symmetric operator. To prove that $\mathcal{L}$ is selfadjoint, we need to show that $\mathcal{L}^* \subseteq \mathcal{L}$. Take $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathcal{L}^*)$. Let $\mathcal{L}^* g = g^* = \langle \psi_-^*, z^*, \psi_+^* \rangle \in \mathcal{H}$, so that

$$(\mathcal{L}f, g)_{\mathcal{H}} = (f, \mathcal{L}^* g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}. \tag{9}$$

By choosing elements with suitable components as $f \in D(\mathcal{L})$ in (9), it is not difficult to show that $\psi_- \in W_2^1(-\infty, 0)$, $\psi_+ \in W_2^1(0, \infty)$, $g \in D(\mathcal{L})$ and $g^* = \mathcal{L}g$, the operator $\mathcal{L}$ is defined (8). Therefore (9) is obtained from $(\mathcal{L}f, g)_{\mathcal{H}} = (f, \mathcal{L}g)_{\mathcal{H}}$ for all $f \in D(\mathcal{L}^*)$. Furthermore, $g \in D(\mathcal{L}^*)$ satisfies the conditions

$$\begin{aligned} y(a) - h_1 p(t) y^\Delta(a) &= C\varphi_-(0), \\ y(a) - \overline{h_1} p(t) y^\Delta(a) &= C\varphi_+(0). \end{aligned}$$

Hence, $D(\mathcal{L}^*) \subseteq D(\mathcal{L})$, i.e., $\mathcal{L} = \mathcal{L}^*$.

The selfadjoint operator $\mathcal{L}$ generates on $\mathcal{H}$ a unitary group $U_t = \exp(i\mathcal{L}t)$ ($t \in \mathbb{R}_+ = (0, \infty)$). Let us denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mapping acting according to the formulae $P : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$. Let $Z_t := PU_tP_1,\ t \geq 0$, by using U_t. The family $\{Z_t : t \geq 0\}$ of operators is a strongly continuous semigroup of completely non unitary contraction on H. Let us denote by B_G the generator of this semigroup: $B\ y = \lim\limits_{t \to +0} (it)^{-1} (Z_t y - y)$. The domain of B consists of all the vectors for which the limit exists. The operator B is dissipative. The operator $\mathcal{L}$ is called the selfadjoint dilation of B (see [5, 22, 25]). We show that $B = \widetilde{L}$, hence $\mathcal{L}$ is selfadjoint dilation of B. To show this, it is sufficient to verify the equality

$$P (\mathcal{L} - \lambda I)^{-1} P_1 y = \left(\widetilde{L} - \lambda I\right)^{-1} y, y \in H,\ \mathrm{Im} h < 0. \tag{10}$$

For this purpose, we set $(\mathcal{L} - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$ which implies that $(\mathcal{L} - \lambda I) g = P_1 y$, and hence $l(z) - \lambda z = y, \psi_- (\xi) = \psi_- (0) e^{-\mathrm{i}\lambda\xi}$ and $\psi_+ (\xi) = \psi_+ (0) e^{-\mathrm{i}\lambda\xi}$. Since $g \in D(\mathcal{L})$, then we have $\psi_- \in W_2^1 (-\infty, 0)$, and it follows that $\psi_- (0) = 0$, and consequently z satisfies the boundary conditions $y(a) - h_1 p(t) y^\Delta (a) = 0,\ y(b) - h_2\ p(t) y^\Delta (b) = 0$. Therefore $z \in D\left(\widetilde{L}\right)$, and since the point λ with $\mathrm{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = \left(\widetilde{L} - \lambda I\right)^{-1} y$. Thus

$$(\mathcal{L} - \lambda I)^{-1} P_1 y = \langle 0, \left(\widetilde{L} - \lambda I\right)^{-1} y, C^{-1} \left(y(a) - \overline{h_1} p(t) y^\Delta (a)\right) \mathrm{e}^{-\mathrm{i}\lambda\xi} \rangle$$

for y and $\mathrm{Im}\lambda < 0$. On applying the mapping P, we obtain (10), and

$$\begin{aligned} \left(\widetilde{L} - \lambda I\right)^{-1} &= P (\mathcal{L} - \lambda I)^{-1} P_1 = -\mathrm{i} P \int_0^\infty U_t \mathrm{e}^{-\mathrm{i}\lambda t} \mathrm{dt} P_1 \\ &= -\mathrm{i} \int^\infty Z_t \mathrm{e}^{-\mathrm{i}\lambda t} \mathrm{dt} = (B - \lambda I)^{-1}, \mathrm{Im}\lambda < 0, \end{aligned}$$

so this clearly shows that $\widetilde{L} = B$. □

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips [24]. i.e., it has orthogonal incoming and outcoming subspaces $D_- = \langle L^2 (-\infty, 0), 0, 0 \rangle$ and $D_+ = \langle 0, 0, L^2 (0, \infty) \rangle$ having the following properties:

