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OLD AND RECENT RESULTS ON FINITE BOLYAI-LOBACHEVSKY PLANES

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Abstract. The revolutionary ideas of János Bolyai opened the way for a far more general and abstract approach to geometry than had previously been pursued. In the spirit of Bolyai's ideas, axioms with their mutual relationships and impacts on geometry had intensively been studied and discussed for a long time. The historical development is treated in the expository papers appeared in the volume [75] which commemorated the 200th anniversary of the birth of János Bolyai, written by leading scientists of non-Euclidean geometry, its history, and its applications. A recent survey on Bolyai's work is also found in the survey paper [44].

Axiom systems proposed for general Bolyai-Lobachevsky planes appeared in the literature for the first time in the 1940's. These attempts were strongly influenced by the classical point of view in geometry in that time, and the proposed definitions included sufficiently many postulates to exclude finite planes, that is, geometries on a finite set of points; see Topel [88], De Baggis [33] and Baer [5].

It was only in 1962 Graves' paper [43] that an axiom system for Bolyai-Lobachevsky planes was proposed that admitted finite geometry. Since then, a number of models for finite Bolyai-Lobachevsky planes have been constructed; some of them present interesting properties from different points of view.

The present paper is an account of the known results on finite Bolyai-Lobachevsky planes. We focus on the finite analogs of the well known models of the classical Bolyai-Lobachevsky plane, and show that the finite Beltrami-Cayley and Poincaré models are related to current research in Finite geometry. We also discuss some more, typically finite, models arising from unitary polarities and maximal (k,n)-arcs of finite projective planes. In this context, we investigate those models which have a large symmetry group. An extensive list of bibliographic references on finite Bolyai-Lobachevsky planes is also provided.

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1. INTRODUCTION

The finite Bolyai-Lobachevsky planes together with other types of finite planes are considered as a part of Finite geometry which is the general theory of geometric structures defined on a finite number of points.

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Perhaps the most important and certainly the best known objects in Finite geometry are the finite affine and projective planes. The concept of a finite affine plane is the discrete analog of the concept of a classical affine plane. A finite affine plane is defined by the following postulates:

(i) given any two distinct points, there is exactly one line that is incident with both points;
(ii) [Parallel postulate] given a line ℓ and a point P not incident with ℓ, there exists exactly one line incident with P parallel to ℓ (here, two distinct lines are parallel if either they coincide or no point is incident with both lines);
(iii) there exists a set of four points, no three of which are incident with the same line;
(iv) there is a line incident with only a finite number of points.

In Finite geometry, we usually adopt a "geometric language" and identify any line with the set of points which are incident with a line. A finite affine plane has *order* n if at least one line is incident with exactly n points. Then the plane has n^2 points and $n^2 + n$ lines; each line is incident with n points, and each point is incident with $n + 1$ lines.

In the usual way, any affine plane can be completed to a projective plane by adding the infinite line incident with $n + 1$ infinite points. So the parallel postulate does not hold in a projective plane and hence Axiom (ii) must be replaced by the following one:

(ii-Pr) any two distinct lines have exactly one common point.

It is a simple matter to construct examples of finite projective and affine planes in terms of coordinates in a Galois field. Such planes are called the Galois planes and are of great interest from various points of view; combinatorics, finite fields, group theory and algebraic curves in positive characteristics are the main ingredients in their extensive study. There exists a well developed and coherent theory of finite affine and projective planes treated in a series of monographs; see for instance [50, 52, 55] and the bibliographic references therein.

Finite geometry is a relatively young discipline. The systematic study of finite affine and projective planes started in the 1940's; and later it was extended to finite Möbius, Laguerre, Minkowski planes [14, 24, 37, 65], and Sperner spaces [79, 80, 81, 82, 83].

