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A CLASS OF UNIVALENT CLOSE-TO-CONVEX FUNCTIONS

B. USNA BANU and G.P. YOUVARAJ

Abstract. We introduce and study a new class of univalent close-to-convex functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We obtain an integral representation and coefficient bounds for these functions.
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1. INTRODUCTION

Let $\mathcal{S}$ denote the class of all univalent (one-to-one and analytic) functions f defined on $\mathbb{D}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the subclasses of $\mathcal{S}$ containing the convex, the starlike with respect to the origin, and the close-to-convex functions. It is well known that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

In 1969, P. Mocanu [6] introduced the concept of α-starlike functions, for $\alpha \in [0, 1]$, which satisfy

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0,\ z \in \mathbb{D} \setminus \{0\},$$

and proved that they belong to $\mathcal{S}^*$.

Following this, in 1977, P.N. Chichra [1] introduced a new subclass of close-to-convex functions, namely the α-close-to-convex functions, for each $\alpha \in [0, 1]$, by requiring that, for some $\phi \in \mathcal{S}^*$,

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(\frac{(zf'(z))'}{\phi'(z)}\right)\right) > 0,\ z \in \mathbb{D} \setminus \{0\}.$$

In 2003, H. Irmak and R.K. Raina [3] introduced a class of $p-$valent starlike and convex functions by requiring that

$$\operatorname{Re}\left(\frac{(1-\lambda)zf'(z) + \lambda z(zf'(z))'}{(1-\lambda)f(z) + \lambda zf'(z)}\right) > \alpha,\ z \in \mathbb{D} \setminus \{0\},$$

where $0 \le \lambda \le 1$, $0 \le \alpha < p$, $p \in \mathbb{N}$, and f is of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n.$$

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For $\alpha, \lambda \in [0,1]$ and $\alpha \neq 1$, we introduce a new class $\mathcal{K}_\alpha^\lambda$ of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ by requiring that, for some $\phi \in S^*$,

$$\operatorname{Re}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)\phi(z) + \lambda z\phi'(z)}\right) > \alpha,\ z \in \mathbb{D} \setminus \{0\}.$$

We prove that it is a subclass of close-to-convex functions. For simplicity, we denote $\mathcal{K}_0^\lambda$ by $\mathcal{K}^\lambda$.

2. PRELIMINARIES

Definition 2.1. A domain $\Omega \subset \mathbb{C}$ is said to be *starlike with respect to* $z_0 \in \Omega$ if, for each $z \in \Omega$, the segment $[z_0, z]$ joining z_0 and z lies in Ω. An analytic univalent function f defined on $\mathbb{D}$ is said to be *starlike* if $f(\mathbb{D})$ is starlike with respect to $f(z_0) = w_0 \in f(\mathbb{D})$ for some $z_0 \in \mathbb{D}$. In this case we also say that f is *starlike with respect to* z_0.

It is well-known (see [2]) that a function $f \in \mathcal{S}$ is starlike with respect to $w_0 \in f(\mathbb{D})$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - w_0}\right) > 0,\ z \in \mathbb{D} \setminus \{0\}.$$

If f is starlike with respect to the origin, then we say that f is a starlike function. A function $f \in \mathcal{S}$ is convex if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}.$$

Thus $f(z)$ is convex in $\mathbb{D}$ if and only if $zf'(z)$ is starlike in $\mathbb{D}$.

Definition 2.2. A function $f \in \mathcal{S}$ is said to be *close-to-convex* if there exists a $\phi \in \mathcal{S}^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) > 0,\ z \in \mathbb{D}.$$

Definition 2.3. Given two functions f and g that are analytic in $\mathbb{D}$, we say *g is subordinate to f* in $\mathbb{D}$, denoted by $g \prec f$, if there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $g(z) = f(\omega(z)),$ for all $z \in \mathbb{D}$.

3. MAIN THEOREMS

The following lemma can be found in [5].