(i) $U_t D_- \subset D_-,\ t \leq 0$ and $U_t D_+ \subset D_+,\ t \geq 0$;

(ii) $\bigcap\limits_{t \leq 0} U_t D_- = \bigcap\limits_{t \geq 0} U_t D_+ = \{0\}$;

(iii) $\overline{\bigcup\limits_{t \geq 0} U_t D_-} = \overline{\bigcup\limits_{t \leq 0} U_t D_+} = \mathcal{H}$;

(iv) $D_- \perp D_+$.

To be able to prove property (i) for D_+ (the proof for D_- is similar), we set $\mathcal{R}_\lambda = (\mathcal{L} - \lambda I)^{-1}$. For all λ, with $\mathrm{Im}\lambda < 0$ and for any $f = \langle 0, 0, \varphi_+ \rangle \in D_+$, we have

$$\mathcal{R}_\lambda f = \langle 0, 0, -\mathrm{i}\mathrm{e}^{-\mathrm{i}\lambda\xi} \int_0^\xi \mathrm{e}^{\mathrm{i}\lambda s} \varphi_+ (s) \,\mathrm{d}s\rangle.$$

So we have $\mathcal{R}_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -\mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t} (U_t f, g)_{\mathcal{H}} \,\mathrm{dt},\ \mathrm{Im}\lambda < 0,$$

which implies that $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \geq 0$. Hence, for $t \geq 0$, $U_t D_+ \subset D_+$, and property (1) has been proved.

In order to prove the property (ii), we define the mappings $P^+ : \mathcal{H} \to L^2 (0, \infty)$ and $P_1^+ : L^2 (0, \infty) \to D_+$ as $P^+ : \langle \varphi_-, \widehat{y}, \varphi_+ \rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi \rangle$, respectively. We take into consider that the semigroup of isometries $U_t^+ := P^+ U_t P_1^+ \quad (t \geq 0)$ is a one-sided shift in $L^2 (0, \infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2 (0, \infty)$ is the differential operator $i \left(\frac{d}{\mathrm{d}\xi}\right)$ with the boundary condition $\varphi (0) = 0$. On the other hand, the generator S of the semigroup of isometries U_t^+ $(t \geq 0)$ is the operator $S\varphi = P^+ \mathcal{L} P_1^+ \varphi = P^+ \mathcal{L} \langle 0, 0, \varphi \rangle = P^+ \langle 0, 0, i(\frac{d}{\mathrm{d}\xi})\varphi \rangle = i(\frac{d}{\mathrm{d}\xi})\varphi$, where $\varphi \in W_2^1 (0, \infty)$ and $\varphi (0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and, hence,

$$\bigcap_{t \geq 0} U_t D_+ = \langle 0, 0, \bigcap_{t \leq 0} V_t L^2 (0, \infty) \rangle = \{0\},$$

so the proof is complete.

Definition 1. The linear operator A with domain $D (A)$ acting in the Hilbert space H is called *completely nonselfadjoint* (*or simple*) if there is no invariant subspace $M \subseteq D (A)$ $(M \neq \{0\})$ of the operator A on which the restriction A to M is selfadjoint.

To prove the property (iii) of the incoming and outgoing subspaces, let us prove the following lemma.