In the 1940's, possible finite versions of Bolyai-Lobachevsky planes were also considered. The obvious idea was to replace the parallel postulate in the definition of an affine plane with

(ii-BL) given a line ℓ and a point P not incident with ℓ, there exists at least two lines through P which are parallel to ℓ;

Unfortunately, there exist too many and substantially different geometric structures satisfying the postulates (i), (ii-BL) and (iii). In other words, these three postulates define a too general geometric structure, and clearly some restriction is absolutely necessary. How to do this by adding more postulates is quite an interesting question. It was addressed by B. J. Topel [88], H. F. De Baggis [33] and, in particular, by the famous German mathematician R. Baer [5] who introduced the concept of an *abstract Bolyai-Lobachevsky plane.* Motivated by the classical point of view in geometry in that time, the postulates proposed were too strongly related to the more topological concept of betweenness, and therefore they necessarily excluded finite geometric structures.

In 1962 L. M. Graves [43] proposed the following system of postulates for Bolyai-Lobachevsky planes that admits finite geometric structures:

(i) given any two distinct points, there is exactly one line that is incident with both points;

(ii-BL) given a line ℓ and a point P not incident with ℓ, there exists at least two lines through P which are parallel to ℓ;

(iii) there exists a set of four points, no three of which are incident with the same line;

(iv) there exists at least two points on each line;

(v) given any triangle Δ, the transversal lines of Δ cover all points, that is, for every point P there exists a line ℓ such that $P \in \ell$ and $|\ell \cap \Delta| = 2$).

Essentially, L. M. Graves added two postulates to get rid of trivial examples and degenerate geometric structures. Nevertheless, the geometric structure defined by such five postulates still remains too general even in the finite case, a number of interesting but somewhat unrelated examples being available in the literature. The usual restriction in Finite geometry that allows to incorporate interesting incidence structures in a systematic frame-work is to require the incidence structure to have some sort of regularity or homogeneity. We discuss briefly how this can be done for finite Bolyai-Lobachevsky planes.

From a combinatorial point of view, such a symmetry may be either *line-regularity*, that is, every line is incident with a constant number of points or, dually, *point-regularity* if every point is incident with a constant number of lines. A finite incidence structure which is both line and point regular is called *regular.*

From a geometric point of view, the usual requirement is *point-homogeneity*, that is, the automorphism group is transitive on the set of points, or *line-homogeneity*, that is, the automorphism group is transitive on the set of lines. An incidence structure which is both point and line homogeneous is called *homogeneous.* A stricter requirement is *flag-transitivity*, that is the automorphism group is transitive on the set of incident point-line pairs. It may be observed that doubly transitivity on points implies flag-transitivity.

Under such restrictions, deeper results and classification theorems on finite Bolyai-Lobachevsky planes have been obtained. A survey is given in the present paper which also includes some recent contributions; see [63]. Our notation and terminology are standard; see for instance [50].

2. LINEAR SPACES AND BOLYAI-LOBACHEVSKY PLANES

A linear space is an incidence structure $\mathfrak{L} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ consisting of "points" and "lines", with incidence I, satisfying the following axioms:

L1 each two distinct points are incident to exactly one line;
L2 each line is incident to at least two points.

Our notation is standard; see [10] for a detailed description of linear spaces.

If $\mathfrak{L}$ has a finite number of points, then we call it a finite linear space. For finite linear spaces, one can further define

- $r(P) = |\{\, \ell \in \mathcal{L} \mid P \mathrm{~I~} \ell \,\}|$ for each $P \in \mathcal{P}$,
- $k(\ell) = |\{\, P \in \mathcal{P} \mid P \mathrm{~I~} \ell \,\}|$ for each $P \in \mathcal{P}$,
- $k_m = \min |\{\, k(\mathcal{L}) \mid \ell \in \mathcal{L} \,\}|$,
- $k_M = \max |\{\, k(\mathcal{L}) \mid \ell \in \mathcal{L} \,\}|$,
- $r_m = \min |\{\, r(P) \mid P \in \mathcal{P} \,\}|$,
- $r_M = \max |\{\, r(P) \mid P \in \mathcal{P} \,\}|$.

