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SOME REMARKS ON UNIVERSAL COVERS AND GROUPS

RENATA GRIMALDI and CORRADO TANASI

Abstract. We give a quick review of problems concerning the topological behavior of contractible covering spaces, from the point of view of the topology at infinity. In particular, we briefly describe the evolution of the notion of the simple connectivity at infinity, from low-dimensional topology to the asymptotic topology of discrete groups, also highlighting the work done by V. Poénaru on these topics.

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1. INTRODUCTION

In this short note we will present a concise panoramic view of some more or less recent research on the geometry and topology at infinity of open manifolds and discrete groups, with an emphasis on universal covering spaces of closed *aspherical* (3)-manifolds M (i.e. with $\pi_i(M) = 0, i > 1$).

More precisely, we will introduce and review some results concerning the well-known (3-dimensional) Universal Covering Conjecture, and the related problems about the asymptotic topology of universal covers and finitely presented groups, also following the line of V. Poénaru's research and results in the last decades. In particular, we will briefly explain Poénaru's contributions (of around the 80's) to the above mentioned Conjecture, as well as his more recent progresses, which have largely generalized the question.

2. CONTRACTIBLE UNIVERSAL COVERS

First of all, let us say that, what we call Universal Covering Conjecture is actually a Theorem nowadays, thanks to the impressive recent advances in 3-dimensional geometry and topology after Perelman's works [30, 31, 32] proving both the *Poincaré Conjecture* and the *Thurston's Geometrization Conjecture* [49] (for a detailed proof see [1]).

Before going on, we state what the Conjecture affirms:

Conjecture 1 (Theorem). *The universal covering space of a closed, orientable, aspherical* 3*-manifold* M *is homeomorphic to the Euclidian space* $\mathbb{R}^3$.

The authors wish to thank V. Poénaru for his constant support, friendship and help during their careers, as well as L. Funar for his suggestions and remarks.

Here some comments are needed. First of all, observe that the previous Conjecture was indeed one of the main problems in 3-dimensional topology, together with Poincaré Conjecture. On the other hand, in higher dimensions, this problem-conjecture was studied as well for several years, but without finding a complete solution (in positive or negative) until the 80's (see e.g. [22] or [26]).

The far more general origin of the issue of understanding the topology of closed aspherical manifolds actually goes back to the 50's, when A. Borel asked the question whether two aspherical manifolds with isomorphic fundamental groups are homeomorphic or not.

Notice that the class of *aspherical* (closed) manifolds (i.e. manifolds with a contractible universal covering space) plays an important rôle in various fields of geometry and topology: aspherical manifolds are, doubtless, very interesting and rich objects.

Some of the main natural questions about them are the following ones:

- Which manifolds admit a **contractible** universal covering space?
- Which topological notion may 'detect' the class of contractible universal covering spaces?
- Which groups occur as fundamental groups of aspherical closed manifolds?

On the other hand, it is known that, topologically, there exist uncountably many distinct contractible open topological manifolds of any dimension ≥ 3 [16] (but, of course, not all of them may support a "nice" group action!).

Obviously, Euclidean spaces $\mathbb{R}^n$ are examples of both open contractible manifolds and universal covering spaces. Hence, one may ask the following two correlated problems:

- Are they the only open manifolds of such a type (namely, which are both contractible and universal covers)?
- If not, is it possible to find some necessary and sufficient topological conditions for an open manifold to be homeomorphic to $\mathbb{R}^n$?

The first attempts for 'detecting' Euclidean spaces among open contractible topological manifolds go back to L. Siebenmann's Thesis in the 60's. In [44], he eventually managed to prove that $\mathbb{R}^n$, for $n \geq 5$, may be characterized by a homotopy criterion: the *simple connectivity at infinity* (which generalizes the condition that complements of large compacts are simply connected). To be more precise, before this theorem, J. Stallings had also proved a similar result in [45], but for PL and differentiable manifolds (and using a slightly stronger topological condition at infinity). Notice that the same conclusion holds true in dimension 4 too, but only topologically (and **not** in the PL or differentiable categories): this is a deep result by Freedman in [7]. Finally, in dimension 3, and assuming the validity of Poincaré Conjecture (and hence simplifying considerably the task) the same result was proved in [5].