Lemma 3.1 (Jack's Lemma). *Let $\omega(z)$ be analytic in $\mathbb{D}$ with $\omega(0) = 0$. If $|z_0| = r < 1$ and $|w(z_0)| = \max_{|z|=r} |w(z)|$, then*

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k \geq 1.$$

LEMMA 3.2. *Let N be analytic in $\mathbb{D}$ such that $N(0) = 0 = N'(0) - 1$ and let D be analytic in $\mathbb{D}$ satisfying the conditions $D(0) = 0 = D'(0) - 1$ and, for some $0 < \delta < 1$,*

$$\left|1 - \lambda + \frac{\lambda z D'(z)}{D(z)} - \frac{1}{2\delta}\right| < \frac{1}{2\delta}, \text{ for all } z \in \mathbb{D}. \tag{1}$$

If

$$\operatorname{Re}\left(\frac{(1-\lambda)N(z) + \lambda z N'(z)}{(1-\lambda)D(z) + \lambda z D'(z)}\right) > \alpha - \frac{(1-\alpha)\lambda\delta}{2}, \text{ for all } z \in \mathbb{D},$$

then the inequality $\operatorname{Re}\left(\frac{N(z)}{D(z)}\right) > \alpha$ *holds for all* $z \in \mathbb{D}$.

Proof. Let $\omega(z)$ be an analytic function defined by

$$\frac{N(z)}{D(z)} - \alpha = \frac{1 - \omega(z)}{1 + \omega(z)}(1 - \alpha). \tag{2}$$

Clearly $\omega(0) = 0$. If we have $|\omega(z)| < 1$, $\forall\ z \in \mathbb{D}$, then $\frac{N(z)}{D(z)} - \alpha$ is subordinate to $(1-\alpha)\frac{1-z}{1+z}$, implying that $\frac{N(z)}{D(z)} - \alpha$ maps the unit disc onto a region contained in the right half-plane. Thus $\operatorname{Re}\left(\frac{N(z)}{D(z)}\right) > \alpha$, for all $z \in \mathbb{D}$.

In order to prove $|\omega(z)| < 1$, we argue by contradiction. Then there exists a $z' \in \mathbb{D}$ such that $|\omega(z')| \geq 1$, and therefore we can find, according to Jack's Lemma, a $z_0 \in \mathbb{D}$ such that $|\omega(z_0)| = 1$ and $\frac{z_0 \omega'(z_0)}{\omega(z_0)} = k \geq 1$.

From (2), we have that

$$\begin{aligned} N(z) &= \alpha D(z) + \frac{1-\omega(z)}{1+\omega(z)}(1-\alpha)D(z), \\ N'(z) &= \alpha D'(z) + (1-\alpha)\left(\frac{1-\omega(z)}{1+\omega(z)}D'(z) - \frac{2\omega'(z)}{(1+\omega(z))^2}D(z)\right). \end{aligned}$$

Let

$$\begin{aligned} \psi(z) &:= \frac{(1-\lambda)N(z) + \lambda z N'(z)}{(1-\lambda)D(z) + \lambda z D'(z)} \\ &= \alpha + (1-\alpha)\left(\frac{1-\omega(z)}{1+\omega(z)} - \frac{2\lambda z \omega'(z)}{(1+\omega(z))^2}\frac{D(z)}{(1-\lambda)D(z) + \lambda z D'(z)}\right). \end{aligned}$$

On the other hand, we have that $\omega(z_0) = \mathrm{e}^{\mathrm{i}\theta}$ and $\dfrac{z_0\omega'(z_0)}{\omega(z_0)} = k \geq 1$. Hence we get

$$\begin{aligned}\psi(z_0) &= \alpha + (1-\alpha)\left(\frac{1-\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}} - \frac{2\lambda k\mathrm{e}^{\mathrm{i}\theta}}{(1+\mathrm{e}^{\mathrm{i}\theta})^2}\frac{D(z_0)}{(1-\lambda)D(z_0)+\lambda z_0 D'(z_0)}\right)\\ &= \alpha + (1-\alpha)\left(-\frac{2\mathrm{i}\sin(\theta/2)}{2\cos(\theta/2)} - \frac{2\lambda k}{(2\cos(\theta/2))^2}\frac{1}{(1-\lambda)+\lambda[z_0D'(z_0)/D(z_0)]}\right)\\ &= \alpha + (1-\alpha)\left(-\mathrm{i}\tan(\theta/2) - \frac{\lambda k}{2\cos^2(\theta/2)}\frac{1}{(1-\lambda)+\lambda[z_0D'(z_0)/D(z_0)]}\right).\end{aligned}$$