In this context, a finite Bolyai-Lobachevsky plane turns out to be a finite linear space $\mathfrak{L} = (\mathcal{P}, \mathcal{L})$ satisfying Axioms (i)–(v).

In particular, a line symmetric Bolyai-Lobachevsky plane is a Steiner system. Recall that a Steiner system $S(2, k, v)$ is a linear space $\mathfrak{L} = (\mathcal{P}, \mathcal{L})$ consisting of v points, where each line is incident with exactly k points. Steiner systems have been intensively studied; see [4, 12, 13, 53, 66].

An obvious necessary condition for the existence of an $S(2, k, v)$ is the following:

$$|\mathcal{L}| = \frac{v(v-1)}{k(k-1)} \in \mathbb{N}.$$

A standard counting argument shows that the number $r(P)$ of lines through a point P in a Steiner system $S(2, k, v)$ is constant. To show it, we note that for any point P and line ℓ not through P there are exactly k lines through P meeting ℓ. Each of these lines contains $k-1$ points distinct from P, and the remaining $v - \big(k(k-1)+1\big)$ points of our $S(2, k, v)$ are on the other lines; hence

$$\mathrm{r}(P) = k + \frac{v - \big(k(k-1)+1\big)}{k-1},$$

where the number of lines through P which are disjoint from ℓ is

$$s = \frac{v - \big(k(k-1)+1\big)}{k-1}.$$

This gives the following result.

Proposition 2.1. *Every line-regular Bolyai-Lobachevsky plane is a Steiner system $S(2, k, v)$ with $r > k$. In particular, every line-regular Bolyai-Lobachevsky plane is regular.*

Finite Bolyai-Lobachevsky planes with $s = 3$ are the unique $S(2, 2, 6)$, the two known $S(2, 3, 13)$ and a hypotetic $S(2, 6, 46)$ the existence of which is yet an open problem [42].

Finite flag-transitive Bolyai-Lobachevsky planes are line-regular and hence they are Steiner systems. Therefore, the classification of the automorphism groups of flag-transitive Steiner systems, depending on the classification of finite simple groups, has the following corollary; see [25].

THEOREM 2.2. *If G is a flag-transitive automorphism group of a finite Bolyai-Lobachevsky plane $\mathfrak{B}$, then one of the following holds:*

- *Affine case: G has an elementary abelian minimal normal subgroup T, acting regularly on the points of $\mathfrak{B}$.*
- *Almost simple case: G has a nonabelian simple subgroup N such that $N \trianglelefteq G \leq \mathrm{Aut}N$, and the only possibilities are:*
  - *$\mathfrak{B} = \mathrm{PG}(d, q)$ with $d > 2$ over $\mathrm{GF}(q)$ and $N = \mathrm{PSL}(d + l, q)$, except for the sporadic example $\mathfrak{B} = \mathrm{PG}(3, 2)$ with $G = \mathrm{Alt}(7)$;*
  - *$\mathfrak{B}$ is a Hermitian unital $U_H(q)$ and $N = \mathrm{PSU}(3, q)$;*
  - *$\mathfrak{B}$ is a Ree unital $U_H(32e + 1)$ with $e > 1$ and $N = {}^2G_2(32e + 1)$;*
  - *$\mathfrak{B}$ is a Witt-Bose-Shrikhande space $W(2^e)$ with $e > 3$ and $N = \mathrm{PSL}(2, 2^e)$.*

In each ease, the action of N on $\mathfrak{B}$ is the usual one.

In [35] A. Delandtsheer introduced the more general concept of an n-fold finite Bolyai-Lobachevsky space and provided a classification of 2-fold Bolyai-Lobachevsky spaces.

