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ON EXTENDED CONVERGENCE DOMAINS FOR THE NEWTON-KANTOROVICH METHOD

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Abstract. We present results on extended convergence domains and their applications for the Newton-Kantorovich method (NKM), using the same information as in previous papers. Numerical examples are provided to emphasize that our results can be applied to solve nonlinear equations using (NKM), in contrast with earlier results which are not applicable in these cases.

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1. INTRODUCTION

Newton's method is one of the most fundamental tools in Computational Analysis, Operations Research, and Optimization [6, 9, 12, 16, 23, 24, 25, 26, 29]. One can find applications in management science, in industrial and financial research, in data mining, as well in linear and nonlinear programming. In particular, interior point algorithms in convex optimization are based on Newton's method.

The basic idea of Newton's method is linearization. Given a differentiable function $F : \mathbb{R} \to \mathbb{R}$, we formulate the equation

$$F(x) = 0. \tag{1.1}$$

Starting from an initial guess, we consider the linear approximation of $F(x)$ in a neighborhood of $x_0 : F(x_0 + s) \approx F(x_0) + F'(x_0)s$, and solve the resulting linear equation $F(x_0) + F'(x_0)s = 0$, leading to the recurrence formula

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0). \tag{1.2}$$

This is Newton's method as proposed in 1669 by I.Newton (for polynomials). One can also use the slower modified Newton-Kantorovich method (MNKM)

$$y_{n+1} = y_n - F'(y_0)^{-1}F(y_n) \quad (n \geq 0). \tag{1.3}$$

It was J.Raphson, who proposed the usage of Newton's method for general functions F. That is why the method is often called the Newton-Raphson method.

Later in 1818, Fourier proposed that the method converges quadratically in a neighborhood of the root, while Cauchy (1829, 1847) provided the multidimensional extension of Newton's method (1.2). In 1948, L.V.Kantorovich published an important paper [23], extending Newton's method for functional spaces (the Newton-Kantorovich method (NKM)), i.e., $F : D \subseteq X \to Y$, where

X, Y are Banach spaces, and D is an open convex set [6, 23, 26, 29]. Since then thousands of papers have been written in the Banach space setting for the (NKM) as well as for Newton-type methods and their applications. We refer the reader to the recent results (see also, the references therein) [1]–[45].

It is stated in the (NKT) theorem that (NKM) (1.2) converges provided the Kantorovich hypothesis (KH) (see (C6)$'$), which is famous for its simplicity and clarity, is satisfied. (KH) uses the information (x_0, F, F'). Any successful attempt for weakening (KH) under the same information is extremely important in computational mathematics, since this will imply the extension of the applicability of (NKM) (1.2). We have already provided conditions weaker than (KH), [2]–[12] by introducing the center Lipschitz condition, which is a special case of the Lipschitz condition.

In this study we provide new sufficient convergence conditions for Newton's method, weaker than (KH). Moreover, there are presented numerical examples, where our results can be applied to solve nonlinear equations, but earlier results are not applicable.

2. CONVERGENCE ANALYSIS FOR (NKM) AND (MNKM)

The following semilocal convergence theorem for the (NKM) and (MNKM) methods can be found in [23]:

Theorem 2.1. (Newton-Kantorovich Theorem for Solving Nonlinear Equations) *Let $F : D \subseteq X \to Y$ be differentiable. Assume that there exist $x_0 \in D$ and constants $b > 0, L > 0, \eta > 0$ such that*

$$F'(x_0)^{-1} \in L(Y, X), \;\; \|F'(x_0)^{-1}\| \le b,$$
$$\|F'(x_0)^{-1}F(x_0)\| \le \eta,$$
$$\|F'(x) - F'(y)\| \le L\|x - y\| \quad \text{for all} \quad x, y \in D,$$
$$h_* = 2bL\eta \le 1,$$

and

$$\overline{U}(x_0, s^*) \subseteq D,$$

where

$$s^* = \frac{1 - \sqrt{1 - h_*}}{Lb}.$$

Then the sequences $\{y_n\}, \{x_n\}$ are well-defined, remain in $\overline{U}(x_0, s^*)$ for all $n \ge 0$, and converge to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, s^*)$. Moreover, the following estimates hold:

$$\|y_{n+1} - y_n\| \le q^n\|y_1 - y_0\| \le q^n\eta,$$
$$\|y_n - x^*\| \le \frac{q^n}{1 - q}\eta,$$
$$\|x_{n+2} - x_{n+1}\| \le \frac{Lb(s_{n+1} - s_n)^2}{2(1 - Lbs_{n+1})},$$

and

$$\|x_n - x^*\| \leq s^* - s_n, \;\; s^* = \lim_{n\to\infty} s_n,$$

where

$$s_0 = 0,\; s_1 = \eta,\; s_{n+2} = s_{n+1} + \frac{Lb(s_{n+1} - s_n)^2}{2(1 - Lbs_{n+1})} \;\; (n \geq 0),$$

and

$$q = 1 - \sqrt{1 - h_*}.$$

Let us provide a numerical example where the main hypothesis in Theorem 2.1 is violated.

EXAMPLE 2.2. Let $X = Y = \mathbb{R}, D = \overline{U}(1, 1 - \frac{a}{2}), a < 2$, and define the scalar function F on D by

$$F(x) = \frac{1}{5}x^3 - a. \tag{2.4}$$

Using Theorem 2.1, and (2.4), we get $b = \frac{5}{3}$, $\eta = \frac{5}{3}|\frac{1}{5} - a|$, and $L = \frac{3}{5}(4 - a)$. Let $a = 0.1226$. Then

$$h_* = 2Lb\eta = 1.0003692 > 1.$$

That is there is no guarantee that by (NKM) the sequence $\{x_n\}$ converges to $x^* = 0.849480652$, starting at $x_0 = 1$.

REMARK 2.3. There is a plethora of estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$, $\|y_{n+1} - y_n\|$, $\|y_n - x^*\|$ $(n \geq 0)$, [1]–[45]. However we decided to list only the estimates related to what we need in this study. In the case of Newton's method, the following improvement of Theorem 2.1 was proved in [2]-[6], [11, 12].

THEOREM 2.4. *Let $F : D \subseteq X \to Y$ be differentiable. Assume that there exist $x_0 \in D$ and constants $b > 0, L_0 > 0, \eta \geq 0$ such that*

$$F'(x_0)^{-1} \in L(Y, X), \;\; \|F'(x_0)^{-1}\| \leq b,$$
$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$
$$\|F'(x) - F'(x_0)\| \leq L_0\|x - x_0\| \quad \text{for all} \;\; x \in D,$$
$$\|F'(x) - F'(y)\| \leq L\|x - y\| \quad \text{for all} \;\; x, y \in D,$$
$$h_{AH} = 2bL_1\eta \leq 1, \;\; L_1 = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0L}),$$
$$\overline{U}(x_0, t^*) \subseteq D,$$

where

$$t_0 = 0,\; t_1 = \eta,\; t_{n+2} = t_{n+1} + \frac{Lb(t_{n+1} - t_n)^2}{2(1 - L_0bt_{n+1})} \;\; (n \geq 0),$$

and

$$t^* = \lim_{n\to\infty} t_n \leq \frac{2\eta}{2 - L_2} = t_0^*, \;\; L_2 = \frac{1}{2}\left(-\frac{L}{L_0} + \sqrt{\left(\frac{L}{L_0}\right)^2 + \frac{8L}{L_0}}\right).$$

Then the sequence $\{x_n\}$ $(n \geq 0)$ *generated by Newton's method is well-defined, remains in* $\overline{U}(x_0, t^*)$ *for all* $n \geq 0$, *and converges to a unique solution* x^* *of equation* $F(x) = 0$ *in* $\overline{U}(x_0, t^*)$. *Moreover the following estimates hold for all* $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n,$$

$$\|x_n - x^*\| \leq t^* - t_n,$$

$$t_n \leq s_n, \tag{2.5}$$

$$t_{n+1} - t_n \leq s_{n+1} - s_n, \tag{2.6}$$

and

$$t^* - t_n \leq s^* - s_n. \tag{2.7}$$

Remark 2.5. Note also that (2.5) and (2.6) hold as strict inequalities if $L_0 < L$. Moreover, we have:

$$h_* \leq \frac{1}{2} \Rightarrow h_{AH} \leq \frac{1}{2}, \tag{2.8}$$

but not vice versa unless $L_0 = L$. That is, under the same computational cost we managed to weaken the (KH) condition, since in practice the computation of L also requires the computation of L_0. In particular, $\frac{h_{AH}}{h_K} \to \frac{1}{4}$ as $\frac{L_0}{L} \to 0$. Hence, Theorem 2.4 quadruples (at most) the application of (NKM).