To sum up, we can now state the following very deep and difficult result:

Theorem 1 (J. Stallings, [45]; L. Siebenmann, [44]; M. Freedman, [7]; G. Perelman, [30, 31, 32], [1]). *For any $n \geq 3$, the simple connectivity at infinity characterizes Euclidean spaces $\mathbb{R}^n$ among open contractible n-manifolds. [Only topologically for $n = 4$, and, for $n = 3$ the manifold has to be irreducible.]*

At this point it is worthy to recall that a feature of 3-manifolds, in contrast to higher dimensions, is that there is no essential difference between smooth, piecewise linear (PL), and topological manifolds. For instance, it was shown by Bing and Moise in the 50's that every topological 3-manifold has a unique smooth structure, and the classifications up to diffeomorphism and homeomorphism coincide.

Coming back to the Conjecture 1, the origin of the systematic study of the topology at infinity of the universal covering spaces of closed aspherical n-manifolds, actually dates back to the late 60's, when S. Novikov asked a number of related questions concerning closed n-manifolds M^n which are $K(\pi, 1)$: he was maybe the first who started wondering about the simple connectivity at infinity of theirs universal covers.

Then, in [20], it was conjectured for the first time that Conjecture 1 may possibly be true in all dimensions, and, in the subsequent [21], even some partial results of this "generalized conjecture" were proved.

The reason why people at that time thought that, for any $n \geq 3$, the only contractible universal cover may be $\mathbb{R}^n$, is indeed very simple: for a long time Euclidean spaces were the **only** known examples of such type; and so was actually until the 80's (see e.g. [22]). Eventually, the story of aspherical manifolds of higher dimension changed towards an exotic panorama when M. Davis, in his seminal paper [4], provided the very first construction of closed, aspherical manifolds whose universal covers are not simply connected at infinity. What he proved implies for instance (after Theorem 1) that there are, in every dimension ≥ 4, contractible manifolds **not** homeomorphic to the Euclidean spaces, but still admitting a proper free action of some discrete group (e.g. Coxeter or hyperbolic) with compact orbit space. This result opened a very active research field for understanding topologically the exotic behavior at infinity of contractible manifolds and discrete groups.

Theorem 2 (M. Davis, [4]). *For each $n \geq 4$, there is a closed aspherical n-manifold such that its universal cover is not homeomorphic to $\mathbb{R}^n$.*

Thanks to this Theorem, we know today that Conjecture 1 does not extend in higher dimensions; however, the original 3-dimensional problem remained totally unsolved until Perelman's proof of the full Thurston Geometrization Conjecture (see [1, 30, 31, 32, 49]). But what is even more surprising, is that until now, as far as we know, there is still no "direct" topological proof of this important theorem, i.e. a result which makes no use of the complicated Ricci flow analytic techniques. It is clear that such a proof would be very interesting: V. Poénaru has recently proposed a new strategy for a even more general (conjectural) result (for the announcement see [39], and see also [26]).

2.1. The case of dimension 3. Let us recall now some basics on the topology and geometry of 3-manifolds. A connected closed 3-manifold M is said to be *prime* if it cannot be expressed as a non-trivial connected sum of two 3-manifolds, and it is called *irreducible* if any embedded sphere bounds a ball. Obviously, an irreducible closed 3-manifold is prime, while a prime closed 3-manifold is irreducible, provided it is not an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. Finally, a closed orientable 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group (by the Sphere Theorem, see [49]).