An n-fold Bolyai-Lobachevsky space is a finite linear space for which there is an integer $s > 0$ such that for any set consisting of n mutually disjoint lines $\ell_1, \ell_2, \ldots, \ell_n$ any point lying outside their union $\ell_1 \cup \ell_2 \cup \cdots \cup \ell_n$ is on exactly s lines disjoint from $\ell_1 \cup \ell_2 \cup \cdots \cup \ell_n$.

Recall that a generalised projective space is a linear space satisfying the Veblen-Young axiom, that is, for every couple of lines ℓ and ℓ', with $\ell \cap \ell' = \{P\}$, any two lines not through P and intersecting both ℓ and ℓ' intersect, in turn, at a point Q.

THEOREM 2.3 (A. Delandtsheer). *For a 2-fold finite Bolyai-Lobachevsky space one of the following occurs:*

(a) *$\mathfrak{S}$ is a projective space of dimension at most 3;*
(b) *$\mathfrak{S}$ is either a degenerate projective plane, or a generalised projective space of dimension 3 consisting of two disjoint lines with the same size and all another lines with two points, or a generalised projective space in which all lines have exactly 2 points;*
(c) *$\mathfrak{S}$ is a projective plane of order q from which k points lying on one of its lines are removed, with $0 \leq k \leq q + 1$ and $k \neq q$;*
(d) *$\mathfrak{S}$ is an affine plane from which either one point or one line has been removed.*

3. FINITE ANALOGS OF THE MODELS OF THE CLASSICAL BOLYAI-LOBACHEVSKY PLANE

To pick up the significative aspects of the finite Bolyai-Lobachevsky planes, it is useful to look into the classical models.

The Beltrami-Klein model, also called the projective model, the Klein disk model, and the Cayley-Klein model, is a model of Bolyai-Lobachevsky plane in which points are represented by the points in the interior of the unit disk, and lines are represented by the chords, that is, straight line segments with endpoints on the boundary circle. It made its first appearance in a memoir of the Italian mathematician E. Beltrami published in 1868 [11].

3.1. Finite Bolyai-Lobachevsky planes arising from ovals. In [74] T. G. Ostrom showed that the Beltrami–Klein model can be defined in any finite projective plane of odd order provided that the plane contains an oval. Recall that an oval Ω in a projective plane π of order n is a set of points such that

- no three points of Ω are collinear;
- for each point P of Ω there exists a unique line through P which meets Ω only in P; such a line is the *tangent* to Ω at P.

Well known examples of ovals are the irreducible conics in the projective plane $\mathrm{PG}(2,n)$ defined over the finite field $\mathrm{GF}(n)$, where n is a power of a prime. B. Segre [77] proved that every oval in $\mathrm{PG}(2,n)$ with n odd consists of all points of an irreducible conic.

Assume that n is odd. Then Ω resembles the basic properties of an irreducible conic of the real projective plane. Further, the points of $\mathrm{PG}(2,n)$ fall in three disjoint classes:

- the points in Ω, each lying on exactly one tangent line;
- the exterior points, which lie on exactly two tangent lines;
- the interior points, which lie on no tangent lines.

It follows that

- there are $\frac{1}{2}n(n-1)$ interior points;
- no tangent line contains any interior points;
- each secant line contains $\frac{1}{2}(n-1)$ interior points;
- each exterior line contains exactly $\frac{1}{2}(n+1)$ interior points.

There is a vast literature on ovals and their generalisations; see for instance [50, 62].

A geometric structure arises from the set of interior points of Ω when lines are defined to be the set-theoretic intersections of the non-tangent lines with the interior points. This geometric structure is the Beltrami-Klein model of a finite Bolyai-Lobachevsky plane.

In the case where the plane is a Galois plane of odd order n (and hence the oval is a conic by Segre's theorem [77]), then the Beltrami-Klein model is point-homogeneous, as it was pointed out by T. G. Ostrom in [74].

Finite point-homogeneous Beltrami-Klein models are strongly related with the following result; see [41].