Lemma 1. *The operator $\widetilde{L}$ is completely noselfadjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\widetilde{L}$ induces a selfadjoint operator $\widetilde{L}'$ with domain $D\left(\widetilde{L}'\right) = H' \cap D\left(\widetilde{L}\right)$. If $f \in D\left(\widetilde{L}'\right)$,then $f \in D\left(\widetilde{L}^*\right)$ and $y (a) - h_1 p (t) y^\Delta (a) = 0,\ y (a) - \overline{h_1} p (t) y^\Delta (a) = 0$. It follows that $y (a) = 0,\ y^\Delta (a) = 0$ and $y (\lambda) = 0$ for the eigenvectors $y (\lambda)$ of the operator $\widetilde{L}$ that lie in H' and are eigenvectors of $\widetilde{L}'$. Since all solutions of $l (y) = \lambda y$ belong to L_Δ^2 $(\mathbb{T}^*)$, from this it can be concluded that the resolvent $R_\lambda \left(\widetilde{L}\right)$ is a compact operator, and the spectrum of $\widetilde{L}$ is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the selfadjoint operator $\widetilde{L}'$, we obtain $H' = \{0\}$. Hence the operator $\widetilde{L}$ is simple. The proof is complete. □

Let us define $H_- = \overline{\bigcup_{t\geq 0} U_t D_-}$, $H_+ = \overline{\bigcup_{t\leq 0} U_t D_+}$.

Lemma 2. *The equality $H_- + H_+ = \mathcal{H}$ holds.*

Proof. Considering the property (i) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ $\left(\text{and hence also } H'\right)$ were nontrivial, then the unitary group $\left\{U_t'\right\}$ restricted to this subspace would be a unitary part of the group $\{U_t\}$, and hence, the restriction $\widetilde{L}'$ of $\widetilde{L}$ to H' would be a selfadjoint operator in H'. Since the operator $\widetilde{L}$ is simple, it follows that $H' = \{0\}$. Hence the lemma is proved. □

Assume that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of $l(y) = \lambda y$ satisfying the conditions

$$\varphi(a,\lambda) = \sin\alpha,\ p(t)\,\varphi^{\Delta}(a,\lambda) = -\cos\alpha,\ \psi(a,\lambda) = \cos\alpha,\ p(t)\,\psi^{\Delta}(0,\lambda) = \sin\alpha.$$

Then

$$\psi(x,\lambda) + m(\lambda)\,\varphi(x,\lambda) \in L^2_{\Delta}(\mathbb{T}^*),$$

where $m(\lambda)$ is a Titchmarsh-Weyl function. Let us adopt the following notations (see [20]): $\theta(x,\lambda) = \psi(x,\lambda) + m(\lambda)\,\varphi(x,\lambda)$,

$$S_G(\lambda) = \frac{m(\lambda) - G}{m(\lambda) - \overline{G}}. \tag{11}$$

We set

$$U_\lambda^-(x,\xi,\zeta) = \langle e^{-i\lambda\xi}, (m(\lambda) - G)^{-1}\,\alpha\theta(x,\lambda), \overline{S_G}(\lambda)\,e^{-i\lambda\zeta}\rangle.$$

We note that the vectors $U_\lambda^-(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^-(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}$.

By means of vector $U_\lambda^-(x,\xi,\zeta)$, we define the transformation $F_- : f \to \widetilde{f}_-(\lambda)$ by

$$(F_- f)(\lambda) := \widetilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}\,(f, U_{\overline{\lambda}})_{\mathcal{H}}$$

on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$, $y(x)$ are smooth, compactly supported functions.

Lemma 3. *The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_-$ the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}} = \left(\widetilde{f}_-, \widetilde{g}_-\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\,\overline{\widetilde{g}_-(\lambda)}\,d\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\,U_{\overline{\lambda}}\,d\lambda,$$

where $\widetilde{f_-}(\lambda) = (F_- f)(\lambda)$ *and* $\widetilde{g_-}(\lambda) = (F_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0 \rangle$, $g = \langle \psi_+, 0, 0 \rangle$, with Paley-Wiener theorem, we have

$$\widetilde{f_-}(\lambda) = \frac{1}{\sqrt{2\pi}} (f, U_{\overline{\lambda}})_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi_-(\xi) \, \mathrm{e}^{-\mathrm{i}\lambda\xi} \mathrm{d}\xi \in H_-^2,$$

and by using usual Parseval equality for Fourier integrals,

$$(f, g)_{\mathcal{H}} = \int_{-\infty}^{\infty} \varphi_-(\xi) \, \overline{\psi_-(\xi)} \mathrm{d}\xi = \int_{-\infty}^{\infty} \widetilde{f_-}(\lambda) \, \overline{\widetilde{g_-}(\lambda)} \mathrm{d}\lambda = (F_- f, F_- g)_{L^2},$$

Here, $H_{\pm}^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively.