Remark 2.6. Returning back to Example 2.2, we find $L_0 = \frac{3}{5}(3 - \frac{a}{2}), L_1 = 1.94528028$, and $h_{AH} = 2bL_1\eta = 0.83635 < 1$. That is, Theorem 2.4 guarantees the convergence of (NKM) to x^*.

Theorem 2.7. ([5]) *Let* $F : D \subseteq X \to Y$ *be differentiable. Assume that there exist* $x_0 \in D$ *and constants* $b > 0, L_0 > 0, \eta \geq 0$ *such that*

$$F'(x_0)^{-1} \in L(Y, X), \ \ \|F'(x_0)^{-1}\| \leq b,$$

$$\|F(x_0)\| \leq \eta,$$

$$\|F'(x) - F'(x_0)\| \leq L_0\|x - x_0\| \quad \text{for all} \ \ x \in D,$$

$$h_0 = 2bL_0\eta \leq 1,$$

and

$$\overline{U}(x_0, s_0^*) \subseteq D, \tag{2.9}$$

where

$$s_0^* = \frac{1 - \sqrt{1 - h_0}}{bL_0}.$$

Then the sequence $\{y_n\}$ $(n \geq 0)$ *generated by the modified Newton's method is well-defined, remains in* $\overline{U}(x_0, s_0^*)$ *for all* $n \geq 0$, *and converges to a unique* *solution* x^* *of equation* $F(x) = 0$ *in* $\overline{U}(x_0, s_0^*)$. *Moreover, the following estimates hold for all* $n \geq 0$:

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq tq_0^* \|y_1 - y_0\|, \\ \|y_n - x^*\| &\leq \frac{q_0^n}{1 - q_0}\eta, \end{aligned}$$

where

$$q_0 = 1 - \sqrt{1 - h_0}.$$

REMARK 2.8. If $L_0 = L$, the Theorems 2.4 and 2.7 reduce to Theorem 2.1. In the other cases these theorems constitute improvements of it. Indeed, use (2.5)–(2.8) and notice that

$$q_0 < q$$

and

$$s_0^* < s^*.$$

Notice also that $h_* \leq 1$ implies $h_{AH} \leq 1$ or $h_0 \leq 1$.

In [5], we showed that one can start with method (1.3) and after a finite number of steps continue with the faster method (1.2).

REMARK 2.9. Returning back to Example 2.2, we have: $h_0 = 2bL_0\eta = 0.7581846 < 1$. That is, Theorem 2.7 guarantees the convergence of (MNKM) to x^*.

In order to cover the local convergence of methods (1.2) and (1.3) we state the following theorem.

THEOREM 2.10. ([2, 3, 6, 9, 12]) *Let* $F : D \subseteq X \to Y$ *be differentiable. Assume that there exist* $x^* \in D$ *and constants* $b_* > 0, L > 0, \eta \geq 0$ *such that:*

$$F'(x^*)^{-1} \in L(Y, X), \ \ F(x^*) = 0, \|F'(x^*)^{-1}\| \leq b_*, \tag{2.10}$$

$$\|F'(x) - F'(y)\| \leq L\|x - y\| \ \text{ for all } \ x \in D, \tag{2.11}$$

and

$$\overline{U}(x^*, r_{TR}) \subseteq D, \tag{2.12}$$

where

$$r_{TR} = \frac{2}{3b_* L}, \tag{2.13}$$

then

(a) *the sequence* $\{x_n\}$ *generated by Newton's method* (1.2) *is well-defined, remains in* $\overline{U}(x^*, r_{TR})$ *for all* $n \geq 0$, *converges to* x^* *provided that* $x_0 \in \overline{U}(x^*, r_{TR})$ *and*

$$\|x_{n+1} - x^*\| \leq \frac{Lb_* \|x_n - x^*\|^2}{2(1 - Lb_*\|x_n - x^*\|)} \ \ (n \geq 0). \tag{2.14}$$