By Thurston Geometrization Conjecture, it follows that a closed 3-manifold is aspherical if and only if its universal cover is homeomorphic to $\mathbb{R}^3$. (For more details on the Conjecture see [49], for its proof see [1, 30, 31, 32])

Roughly speaking, Thurston Geometrization Conjecture affirms that every compact, orientable 3-manifold M can be decomposed (canonically) into finitely many pieces P_i, each of which admits a unique geometric structure (as a complete, locally homogeneous, Riemannian manifold); and the decomposition is done by cutting the manifold along certain 2-spheres and tori. Furthermore, the universal cover $\tilde{P}_i$ of each component P_i, carries a complete, homogeneous metric in which the covering transformations are isometries, and hence $\tilde{P}_i$ is called the geometry on which P_i is modelled. And moreover, only eight possible homogeneous geometries can arise in the geometrization conjecture. These are the following ones. Firstly, those of constant curvature, namely: the sphere $\mathbb{S}^3$ of curvature $+1$, the Euclidean space $\mathbb{R}^3$ of curvature 0, and the hyperbolic space $\mathbb{H}^3$ having curvature -1. Then, there are the product metrics on $\mathbb{S}^2 \times \mathbb{R}^1$ and $\mathbb{H}^2 \times \mathbb{R}^1$. Finally, the last three are: the universal cover of $SL(2, \mathbb{R})$, the Heisenberg group $\mathcal{H}$ (of upper triangular 3×3 matrices with diagonal entries 1), and the geometry *Sol*, which, as $\mathcal{H}$, can be thought of as $\mathbb{R}^3$ with specific multiplication and metric.

Thus, the only contractible 3-dimensional covers with compact quotient are all, topologically, the Euclidean 3-space $\mathbb{R}^3$, so that Conjecture 1 may be stated as a Theorem today.

Nevertheless, as one sees, it is needed the full comprehension of the whole word of the topology and the geometry of 3-manifolds, in order to understand "just" the (a priori easier) topological behavior at infinity of covering spaces of aspherical 3-manifolds.

For this reason, the recent strategy by V. Poénaru announced in [39], for possibly showing that **all** finitely presented groups share the same topological condition (namely the *quasi-simple filtration*, or QSF property) should be considered very useful and interesting, besides its own interest *per se* in the world of geometric group theory (for more on the QSF property see [3] as well as [11, 26]). Un to now, there is no written detailed proof of this conjectural statement (only the very first part in [40]), but the whole subject deserves to be studied more and carefully verified (for more see also [27, 28, 29, 43]).

More precisely, if one admits that all groups are QSF, then in particular all 3-manifold groups will be QSF too. Now, it is a feature of dimension 3 that, for open manifolds, being QSF is equivalent, to the property of having an exhaustion by compact and simply connected sub-manifolds (the so-called *weak geometric simple connectivity*), which is, in turn, equivalent (and this again only in dimension 3) to the simple connectivity at infinity (see [3, 34] and also [8, 11, 25]). Hence, the previously discussed topological characterization of $\mathbb{R}^n$ (in dimension at least 3) completes the proof of Conjecture 1.

In this way, the main conjectural result of [39] could furnish the first purely topological proof of Conjecture 1, i.e. bypassing the geometric and analytical methods of Perelman's technology.

Another consequence of Poénaru's conjectural result in the announcement [39], would be the proof of the following statement, which largely generalizes Conjecture 1 (and which was stated as a conjecture in [47, 8, 11, 26]):

CONJECTURE 2 ([47, 8]). *In any dimension, the universal covering space of a closed, aspherical manifold is* QSF.

REMARK 1. Notice there exist infinitely many open contractible manifolds which are not QSF by results in [8, 9]: the so-called Whitehead-types manifolds. These manifolds generalize the classical Whitehead 3-manifold [50], that is a contractible open 3-manifold W which is not simply connected at infinity (nor QSF), and hence not homeomorphic to $\mathbb{R}^3$ (even if, it turns out that the product of W with an open interval, $W \times (0,1)$, is homeomorphic to $\mathbb{R}^4$).

It is still an open problem whether one of these contractible manifolds could cover a closed manifold: this would follow from Poénaru's (conjectural) statement of [39].

We end this section by noticing that another recent reformulation and generalization of Conjecture 1 may be found in [9].

3. END-TOPOLOGY OF GROUPS

What is also interesting in the above mentioned paper [21], is that one may find there some connections between the topology at infinity of the universal cover of a closed manifold M with the end-topology of its fundamental group. More precisely, the main result in [21] (see also [19]) states that the generalized universal covering conjecture is true if and only if the fundamental group of M is *one-ended*, *stable at infinity* and *simply connected at infinity.*

Recall that a non compact space X is said to be *one-ended*, if for every compact subset $C \subset X$, there is another compact D, with $C \subset D \subset X$, such that $X - D$ has only one connected component. [This simply means that there is only one way to move to infinity within the space].

