MATHEMATICA, Tome 56 (79), N° 1, 2014, pp. 41–46

AN ANALOG OF TITCHMARSH'S THEOREM FOR THE DUNKL TRANSFORM IN THE SPACE $\mathrm{L}^p(\mathbb{R}^d, w_k(x)\mathrm{d}x)$

MOHAMED EL HAMMA and RADOUAN DAHER

Abstract. Using a generalized spherical mean operator, we obtain an analog of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (β, k)-Dunkl Lipschitz condition in the space $\mathrm{L}^p(\mathbb{R}^d, w_k(x)\mathrm{d}x)$, where $1 < p \leq 2$ and w_k is a weight function, invariant under the action of an associated Weyl group.

MSC 2010. 42B10, 47B48, 33C52.

Key words. Dunkl operator, Dunkl transform, generalized spherical mean operator.

1. INTRODUCTION AND PRELIMINARIES

The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we mention the exponential function, the Fourier transform and the translation operator. For more details about these operators we refer to [1, 2, 4, 7] and the references therein.

In [10] it is proved the following result.

THEOREM 1. *Let $\alpha \in (0,1)$ and assume that $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalent:*

(1) $\|f(x+h) - f(x)\|_{\mathrm{L}^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,

(2) $\int_{|\lambda| \geq r} |\widehat{f}(\lambda)| d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow +\infty$,

where $\widehat{f}$ stands for the Fourier transform of f.

The aim of this paper is to establish an analog of Theorem 1 for the Dunkl transform in the space $\mathrm{L}^p(\mathbb{R}^d, w_k(x)\mathrm{d}x)$. For this purpose, we use a generalized spherical mean operator.

Let W be a finite reflection group on $\mathbb{R}^d$, associated with a root system R. Let R_+ be the positive subsystem of R (see [1, 3, 4, 5, 8, 9]). We denote by k a nonnegative multiplicity function defined on R with the property that k is W-invariant. We associate with k the index

$$\gamma = \sum_{\xi \in \mathrm{R}_+} k(\xi) \geq 0,$$

and the weight function w_k, defined by

$$w_k(x) = \prod_{\xi \in \mathrm{R}_+} |\langle \xi, x \rangle|^{2k(\alpha)}, \; x \in \mathbb{R}^d.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$.

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplan $\mathrm{H}_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{|\alpha|^2}\alpha.$$

Introduced by C. F. Dunkl in [2], the Dunkl operators D_j, $1 \leq j \leq d$, are defined by

$$\mathrm{D}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathrm{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \; x \in \mathbb{R}^d,$$

where $\alpha_j := \langle \alpha, e_j \rangle$ and $\{e_1, e_2, ..., e_d\}$ is the canonical basis of $\mathbb{R}^d$.

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C. F. Dunkl in [3]. For $y \in \mathbb{R}^d$ the function $x \mapsto E_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} \mathrm{D}_j u(x, y) = y_j u(x, y), & \text{for } \; 1 \leq j \leq d \\ u(0, y) = 1, & \text{for all } \; y \in \mathbb{R}^d. \end{cases}$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. M. Rösler has proved in [7] the following integral representation for the Dunkl kernel

$$\mathrm{E}_k(x, z) = \int_{\mathbb{R}^d} \mathrm{e}^{\langle y, z \rangle} \mathrm{d}\mu_x(y), \; x \in \mathbb{R}^d, \; z \in \mathbb{C}^d,$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $\mathrm{B}(0, |x|)$ of center 0 and radius $|x|$.

We recall from [1] the following result.

Proposition 1. Let $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then the following assertions hold.

(1) $E_k(z, 0) = 1$.
(2) $E_k(z, w) = E_k(w, z)$.
(3) $E_k(\lambda z, w) = E_k(z, \lambda w)$.
(4) For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d, \; x \in \mathbb{R}^d, \; z \in \mathbb{C}^d$, the inequality

$$|\partial_z^\nu E_k(x, z)| \leq |x|^{|\nu|} \exp(|x| \mathrm{Re}\,(z)|)$$

holds, where

$$\partial_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}...\partial z_d^{\nu_d}}, \ |\nu| = \nu_1 + ... + \nu_d.$$

In particular,

$$|\partial_z^\nu E_k(ix, z)| \leq |x|^\nu, \text{ for all } x, \, z \in \mathbb{R}^d.$$

Let η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set

$$\mathrm{d}\eta_k(y) = w_k(y)\mathrm{d}\eta(y).$$

Then η_k is a W-invariant measure on $\mathbb{S}^{d-1}$. Put $d_k := \eta_k(\mathbb{S}^{d-1})$.