We now extend the Parseval equality to the whole of H_-. We consider in H_- the dense set of $H_-^{\prime}$ of the vectors obtained as follows from the smooth, compactly supported functions in $D_- : f \in H_-^{\prime}$ if $f = U_T \ f_0$, $f_0 = \langle \varphi_-, 0, 0 \rangle$, $\varphi_- \in C_0^{\infty}(-\infty, 0)$, where $T = T_f$ is a nonnegative number depending on f. If $f, g \in H_-^{\prime}$, then for $T > T_f$ and $T > T_g$ we have $U_{-T} f, U_{-T} g \in D_-$, moreover, the first components of these vectors belong to $C_0^{\infty}(-\infty, 0)$. Therefore, since the operators U_t $(t \in \mathbb{R})$ are unitary, by the equality

$$F_- U_t f = (U_t f, U_{\overline{\lambda}})_{\mathcal{H}} = \mathrm{e}^{\mathrm{i}\lambda t} \left(f, U_{\overline{\lambda}}^{-} \right)_{\mathcal{H}} = \mathrm{e}^{\mathrm{i}\lambda t} F_- f,$$

we have

$$(f, g)_{\mathcal{H}} = (U_{-T} \ f, U_{-T} \ g)_{\mathcal{H}} = (F_- U_{-T} \ f, F_- U_{-T} \ g)_{L^2}$$

$$(\mathrm{e}^{\mathrm{i}\lambda T} F_- f, \mathrm{e}^{\mathrm{i}\lambda T} F_- g)_{L^2} = \left(\widetilde{f}, \widetilde{g} \right)_{L^2}. \tag{12}$$

By taking the closure (12), we obtain the Parseval equality for the space H_-. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally $F_- H_- = \overline{\underset{t \geq 0}{\cup} F_- U_t D_-} = \overline{\underset{t \geq 0}{\cup} \mathrm{e}^{\mathrm{i}\lambda t} H_-^2} = L^2(\mathbb{R})$, that is F_- maps H_- onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

We set

$$U_{\lambda}^{+}(x, \xi, \zeta) = \langle S_G(\lambda) \, \mathrm{e}^{-\mathrm{i}\lambda\xi}, \left(m(\lambda) - \overline{G} \right)^{-1} \alpha\theta(x, \lambda), \mathrm{e}^{-\mathrm{i}\lambda\zeta} \rangle.$$

We note that the vectors $U_{\lambda}^{+}(x, \xi, \zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_{\lambda}^{+}(x, \xi, \zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}$. With the help of the vector $U_{\lambda}^{+}(x, \xi, \zeta)$, we define the transformation $F_+ : f \to \widetilde{f_+}(\lambda)$ by $(F_+ f)(\lambda) :=$

$\widetilde{f_+}(\lambda) := \frac{1}{\sqrt{2\pi}}\left(f, U_\lambda^+\right)_{\mathcal{H}}$ on the vectors $f = \langle\varphi_-, y, \varphi_+\rangle$ in which $\varphi_-(\xi), \varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

Lemma 4. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = \left(\widetilde{f_+}, \widetilde{g_+}\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f_+}(\lambda)\,\overline{\widetilde{g_+}(\lambda)}\mathrm{d}\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f_+}(\lambda)\, U_\lambda^+ \mathrm{d}\lambda,$$

where $\widetilde{f_+}(\lambda) = (F_+f)(\lambda)$ and $\widetilde{g_+}(\lambda) = (F_+g)(\lambda)$.

Proof. The proof is analogous to that of lemma 3. □

From (11), $|S_G(\lambda)| = 1$ for $-\infty < \lambda < \infty$. Therefore, it explicitly follows from the formulae for the vectors U_λ^- and U_λ^+ that

$$U_\lambda^- = \overline{S}_G(\lambda)\, U_\lambda^+. \tag{13}$$

It follows from Lemmas 3 and 4 that $H_- = H_+$. Together with Lemma 2, this shows that $H_- = H_+ = \mathcal{H}$, therefore the property (iii) above has been proved for the incoming and outgoing subspaces. Finally the property (4) is clear.

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $\mathrm{e}^{\mathrm{i}\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (13) that the passage from the F_- representation of an element $f \in \mathcal{H}$ to its F_+ representation is accomplished as $\widetilde{f_+}(\lambda) = \overline{S_h}(\lambda)\,\widetilde{f_-}(\lambda)$. Consequently, according to [24] we have proved the following.