If one considers discrete groups, the definition above may be adapted as follows: the ends of a finitely generated group are defined to be the ends of the corresponding *Cayley graph*, and this definition is "insensitive" to the choice of the generating set (see [13]). Unlike general topological spaces, every finitely generated group has either 1, 2, or infinitely many ends (this is a classical result by Hopf, see [33]); moreover, Stallings Theorem on ends of groups (see [46]), provides even a decomposition (as amalgamated products or HNN-extensions) for groups with more than one end (see also [33] for a detailed proof and for the connections with the topology of 3-manifolds).

Following Gromov's program on the quasi-isometry classification of discrete groups [17], it was shown in [2] that the number of ends of a group is a geometric invariant (meaning invariant under *quasi-isometry*). Furthermore, it is also possible to "measure the kind of one-endedness" defining the *end-depth* of a one-ended metric space as the growth-rate of the depth of bounded connected components of complements of compacts (see [24]); nevertheless, for finitely presented groups such a function turns out to be always linear [12].

For non-compact topological spaces, since Siebenmann's Thesis (as well as [20]), one started to try to define the notion of the *fundamental group of an end*, by means of inverse limits of (fundamental) groups associated to exhaustions of non compact spaces. Roughly speaking, Siebenmann defined an end of a non-compact space X to be *semi-stable* if, in the tower of fundamental groups $\pi_1(X - K_i)$, associated to the exhaustion of connected compacts $X = \cup K_i$, all the morphisms are surjective. The end is called *stable* if the morphisms are all isomorphisms (for precise definitions and results see [13]).

Successively, Geoghegan and Mihalik in [14], defined and studied the *fundamental group at infinity* for the class of groups which are *semistable at infinity*. For a non-compact complex X, a proper map from $[0, \infty)$ to X is called a *proper ray*, and two proper rays are said to define the *same end* if their restrictions to the subset of natural numbers are properly homotopic. Finally, an end of X is called *semistable at* ∞ if any two proper rays defining this end are properly homotopic.

These notions may be defined for groups too: a finitely presented group G is said semistable at ∞ (respectively, it has semi-stable ends) if there exists a compact polyhedron X, with G as π_1, whose universal covering space is semistable at ∞ (resp. has semistable ends).

Many classes of finitely generated groups are known to be semistable at ∞, and actually, is is still unknown if all finitely presented groups are semistable at ∞ or not. Whenever a finitely presented group G is semistable at ∞, then one can define the fundamental group at an end of G, and this definition is independent of choice of base points and on the rays in some associated space (see [14]). The main results and references concerning the semistability at infinity may be found in [13, 14], but see also the recent interesting paper [9] for connections with 3-dimensional manifolds.

Of course, whenever the fundamental group at ∞ is trivial, the end is called simply connected at infinity. However, there is a simpler and just topological (bur still very useful) notion of *simple connectivity at infinity*, which makes no use of inverse limits and proper rays.

Definition 1. A connected complex X is *simply connected at infinity* (sci) if for each compact $C \subset X$ there exists a compact $D \subset X$ such that loops in $X - D$ are homotopically trivial in $X - C$. (This simply means that any far away loop bounds a disk which is enough far away).

Simple connectivity at infinity implies semistability at ∞, and, as it, the idea of simple connectivity at infinity can be extended from spaces to discrete groups as follows (for an extensive discussion on the subject see [13]).

Definition 2. A finitely presented group G is simply connected at infinity if there exists a compact complex X such that $\pi_1 X = G$ and whose universal covering space is simply connected at infinity.