Suppose that

$$\overline{U}(x^*, r_{TRM}) \subseteq D \tag{2.15}$$

is satisfied, where

$$r_{TRM} = \frac{2}{5b_* L}, \tag{2.16}$$

then

(b) *the sequence* $\{y_n\}$ *generated by the modified Newton method* (1.3) *is well-defined, remains in* $\overline{U}(x^*, r_{TRM})$ *for all* $n \geq 0$, *converges to* x^* *provided that* $x_0 \in \overline{U}(x^*, r_{TRM})$, *and*

$$\|y_{n+1} - x^*\| \leq \frac{Lb_*[\|y_0 - x^*\| + \frac{1}{2}\|y_n - x^*\|]}{(1 - Lb_*\|y_0 - x^*\|}\|y_n - x^*\| \ (n \geq 0). \tag{2.17}$$

Proof. The proof of (a) can be found in [2, 3, 6, 9, 12]. The proof of part (b) is a special case of the proof of part (b) of Theorem 2.11 (see also [41]). □

It follows from (2.11) that there exists $L_0 \in (0, L)$ such that:

$$\|F'(x) - F'(x^*)\| \leq L_0\|x - x^*\| \quad \text{for all } x \in D. \tag{2.18}$$

Then using a combination of conditions (2.11) and (2.18) for method (1.2), and only condition (2.18) for method (1.3) we can show:

THEOREM 2.11. *Let* $F : D \subseteq X \to Y$ *be differentiable. Assume that there exist* $x^* \in D$ *and constants* $b_* > 0, L > 0, \eta \geq 0$ *such that:*

$$F'(x^*)^{-1} \in L(Y, X), \ \ F(x^*) = 0, \|F'(x^*)^{-1}\| \leq b_*,$$
$$\|F'(x) - F'(x^*)\| \leq L_0\|x - x^*\| \quad \text{for all} \ \ x \in D,$$
$$\|F'(x) - F'(y)\| \leq L\|x - y\| \quad \text{for all} \ \ x, y \in D,$$

and

$$\overline{U}(x^*, r_{AH}) \subseteq D, \tag{2.19}$$

where

$$r_{AH} = \frac{2}{(2L_0 + L)b_*}. \tag{2.20}$$

Then

(a) *the sequence* $\{x_n\}$ *generated by Newton's method* (1.2) *is well-defined, remains in* $\overline{U}(x^*, r_{AH})$ *for all* $n \geq 0$, *converges to* x^* *provided that* $x_0 \in \overline{U}(x^*, r_{AH})$, *and*

$$\|x_{n+1} - x^*\| \leq \frac{Lb_*\|x_n - x^*\|^2}{2(1 - L_0 b_*\|x_n - x^*\|)} \ (n \geq 0). \tag{2.21}$$

Suppose that

$$\overline{U}(x^*, r_{AM}) \subseteq D \tag{2.22}$$

and hypothesis (2.18) *are satisfied, where*

$$r_{AM} = \frac{2}{5b_* L_0}, \tag{2.23}$$

then

(b) *the sequence* $\{y_n\}$ *generated by* (MNKM) *is well-defined, remains in* $\overline{U}(x^*, r_{TRM})$ *for all* $n \geq 0$*, converges to* x^* *if* $x_0 \in \overline{U}(x^*, r_{TRM})$*, and*

$$\|y_{n+1} - x^*\| \leq \frac{Lb_*[\|y_0 - x^*\| + \frac{1}{2}\|y_n - x^*\|]}{(1 - Lb_*\|y_0 - x^*\|}\|y_n - x^*\| \;\; (n \geq 0). \tag{2.24}$$