Theorem 3.1 (M. R. Enea, G. Korchmáros). *Let G be collineation group of a projective plane of odd order n that fixes an oval Ω and acts transitively on the set of all internal points to Ω. Then n is a prime power and either*

(I) *G is 2-transitive on Ω, the plane is of Galois and Ω is a conic, or*
(II) *G fixes a point $X \in \Omega$ and acts on $\Omega \setminus \{X\}$ as an affine-type primitive permutation group. If, in addition, each involution in G is a homology, then G is 2-transitive on $\Omega \setminus \{X\}$, and one of the following holds:*
   (a) *$n = q$, and $G \leq AGL(1, q)$.*
   (b) *$n = p^2$ with $p \in \{5, 7, 11, 23, 29, 594\}$, and G acts on $\Omega \setminus \{X\}$ as a sharply 2-transitive permutation group arising from an irregular nearfield of order p^2.*

A finite Beltrami-Klein model cannot be line-transitive, as no collineation fixing Ω may take an internal line to an external line.

A finite Beltrami-Klein model is not regular, since a line has either $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$ points according as the line is a secant or an external line.

3.2. Finite Bolyai-Lobachevsky planes arising from triangles. In [76] R. Sandler presented the following construction for a class of finite Bolyai-Lobachevsky planes. Let π be a finite projective plane of order $n \geq 7$, and π_0 the set of points obtained by removing from π the points on three non-concurrent lines, say ℓ_1, ℓ_2 and ℓ_3.

Now consider an incidence structure $\mathfrak{B} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \pi_0$, lines of $\mathcal{L}$ are the lines of π except ℓ_1, ℓ_2 and ℓ_3, with the points contained in ℓ_1, ℓ_2 and ℓ_3 removed. The following facts are obvious.

(1) Every point of $\mathcal{P}$ is in exactly $n+1$ lines of $\mathcal{L}$.
(2) Every line of $\mathcal{L}$ contains either $n-1$ or $n+1$ points, according as whether a line passes through an inersection $\ell_i \cap \ell_j$ or not, with $i, j \in \{1, 2, 3\}$.
(3) Two distinct points of $\mathcal{P}$ determine a unique line of $\mathcal{L}$.
(4) Through each point $P \in \mathcal{P}$ not on a line $\ell \in \mathcal{L}$ there pass at least two lines of $\mathcal{L}$ which do not intersect ℓ.

Actually, properties (3) and (4) are the first two axioms for a finite Bolyai-Lobachevsky plane.

Note that if π is Desarguesian, then the finite Bolyai-Lobachevsky plane $\mathfrak{B}$ so obtained is homogeneous, that is, its collineation group is transitive on its points. In fact, any collineation of π preserving a triangle induces a collineation of $\mathfrak{B}$, and if π is Desarguesian its collineation group is known to be transitive on quadrilaterals. Hence, the subgroup of the collineation group of π fixing the three points $\ell_i \cap \ell_j$, with $i, j \in \{1, 2, 3\}$, will induce a collineation group of $\mathfrak{B}$ which is transitive on $\mathcal{P}$.

Sandler's construction can be extended either to the case where more than three lines are removed from π, or to the case where π is infinite, in order to obtain new examples of (perhaps infinite) Bolyai-Lobachevsky planes.

3.3. Finite Bolyai-Lobachevsky planes arising from inversive planes. In a series of papers [30, 31, 32], D. W. Crowe proposed a finite Poincaré model for Bolyai-Lobachevsky planes based on finite inversive planes.

An inversive plane is an incidence structure $\mathfrak{I}$ consisting of "points" and "circles" satisfying the following axioms.

I1. Any three points lie on exactly one circle.
I2. If $\mathcal{C}$ is a circle, P is a point on $\mathcal{C}$, and Q is a point not on $\mathcal{C}$, then there is exactly one circle $\mathcal{C}'$ containing both P and Q and intersecting $\mathcal{C}$ only in P.
I3. There exist four points not all in the same circle.