It turns out that this definition does not depend neither on the finite presentation of the group nor on the compact space X one may choose: for a topological proof see [48], for a more geometrical proof see [23]). What is also interesting is that the simple connectivity at infinity for groups is an *asymptotic invariant*, in the sense that it is preserved by quasi-isometries (see [2] and, for a geometric group theory proof, [10]). Summarizing up, one has:

Theorem 3 (S. Brick, [2], see also [10, 23, 48]). *The simple connectivity at infinity is a well-defined quasi-isometry invariant of finitely presented groups (i.e. if $\Gamma = \pi_1 X$ with X a finite complex whose universal cover is simply connected at infinity, then any other compact space with Γ as fundamental group has a simply connected at infinity universal cover; moreover, any group H quasi-isometric to Γ, will also be simply connected at infinity).*

Further generalizations were investigated in [10, 12], where the authors somehow "refined" the notion of simple connectivity at infinity, trying to compute it metrically: they defined a function, called the *rate of vanishing of the* sci, or *sci-growth*, that measures the growth rate of how far one needs to go in order to kill loops outside large compacts (i.e., given r, how large needs to be $R(r)$ to be able to kill all loops in $X - B(R)$ outside $B(r)$).

In [10, 12] it is shown that the sci-growth is an asymptotic invariant too, and, moreover, that it is linear (i.e. trivial) in many (geometric) cases, such as 3-manifold groups, (most) lattices in Lie groups, or sci hyperbolic groups. Nevertheless, it is still an open question whether there exist groups with a super-linear sci-growth (or if it is always linear for discrete groups).

To end this section, we state some properties of the simple connectivity at infinity for groups, and we present some examples and classes of sci groups.

First of all, it follows from the definition that, a finitely presented group is sci if and only if a finite index subgroup of it is sci (their respective standard 2-complexes have the same universal cover). Furthermore, the class of groups that are simply connected at infinity is closed under amalgamated free products over one-ended groups (but not over multi-ended groups, see [18]).

Another source of examples of SCI groups comes from extensions: in [19] it is proved that if $1 \to H \to G \to K \to 1$ is a short exact sequence of infinite finitely presented groups and either H or K is 1-ended, then G is SCI. A further implementation of these results may be given as follows (see [13]): if $N \trianglelefteq H \trianglelefteq G$ are groups with N and G finitely presented and N 1-ended, then, if G/H or H/N is infinite, then G is simply connected at infinity.

Other examples of SCI groups are, of course, fundamental groups of closed, aspherical 3-manifolds; on the other hand, the above discussed examples of [4] consist actually of closed manifolds whose fundamental groups are semistable at infinity, but are **not** SCI (and hence these manifolds are not covered by $\mathbb{R}^n$). Note that all of these (fundamental) groups are in fact finite index subgroups of finitely generated Coxeter groups (which are semistable at infinity, see [13]).

4. POÉNARU'S CONTRIBUTION

In this final section we come back to the Universal Covering Conjecture in dimension 3 and to some attempts to prove it.

Despite all the efforts, the understanding of the structure at infinity of open simply connected 3-manifold was not so deep until the 90's. Things then changed when A. Casson and V. Poénaru (see in particular [15] and [37]) initiated (independently of one other) a group-theoretical approach to Conjecture 1 (following Gromov's new theory of the geometry of groups): they developed some ideas about the metric geometry of the Cayley graph of the fundamental group of manifolds to solve the 3-dimensional Conjecture for those closed 3-manifolds whose fundamental groups possess some "nice" geometric conditions. A typical result may be stated as follows:

THEOREM 4 (A. Casson, [15]; V. Poénaru, [34, 35, 37, 41, 42]). *Take $G = \pi_1 M^3$ where M^3 is a closed 3-manifold. Assume that the group G satisfies a "nice geometric condition" such as: hyperbolicity (in the sense of Gromov* [17]*), automaticity, or, more generally, combability (in the sense of Thurston* [49]*), or Casson's condition $\hat{C}_\alpha$ (see* [15]*). Then $\widetilde{M}^3$ is simply connected at infinity. [If, in addition, one assumes that M^3 is irreducible, then $\widetilde{M}^3 = \mathbb{R}^3$].*

Although these results are today superseded by Perelman's breakthrough (as well as by Poénaru's recent announcement [39]), it is still worthy to spend some comments about them in order to understand the main ideas, and to be able to state and insert these results in a more recent viewpoint.