Remark 2.12. In general,

$$L_0 \leq L \tag{2.25}$$

holds and $\frac{L}{L_0}$ can be arbitrarily large ([2, 3, 6, 9, 12]). If $L_0 = L$, Theorem 2.11 reduces to Theorem 2.10. Otherwise, Theorem 2.11 improves Theorem 2.10 under the same hypotheses for method (1.2), and the same or less computational cost for method (1.3); finer estimates on the distances $\|x_n - x^*\|$ $(n \geq 0)$ are obtained and the radius of convergence is enlarged. In particular, we have

$$r_{RN} < r_{AN}, \tag{2.26}$$

$$r_{RM} < r_{AM}. \tag{2.27}$$

Moreover, since

$$r_{RM} < r_{RN}, \tag{2.28}$$

iterates from method (1.3) cannot be used to find the initial guess x_0 for the faster method (1.2). The convergence domain in the Newton-Kantorovich Theorem 2.1 can be extended if

$$\frac{1}{2Lb} < \eta \leq \frac{1}{2L_0 b}. \tag{C1}$$

Indeed, according to Theorem 2.7 there exists a solution x^* of equation $F(x) = 0$, which can be found as the limit of (MNKM). In this case we can only show linear convergence. However, if

$$\|x_0 - x^*\| \leq r_{TR} \;\; (\text{or} \leq r_{AH}), \tag{C2}$$

then according to Theorem 2.10, the solution x^* can be obtained as the limit of the quadratically convergent (NKM). It follows from Theorem 2.7 (since $x^* \in \overline{U}(x^*, s_0^*)$) that conditions (C1) and (C2) can be replaced by

$$\frac{1}{2bL} < \eta \leq \frac{1}{2bL_0}\left[1 - (1 - \frac{2L_0 b}{3Lb_*})^2\right] = \frac{2}{3Lb_*}\left(1 - \frac{L_0 b}{3Lb_*}\right) \tag{C3}$$

and

$$\frac{2bL_0}{3Lb_*} \leq 1$$

or

(C4) $$\frac{1}{2bL} < \eta \le \frac{1}{2bL_0}$$

and

$$\frac{2L_0 b}{3Lb_*} > 1$$

or

(C5) $$\frac{1}{2bL} < \eta \le \frac{1}{2bL_0}\left[1-\left(1-\frac{2L_0 b}{(2L_*+L)\,b_*}\right)^2\right]$$

and

$$\frac{2L_0 b}{(2L_*+L)b_*} \le 1$$

or

(C6) $$\frac{1}{2bL} < \eta \le \frac{1}{2bL_0}$$

and

$$\frac{2L_0 b}{3Lb_*} > 1.$$

In an analogous way the convergence domain of Theorem 2.4 can be extended, if

(H1) $$\frac{1}{2bL_1} < \eta \le \frac{1}{2bL_0}$$

and (C2) hold or

(H2) $$\frac{1}{2bL_1} < \eta \le \frac{1}{2bL_0}$$

and

$$\frac{2L_0 b}{3Lb_*} > 1$$

or

(H3) $$\frac{1}{2bL_1} < \eta \le \frac{1}{2bL_0}\left[1-\left(1-\frac{2L_0 b}{3Lb_*}\right)^2\right]$$

and

$$\frac{2L_0 b}{3Lb_*} \le 1,$$

or

(H4) $$\frac{1}{2bL_1} < \eta \le \frac{1}{2bL_0}$$

and

$$\frac{2L_0 b}{3Lb_*} > 1,$$

or

(H5) $$\frac{1}{2bL_1} < \eta \leq \frac{1}{2bL_0}\left[1 - \left(1 - \frac{2L_0 b}{(2L_* + L)b_*}\right)^2\right]$$

and

$$\frac{2L_0 b}{(2L_* + L)b_*} \leq 1,$$

or

(H6) $$\frac{1}{2bL_1} < \eta \leq \frac{1}{2bL_0}$$

and

$$\frac{2L_0 b}{3Lb_*} \geq 1.$$

Remark 2.13. Returning back to Example 2.2, we see that $b_* = 2.309626568$, $\|x_0 - x^*\| = 0.150519348$, and $r_{AH} = 0.154668289 > \|x_0 - x^*\|$. Hence, conditions (C1) and (C2) hold.

Remark 2.14. In practice, we shall test the (C) or (H) conditions (or the conditions of Theorems 2.1, 2.4, 2.7) to which ones apply. Finally, the results obtained here can be given in an affine invariant form if the hypotheses hold for the operator $F'(x_0)^{-1}F$ in the semilocal case, and for $F'(x^*)^{-1}F$ in the local case. Other numerical examples can be found in [2, 3, 6, 9, 12].

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