If there exists an integer n such that each circle of $\mathfrak{I}$ contains exactly $n+1$ points, then $\mathfrak{I}$ is called a finite inversive plane of order n. A comprehensive account on inversive planes can be found in [37].

An orthogonality relation on an inversive plane $\mathfrak{I}$ is a symmetric binary relation $\perp$ on circles of $\mathcal{I}$ with the following properties.

O1 If P and Q are two distinct points and $\mathcal{C}$ is a circle through P, then there is exactly one circle $\mathcal{C}'$ through both P and Q such that $\mathcal{C} \perp \mathcal{C}'$.
O2 If $\mathcal{C}$ and $\mathcal{C}'$ are two distinct circles, each through the distinct points P and Q, and if $\mathcal{C}''$ is a circle such that $\mathcal{C} \perp \mathcal{C}' \perp \mathcal{C}''$ then $\mathcal{C}'' \perp \mathcal{X}$ for all circles $\mathcal{X}$ through P and Q.

Recall that a finite inversive plane can have at most one orthogonality relation; see [37]. Let $\mathfrak{I}$ be a finite inversive plane. If the order of $\mathfrak{I}$ is even, then orthogonality coincides with tangency. If instead $\mathfrak{I}$ has odd order, then two orthogonal circles in $\mathfrak{I}$ are either disjoint or have two common points.

Further, the most useful combinatorial tool in this context is represented by the following result; see [32, 85].

Theorem 3.2. *Let r_m, r_M, k_m and k_M be as in Page 62. Then, any finite linear space satisfying*

$$r_m \geq k_M + 2,$$
$$k_m(k_m - 1) \geq r_M,$$

is a hyperbolic plane.

Every finite inversive plane equipped with an orthogonality gives rise to a finite Poincaré model of a Bolyai-Lobachevsky plane. In order to describe Crowe's construction we need to recall the following result; see [38].

Lemma 3.3. *Let P be a point and $\mathcal{C}$ a circle not through P in a finite inversive plane equipped with orthogonality. Then there exists a point $\overline{P} \neq P$*

not on $\mathcal{C}$ such that for all circles $\mathcal{X}$ through P the relation $\mathcal{X} \perp \mathcal{C}$ holds if and only in $\overline{P}$ is not on $\mathcal{X}$.

Now let $\mathfrak{I}$ be a finite inversive plane of order n with orthogonality and $\mathcal{C}$ a fixed circle in $\mathfrak{I}$. Define an incidence structure $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ of "points" and "lines" as follows:

- points are the sets $\{P, \overline{P}\}$ where P is not on $\mathcal{C}$ as in Lemma 3.3;
- lines are the circles $\mathcal{L}$ which are orthogonal to $\mathcal{C}$.

Note that the point $\{P, \overline{P}\}$ is on $\mathcal{L}$ if and only if either P or $\overline{P}$ is on $\mathcal{L}$ (and hence both are). Further, if $\{P, \overline{P}\}$ and $\{Q, \overline{Q}\}$ are distinct points, then in $\mathfrak{I}$ there is exactly one circle $\mathcal{L}$ containing P, $\overline{P}$ and Q (hence also $\overline{Q}$). So, $\mathcal{L}$ is the unique line of $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ through the distinct points $\{P, \overline{P}\}$ and $\{Q, \overline{Q}\}$. Thus $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ is a finite linear space. If n is even, then $k = \frac{1}{2}n$, while if n is odd then $k_m = \frac{1}{2}(n-1)$ and $k_M = \frac{1}{2}(n+1)$. Hence, by Theorem 3.2, $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ is a hyperbolic plane for all $n > 7$. It can be shown directly that $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ is a hyperbolic plane also for $n = 7$, but not for $n < 7$. Further, $\mathfrak{F}_{\mathcal{C}}(\mathfrak{I})$ is regular if and only if n is even.