Taking real parts on both sides, we obtain

$$\mathrm{Re}\,(\psi(z_0)) = \alpha - (1-\alpha)\frac{\lambda k}{2\cos^2(\theta/2)}\mathrm{Re}\left(\frac{1}{1-\lambda+\lambda[z_0D'(z_0)/D(z_0)]}\right).$$

By (1) we have that $\mathrm{Re}\left(\dfrac{1}{1-\lambda+\lambda[z_0D'(z_0)/D(z_0)]}\right) > \delta$. This yields

$$\mathrm{Re}\,(\psi(z_0)) < \alpha - \frac{(1-\alpha)\lambda\delta}{2},$$

contradicting our hypothesis. Hence $|\omega(z)| < 1$, for all $z \in \mathbb{D}$. □

THEOREM 3.1. *If $f \in \mathcal{K}_\alpha^\lambda$, for $\alpha, \lambda \in [0,1]$, and $\alpha \neq 1$, then f is close-to-convex.*

Proof. Taking $N(z) = zf'(z)$ and $D(z) = \phi(z)$, for some $\phi \in S^*$, in the above lemma, we complete the proof, since δ can approach 0^+. □

THEOREM 3.2 (Integral representation of the class). *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}_\alpha^\lambda$, for some λ, $\alpha \in [0,1]$ and $\alpha \neq 1$, if and only if there exist an analytic function P with positive real part and a starlike function ϕ such that*

$$f'(z) = ((1-\alpha)P(z)+\alpha)\frac{\phi(z)}{z} - \frac{(1-\alpha)}{z^{c+1}}\int_0^z t^c\phi(t)P'(t)\mathrm{d}t,$$

where $c = \dfrac{1}{\lambda} - 1$, for $\lambda \neq 0$, and $f'(z) = [(1-\alpha)P(z)+\alpha]\dfrac{\phi(z)}{z}$, if $\lambda = 0$.

Proof. Let $f \in \mathcal{K}_\alpha^\lambda$ and define

$$P(z) := \frac{1}{(1-\alpha)}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)\phi(z)+\lambda z\phi'(z)} - \alpha\right).$$

Since $f \in \mathcal{K}_\alpha^\lambda$, we have that $\mathrm{Re}(P(z)) > 0$ and $P(0) = 1$. Moreover,

$$zf'(z) + \lambda z^2 f''(z) = ((1-\lambda)\phi(z)+\lambda z\phi'(z))((1-\alpha)P(z)+\alpha).$$

Thus, for $\lambda \neq 0$ and $z \in \mathbb{D}$, we have that

$$\frac{1}{\lambda} z f'(z) + z^2 f''(z) = \left[\left(\frac{1}{\lambda} - 1\right)\phi(z) + z\phi'(z)\right]((1-\alpha)P(z) + \alpha), \text{i.e.,}$$

$$(c+1)zf'(z) + z^2 f''(z) = [c\phi(z) + z\phi'(z)]((1-\alpha)P(z)+\alpha)$$

where $c = \dfrac{1}{\lambda} - 1$. Multiplying throughout by z^{c-1}, we obtain

$$(c+1)z^c f'(z) + z^{c+1} f''(z) = [(cz^{c-1}\phi(z) + z^c\phi'(z)]((1-\alpha)P(z)+\alpha).$$

Therefore,

$$\mathrm{d}(z^{c+1}f'(z)) = ((1-\alpha)P(z)+\alpha)\mathrm{d}(z^c\phi(z)).$$

By integrating both sides, we get that

$$z^{c+1}f'(z) = ((1-\alpha)P(z)+\alpha)z^c\phi(z) - (1-\alpha)\int_0^z t^c\phi(t)P'(t)\mathrm{d}t.$$

For $\lambda = 0$, we have $\mathrm{Re}\left(\dfrac{zf'(z)}{\phi(z)}\right) > \alpha$ and therefore

$$f'(z) = ((1-\alpha)P(z)+\alpha)\frac{\phi(z)}{z}.$$

This proves the "necessary"-part of the theorem. For the "sufficiency"-part we have to retrace the steps backward. □

Corollary 3.1. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}^\lambda$ if and only if there exists an analytic function P with positive real part and a starlike function ϕ on $\mathbb{D}$ such that*

$$f'(z) = P(z)\frac{\phi(z)}{z} - \frac{1}{z^{c+1}}\int_0^z t^c\phi(t)P'(t)\mathrm{d}t,$$

for $\lambda \neq 0$, and $f'(z) = P(z)\phi(z)/z$, if $\lambda = 0$.