We denote by $\mathrm{L}_{p,k} := \mathrm{L}^p(\mathbb{R}^d, w_k(x)\mathrm{d}x)$, $1 < p \leq 2$, the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,k} = \left(\int_{\mathbb{R}^d} |f(x)|^p w_k(x)\mathrm{d}x\right)^{1/p} < +\infty.$$

The Dunkl transform is defined for $f \in \mathrm{L}_{p,k}$ by

$$\mathcal{F}_k(f)(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x)E_k(-\mathrm{i}\xi, x)w_k(x)\mathrm{d}x,$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} \mathrm{e}^{\frac{-|z|^2}{2}} w_k(z)\mathrm{d}z.$$

The inversion formula is then

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi)E_k(\mathrm{i}x, \xi)w_k(\xi)\mathrm{d}\xi, \ x \in \mathbb{R}^d.$$

By Plancherel's theorem and the Marcinliewicz interpolation theorem (see [10]) we get, for $f \in \mathrm{L}_{p,k}$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_k(f)\|_{q,k} \leq C\|f\|_{p,k}, \tag{1}$$

where C is a positive constant.

K. Trimèche has intoduced in [11] the Dunkl translation operators T_h on $\mathrm{L}_{p,k}$. For $f \in \mathrm{L}_{p,k}$ we have

$$\mathcal{F}_k(\mathrm{T}_h(f))(\xi) = E_k(\mathrm{i}x, \xi)\mathcal{F}_k(f)(\xi)$$

The generalized spherical mean operator of $f \in \mathrm{L}_{p,k}$ is defined by

$$\mathrm{M}_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \mathrm{T}_h(f)(hy)\mathrm{d}\eta_k(y), \text{ for } x \in \mathbb{R}^d \text{ and } h > 0.$$

For $\alpha \geq -\frac{1}{2}$, we introduce the normalized Bessel function of the first kind j_α by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{z}{2})^{2j}}{j!\Gamma(j+\alpha+1)},\ z \in \mathbb{C}. \tag{2}$$

We see from (2) that

$$\lim_{z\to 0}\frac{j_{\gamma+\frac{d}{2}-1}(z)-1}{z^2} \neq 0,$$

hence, there exist $c > 0$ and $\eta_1 > 0$ such that

$$|z| \leq \eta_1 \Longrightarrow |j_{\gamma+\frac{d}{2}-1}(z) - 1| \geq c|z|^2. \tag{3}$$

LEMMA 1. *Let $f \in \mathrm{L}_{p,k}$ and fix $h > 0$. Then $\mathrm{M}_h f \in \mathrm{L}_{q,k}$ and*

$$\mathcal{F}_k(\mathrm{M}_h f)(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\mathcal{F}_k(f)(\xi),$$

where $\frac{1}{p}+\frac{1}{q} = 1$

Proof. Analog to the proof of Proposition 1.3 in [6]. □

The first and higher order finite differences of $f(x)$ are defined as follows:

$$\begin{aligned} &\Delta_h f(x) = \mathrm{M}_h f(x) - f(x) = (\mathrm{M}_h - \mathrm{I})f(x), \\ &\Delta_h^m f(x) = \Delta_h(\Delta_h^{m-1}f(x)) = (\mathrm{M}_h - \mathrm{I})^m f(x) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\mathrm{M}_h^i f(x), \end{aligned} \tag{4}$$

where $\mathrm{M}_h f(x) = f(x)$, $\mathrm{M}_h^i f(x) = \mathrm{M}_h(\mathrm{M}_h^{i-1}f(x))$, $i \in \{1, 2, \dots, m\}$, m is a positive natural number, and I is the unit operator in the space $\mathrm{L}_{p,k}$.

2. MAIN RESULT

In this section we give the main result of this paper. We need first to define the (β, k)-Dunkl Lipschitz class.

DEFINITION 1. Let $\beta > 0$. A function $f \in \mathrm{L}_{p,k}$ is said to be in the (β, k)-Dunkl Lipschitz class, denoted by $Lip(\beta, p, k)$, if

$$\|\Delta_h^m f(x)\|_{p,k} = O(h^\beta) \ as \ h \longrightarrow 0,$$

where $m \in \{1, 2, \dots\}$.