Finite inversive planes arise from ovoids, in particular from quadrics in a three-dimensional projective space over a Galois field. Interestingly, in the even order case there exist ovoids other than quadrics. They are called Tits ovoids, and are strongly related to the simple Suzuki groups; see [67, Chapter IV]. It would be interesting to investigate the geometric properties of these particular finite Poincaré models.

3.4. Finite Bolyai-Lobachevsky planes arising from unitals. Bumcrot pointed out in [26] that the set of all absolute points of a unitary polarity provides an example of a finite regular Bolyai-Lobachevsky plane. Such a set is called classical unital. Here, the projective plane π must be a Galois plane of order $n = m^2$, and a canonical form of the classical unital is

$$X^{m+1} + Y^{m+1} + Z^{m+1} = 0.$$

This model of a finite regular Bolyai-Lobachevsky plane is called the unitary model. This model is as nice as possible, since its symmetry group is doubly point-transitive—hence line-homogeneous—and regular.

The classical unital consists of $m^3 + 1$ points. It has exactly one tangent at each of its points, and each other line is an $(m+1)$-secant. Therefore, the unitary unital model has $m^3 + 1$ points, $m^2(m^2 - m + 1)$ lines, each line is incident with $m + 1$ points and each point is incident with m^3 lines.

Actually, to show that a classical unital gives rise to Bolyai-Lobachevsky plane, the above combinatorial properties are enough; the fact that they are absolute points of an unitary polarity is unnecessary. So, a unital model of a Bolyai-Lobachevsky plane exists in every projective plane of order m^2 containing an unital. Unitals other than the classical ones exist even in finite Galois planes; see [2, 23, 51, 70]. In fact, one of the main problems in Finite

Geometry is to classify unitals in Galois planes. There exists a vast literature on unitals; see the monograph [9].

The inherited symmetry group of a finite Bolyai-Lobachevsky arising from a unital $\mathcal{U}$ is the subgroup G of the collineation group of the plane which preserves $\mathcal{U}$. In this context, "line-homogeneity" is meant "transitivity of G on the set of all secants to $\mathcal{U}$". From a result by M. Biliotti and the first author; see [18, 19], a line-homogeneous unital model of a Bolyai-Lobachevsky plane is necessarily a unitary model. For more results on collineation groups preserving unitals see [28, 29].

3.5. Finite Bolyai-Lobachevsky planes arising from maximal (k, n)-arcs. Other nice models of finite regular Bolyai-Lobachevsky planes can be obtained from particular configurations in a finite projective plane, called maximal (k, n)-arcs. A (k, n)-arc in a projective plane π of order q is a set of k points meeting each line in at most n points and some line in exactly n points. A (k, n)-arc has at most $(n-1)q+n$ points and it is said to be maximal if it has that many. If this is the case then $k|n$. The dual of a maximal (k, n)-arc Ω, that is, the set of external lines to Ω together with the points outside Ω, is a $((q+1-n)q/n, q/n)$-arc in the dual plane π^* of π. Formally, any point is a maximal $(1, 1)$-arc, and its dual, that is, the complement of a line, is a maximal (q^2, q)-arc. However they behave as trivial examples in the context of (k, n)-arcs and are not considered in our discussion.

For $n > 2$, the set of all points of a maximal (k, n)-arc together with its secants gives a further example of a finite regular Bolyai-Lobachevsky plane. Such a model has $(n-1)q+n$ points, $(q^2+q+1)k/n$ lines, each line is incident with n points and each point is incident with q^2+q+1 lines.

Maximal arcs have been used to construct interesting new partial geometries, strongly regular graphs, 2-weight codes, and resolvable Steiner 2-designs.

It was proved by S. Ball, A. Blokhuis and F. Mazzocca [7] that maximal arcs do not exist in any Galois plane of odd order.