Theorem 3.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{K}_\alpha^\lambda$, then the following inequality holds for $n \geq 2$*

$$|a_n| \leq \frac{3[1+(n-1)(\lambda+(1-\alpha)(1-\lambda))] + \lambda(1-\alpha)(n-1)(2n-1)}{3(1+(n-1)\lambda)}.$$

Proof. Suppose that $f \in \mathcal{K}_\alpha^\lambda$. Then

$$P(z) = \frac{1}{(1-\alpha)}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)\phi(z) + \lambda z\phi'(z)} - \alpha\right)$$

has positive real part and $P(0) = 1$, for some $\phi(z) \in S^*$.

Let $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$.

Since $\phi \in \mathcal{S}^*$ we have that $|b_n| \le n$. As $P(z)$ is an analytic function with positive real part, we get $|c_n| \le 2$. Also, $zf'(z) = \sum_{n=1}^{\infty} na_n z^n$ and $z^2 f''(z) = \sum_{n=1}^{\infty} n(n-1)a_n z^n$. Substituting these in the equality

$$zf'(z) + \lambda z^2 f''(z) = ((1-\alpha)P(z) + \alpha)((1-\lambda)\phi(z) + \lambda z\phi'(z)),$$

we obtain

$$\sum_{n=1}^{\infty} na_n z^n + \lambda \sum_{n=1}^{\infty} n(n-1)a_n z^n = \left((1-\alpha)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) + \alpha\right) \left((1-\lambda)\sum_{n=1}^{\infty} b_n z^n + \lambda \sum_{n=1}^{\infty} nb_n z^n\right).$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty} n(1+(n-1)\lambda)a_n z^n =&[(1-\alpha)(1+\sum_{n=1}^{\infty} c_n z^n) + \alpha](\sum_{n=1}^{\infty}(1+(n-1)\lambda)b_n z^n)\\ =&\sum_{n=1}^{\infty}(1+(n-1)\lambda)b_n z^n\\ &+(1-\alpha)\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n-1}(1+(k-1)\lambda)b_k c_{n-k}\right] z^n.\end{aligned}$$

Equating the coefficient of $z^n, n > 1$, we obtain

$$n(1+(n-1)\lambda)a_n = (1+(n-1)\lambda)b_n + (1-\alpha)\sum_{k=1}^{n-1}(1+(k-1)\lambda)b_k c_{n-k}.$$

Therefore,

$$\begin{aligned}n(1+(n-1)\lambda)|a_n| &\le (1+(n-1)\lambda)|b_n| + (1-\alpha)\sum_{k=1}^{n-1}(1+(k-1)\lambda)|b_k||c_{n-k}|\\ &\le (1+(n-1)\lambda)n + 2(1-\alpha)\sum_{k=1}^{n-1}(1+(k-1)\lambda)k\\ &= (1+(n-1)\lambda)n + 2(1-\alpha)\left(\sum_{k=1}^{n-1} k + \lambda\sum_{k=1}^{n-1}(k^2-k)\right)\\ &= n\left(1+(n-1)\lambda + (1-\alpha)\left[(1-\lambda)(n-1) + \lambda\frac{(n-1)(2n-1)}{3}\right]\right)\\ &= n\frac{3[1+(n-1)(\lambda+(1-\alpha)(1-\lambda))] + \lambda(1-\alpha)(n-1)(2n-1)}{3}.\end{aligned}$$

Hence

$$|a_n| \leq \frac{3[1+(n-1)(\lambda+(1-\alpha)(1-\lambda))]+\lambda(1-\alpha)(n-1)(2n-1)}{3(1+(n-1)\lambda)}$$

which completes the proof. □

COROLLARY 3.2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to* $\mathcal{K}^\lambda$, *then the following inequality holds for* $n \geq 2$

$$|a_n| \leq \frac{3n+\lambda(n-1)(2n-1)}{3(1+(n-1)\lambda)}.$$

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University of Madras
Ramanujan Institute for Advanced Study in Mathematics
Chepauk, Chennai 600 005, India
E-mail: usnabanu1005@gmail.com
E-mail: youvarajgp@yahoo.com