Theorem 4. *The function $\overline{S}_G(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}$).*

Let $S(\lambda)$ be an arbitrary non constant inner function (see [25]) on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S_G(\lambda)| \le 1$ for all $\lambda \in \mathbb{C}_+$ and $|S_G(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H_+^2 \ominus SH_+^2$. Then $K \ne \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \ge 0)$ acting in K according to the formula $Z_t\varphi = P\left[\mathrm{e}^{\mathrm{i}\lambda t}\varphi\right]$, $\varphi = \varphi(\lambda) \in K$, where P is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0}(\mathrm{i}t)^{-1}(Z_t\varphi - \varphi),$$

where T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax-Phillips [24], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [25]). The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K=\langle 0,H,0\rangle$, so that $\mathcal{H}=D_-\oplus K\oplus D_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_- that

$$\begin{aligned} \mathcal{H} &\rightarrow L^2(\mathbb{R}), \quad f\rightarrow \widetilde{f_-}(\lambda)=(F_-f)(\lambda), \; D_-\rightarrow H_-^2, \quad D_+\rightarrow S_G H_+^2, \\ K &\rightarrow H_+^2\ominus S_G H_+^2, \; U_t\rightarrow \left(F_-U_tF_-^{-1}\widetilde{f_-}\right)(\lambda)=\mathrm{e}^{\mathrm{i}\lambda t}\widetilde{f_-}(\lambda). \end{aligned} \tag{14}$$

The formulas (14) show that the operator $\widetilde{L}$ is unitarily equivalent to the model dissipative operator with the characteristic function $S_G(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide (see [25]), we have thus proved the following theorem.

THEOREM 5. *The characteristic function of the maximal dissipative operator $\widetilde{L}$ coincides with the function $S_G(\lambda)$ defined* (11).

Using the characteristic function, the spectral properties of the maximal dissipative operator $\widetilde{L}$ can be investigated. The characteristic function of the maximal dissipative operator $\widetilde{L}$ is known to lead to information of completeness about the spectral properties of this operator. For instance, the absence of a singular factor $s(\lambda)$ of the characteristic function $S_G(\lambda)$ in the factorization $detS_G(\lambda)=s(\lambda)B(\lambda)$ ($B(\lambda)$ is a Blaschke product) ensures the completeness of the system of eigenvectors and associated vectors of the operator $\widetilde{L}$ in the space $L_2(0,\infty)$ (see [5, 22, 25]).

THEOREM 6. *For all values of G with* $\mathrm{Im}G>0$, *except possibly for a single value $G=G_0$, the characteristic function $S_G(\lambda)$ of the maximal dissipative operator $\widetilde{L}$ is a Blaschke product. The spectrum of $\widetilde{L}$ is purely discrete and belongs to the open upper half-plane. The operator $\widetilde{L}$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors of the operator $\widetilde{L}$ is complete in the space H.*

Proof. From (11), it is clear that $S_G(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be factored in the form

$$S_G(\lambda)=\mathrm{e}^{\mathrm{i}\lambda c}B_G(\lambda), \; c=c(G)\geq 0, \tag{15}$$

where $B_G(\lambda)$ is a Blaschke product. It follows from (15) that

$$|S_G(\lambda)|\leq \mathrm{e}^{-c(G)\mathrm{Im}\lambda}, \; \mathrm{Im}\lambda\geq 0. \tag{16}$$

Further, for $m(\lambda)$ in terms of $S_G(\lambda)$, we find from (11) that

$$m(\lambda) = \frac{\overline{G} S_G(\lambda) - h}{S_G(\lambda) - 1}. \tag{17}$$

If $c(G) > 0$ for a given value G ($\mathrm{Im} G > 0$), then (16) implies that $\lim_{t\to+\infty} S_G(it) = 0$, and then (17) gives us that $\lim_{t\to+\infty} M(it) = -G$. Since $M(\lambda)$ does not depend on G, which implies that $c(G)$ can be nonzero at not more than a single point $G = G_0$ (and further $G_0 = -\lim_{t\to+\infty} m(it)$). Hence the proof is complete. $\square$

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Mehmet Akif Ersoy University
Department of Mathematics
Burdur, Turkey
E-mail: hustuna@gmail.com