Another known (sporadic) example of a point-homogeneous dual hyperoval arises from the Lunelli-Sce hyperoval in $\mathrm{PG}(2, 16)$; see [58]. Other families of maximal (k, n)-arcs in Galois planes of even order were constructed by R. H. F. Denniston, R. Mathon and N. Hamilton; see [39, 45, 46, 69]. Apart from hyperovals and their duals, there are three classes of non-trivial maximal arcs known in $\mathrm{PG}(2, q)$, they are those due to R. H. F. Denniston and two classes due to J. A. Thas; see [86, 87] Maximal (k, n)-arcs are also related with algebraic plane curves with many rational points; see [1].

In [47] Hamilton and Penttila investigated collineation groups of maximal (k, n)-arcs in a Galois plane of even order.

Theorem 3.4 (N. Hamilton, T. Penttila). *Let K be a maximal arc in $\mathrm{PG}(2, q)$ with $q = 2^h$ and $h \geq 2$. If a collineation group G preserving K acts transitively on the points of K then one of the following cases occur:*

(i) *K is a regular hyperoval in* PG(2, 2)*, or PG*(2, 4)*, or a Lunelli-Sce hyperoval in* PG(2, 16).
(ii) *K is the dual of a translation oval in* PG(2, q) *for any even q.*

We state a corollary on Bolyai-Lobacevsky planes.

Theorem 3.5. *Let K be a maximal (k, n)-arc with $n > 2$ in a Galois plane of order $q = 2^h$ such that the finite Bolyai-Lobacevsky plane arising from K is point-homogeneous as it has an inherited point-transitive symmetry group G. Then K is a the dual of a hyperoval Ω. Furthermore, Ω is either a regular hyperoval, or a translation oval. In the latter case G preserves the special secant ℓ of Ω and acts on the points of Ω outside ℓ as a sharply 2-transitive permutation group.*

The following result provides a useful classification of collineation groups preserving maximal arcs; see [36].

Theorem 3.6 (A. Delandtsheer, J. Doyen). *Let K be a maximal (k, n)-arc in a projective plane of order q. If a collineation goup G preserving K acts transitively on the set of all secant lines to K then one of the following holds:*
(i) *$q = 2^h$, $h \geq 3$, $n = q/2$, K is the dual of a regular hyperoval in* $\mathrm{PG}(2, 2^h)$*, and* $\mathrm{PSL}(2, 2^h) \leq G \leq \mathrm{P\Gamma L}(2, 2^h)$.
(ii) *$q = 4$, $n = 2$, K is a regular hyperoval in* PG(2, 4) *and G is isomorphic to one of the groups* Alt(6)*,* Sym(6)*,* Alt(5)*,* Sym(5).

As a corollary we have the following result on Bolyai-Lobacevsky planes.

Theorem 3.7. *Let K be a maximal (k, n)-arc with $n > 2$ in a projective plane of even order q such that the finite Bolyai-Lobacevsky plane arising from K is line-homogeneous as it has an inherited line-transitive symmetry group G. Then* (i) *of Theorem 3.6 holds.*

We end our discussion by a generalization of Theorem 3.4 to any projective plane of order $q = 2^h$; see [63].

Theorem 3.8. *Let π be a projective plane of order $q = 2^h$ containing a hyperoval Ω. Let G be a collineation group of π preserving Ω that acts transitively on the set of all external lines to Ω. If G contains no planar collineation, then either π is a Galois plane and Ω is regular hyperoval, or Ω is a translation oval, G preserves the special secant ℓ of Ω and acts on the points of Ω outside ℓ as a sharply 2-transitive permutation group.*

Corollary 3.9. *Let Ω be a hyperoval in a projective plane of order $q = 2^h$ such that the finite Bolyai-Lobacevsky plane arising from the dual of Ω is point-homogeneous as it has an inherited point-transitive symmetry group G. If G contains no planar collineations then Ω is either a Galois plane and Ω is regular hyperoval, or Ω is a translation oval, G preserves the special secant ℓ of Ω and acts on the points of Ω outside ℓ as a sharply 2-transitive permutation group.*

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