THEOREM 2. *Let $f(x)$ belong to $Lip(\beta, p, k)$. Then*

$$\int_{|\xi|\geq r}|\mathcal{F}_k(f)(\xi)|^q w_k(\xi)\mathrm{d}\xi = O(r^{-q\beta}) \ as \ r \longrightarrow +\infty.$$

Proof. Let $f \in Lip(\beta, p, k)$. Then we obtain that

$$\|\Delta_h^m f(x)\|_{p,k} = O(h^\beta) \text{ as } h \longrightarrow 0.$$

The formulas (1) and (4) yield

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qm} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi \leq C^q \|\Delta_h^m f(x)\|_{p,k}^q.$$

We use now formula (3) in order to obtain the inequality

$$\int_{\frac{\eta_1}{2h} \leq |\xi| \leq \frac{\eta_1}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qm} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi \geq \frac{c^{qm}\eta_1^{2qm}}{2^{2qm}} \int_{\frac{\eta_1}{2h} \leq |\xi| \leq \frac{\eta_1}{h}} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi.$$

Thus there exists $K > 0$ such that

$$\begin{aligned}
\int_{\frac{\eta_1}{2h} \leq |\xi| \leq \frac{\eta_1}{h}} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi
&\leq K \int_{\frac{\eta_1}{2h} \leq |\xi| \leq \frac{\eta_1}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qm} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qm} |\mathcal{F}_k(f)(\xi)|^2 w_k(\xi) \mathrm{d}\xi \\
&\leq KC^q \|\Delta_h^m f(x)\|_{p,k}^q \\
&\leq C_1 h^{q\beta},
\end{aligned}$$

where $C_1 := KC^q$. Thus

$$\int_{r \leq |\xi| \leq 2r} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi = O(r^{-q\beta}).$$

Furthermore, we have that

$$\begin{aligned}
\int_{|\xi| \geq r} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi
&= \left[\int_{r \leq |\xi| \leq 2r} + \int_{2r \leq |\xi| \leq 4r} + \int_{4r \leq |\xi| \leq 8r} \cdots \right] |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi \\
&\leq C_1 r^{-q\beta} + C_1 (2r)^{-q\beta} + C_1 (4r)^{-q\beta} + \ldots \\
&\leq C_1 r^{-q\beta} (1 + 2^{-q\beta} + (2^{-q\beta})^2 + (2^{-q\beta})^3 + \ldots) \\
&\leq C_1 C_2 r^{-q\beta},
\end{aligned}$$

where $C_2 := (1 - 2^{-q\beta})^{-1}$. This proves that

$$\int_{|\xi| \geq r} |\mathcal{F}_k(f)(\xi)|^q w_k(\xi) \mathrm{d}\xi = O(r^{-q\beta}) \text{ as } r \longrightarrow +\infty,$$

which finishes the proof. □

REFERENCES

[1] De Jeu, M.F.E., *The Dunkl transform*, Invent. Math., **113** (1993), 147–162.
[2] Dunkl, C.F., *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc., **311** (1989), 167–183.
[3] Dunkl, C.F., *Integral Kernels with reflection group invariance*, Canad. J. Math., **43** (1991), 1213–1227.
[4] Dunkl, C.F., *Hankel transforms associated to finite reflection groups*, Contemp. Math., **138** (1992), 123–138.
[5] Dunkl, C.F. and Xu, Y., *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, **81**, Cambridge University Press, Cambridge, 2001.
[6] Maslouhi, M., *An analog of Titchmarsh's Theorem for the Dunkl transform*, Integral Transforms Spec. Funct., **21** (2010), 10, 771–778.
[7] Rösler, M., *Positivity of Dunkl's intertwining operator*, Duke Math. J., **98** (1999), 445–463, [arxiv.org/abs/q-alg/9710029].
[8] Rösler, R. and Voit, M., *Markov processes with Dunkl operators*, Adv. in Appl. Math., **21** (1998), 575–643.
[9] Thangavelyu, S. and Xu, Y., *Convolution operator and maximal function for Dunkl transform*, J. Anal. Math., **97** (2005), 25–55.
[10] Titchmarsh, E.C., *Introduction to the Theory of Fourier Integrals*, Oxford University Press, Amen House, London, E.C. 4, 1948.
[11] Trimèche, K., *Paley-Wiener theorems for the Dunkl transform and Dunkl transform operators*, Integral Transforms Spec. Funct., **13** (2002), 17–38.

Received April 15, 2013
Accepted April 2, 2014

University of Hassan II
Faculty of sciences Aïn Chock
Department of Mathematics
Casablanca, Morocco
E-mail: m_elhamma@yahoo.fr
E-mail: rjdaher024@gmail.com