More precisely, in [35, 37], Poénaru illustrated several conditions on $\pi_1 M^3 = G$, each implying that the universal cover of M^3 is simply connected at infinity. A first condition is that G should be *almost convex* (in the sense of Cannon) with respect to a finite set of generators (and this roughly means that the "curvature" of the n-sphere in the Cayley graph of the group is bounded from below, independently of the radius n), while the second condition is that G should admit a *quasi-Lipschitz combing* in the sense of Thurston. [Note that this implies, in particular, that the result holds true both for *Gromov-hyperbolic* groups and for *automatic* groups, see [6]].

Both results were then extended by Poénaru and Tanasi, respectively in [41] for more general combings (called *Hausdorff combings*) and in [42], where they defined the class of *weakly almost convex groups*, which generalizes Cannon's original almost convexity condition.

More interestingly, in [34], the first of this series of papers concerning this issue, Poénaru also related the problem of the simple connectivity at infinity of an open simply connected 3-manifold to the *geometric simple connectivity* (i.e. the condition of possessing a handlebody decomposition without handles of index one) of its stabilization with a ball. In particular he proved that:

THEOREM 5 (V. Poénaru, [34]). *Let V^3 be a smooth open 3-manifold. Then, the existence of a handlebody decomposition without handles of index one for $V^3 \times B^p$, $p \geq 1$, implies the simple connectivity at infinity of the manifold V^3.*

Notice that, in order to be able to apply the techniques and the results of all these papers especially in dimension 3, namely to prove the above mentioned theorems concerning Conjecture 1, Poénaru made use of his own "Dehn-type lemma" (proved in [34]), which is a result of independent interest in dimension 3, but which does not admit a generalization in higher dimensions (and so the above theorems are purely 3-dimensional results). This lemma says, more or less, that an open and simply connected 3-manifold which is *Dehn-exhaustible* (which roughly means that any compact subset of the manifold is homeomorphically contained in the image of a simply connected abstract compact) possesses actually itself an exhaustion by compact and simply connected sub-manifolds (and hence, as already said, being in dimension 3, this implies that the manifold is simply connected at infinity).

However, if one looks more carefully at all these papers, and if one does not pay attention to the (somehow irrelevant) dimension issues, then, in a more appropriate and recent language, what they really do prove is that: "a finitely presented group G which satisfies one of the nice geometric conditions mentioned before (e.g. almost convex, combable, hyperbolic, automatic, etc.), admits an *easy-representation*" (for this condition we refer to [38, 43, 39] and to [26, 27, 29]). Then, thanks to this result, it can be shown that the manifold V^3 is Dehn-exhaustible, and this completes the proof (this is actually the common strategy to all these papers by Poénaru, even if it was not explicitly mentioned in this way).

Thus, the underlying idea of all these papers and proofs is actually this particular notion of *(inverse) representation* (of manifolds and discrete groups G), following the original ideas already developed in [36] (see also [38]).

Roughly speaking, a group G is "presented" as the fundamental group of a compact singular 3-manifold $M^3(G)$, and an inverse *representation* is a certain kind of non-degenerate simplicial map $f : X^2 \to \widetilde{M}^3(G)$, satisfying several topological conditions (e.g. the 2-complex X^2 at the source has to be geometrically simply connected). If, in addition, the set of double points of f, $M_2(f) \subset X^2$, is closed (and this in not always verified), then the (representation and the) group G admitting such a representation is called *easy*.

We do not want to go further into these topics, but let us just remark that a very nice open question now is to understand the relations between this notion of easy-representability and the QSF property, and to see whether or not all groups are easy. Some first results concerning this problems may be found in [27, 28], the general idea being that the QSF should be equivalent, following [11], to the easy-representability condition for groups (see also [26, 29]).

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Università Degli Studi di Palermo
D.E.I.M.
viale delle Scienze, Ed. 9
90128 Palermo, Italy
E-mail: renata.grimaldi@unipa.it

Università Degli Studi di Palermo
Dipartimento di Matematica e Informatica
via Archirafi 34, 90123 Palermo, Italy
E-mail: corrado.tanasi@unipa.it