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GROUP GRADED BIMODULES OVER A COMMUTATIVE G-RING

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Abstract. In this paper we discuss G-graded algebras and G-acted algebras and Morita equivalences over a commutative G-ring Z. We also associate a central simple G-graded algebra over Z to a character of a strongly G-graded algebra over a field of characteristic zero.

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1. INTRODUCTION AND PRELIMINARIES

Let G be a finite group, Z a commutative G-ring, and let $F = Z^G$. Let A and B be two G-acted F-algebras such that $Z \to Z(A)$ and $Z \to Z(A)$ are G-ring homomorphisms. Then the tensor product over Z of A and B is again a G-acted F-algebra over Z. Motivated by the study of Clifford theory in combination with Schur indices, Turull has introduced an equivalence relation between simple algebras of this kind, which comes down to the notion of equivariant Morita equivalence over Z between them.

However, strongly graded algebras are natural for Clifford theory. So let R and S be two strongly G-graded algebras such that $Z \to Z(R_1)$ and $Z \to Z(S_1)$ are G-ring homomorphisms.

Turull's equivalence classes over F (see [6]) can be generalized to the case of strongly G-graded algebras (see Marcus [4], [5]). The problem is that the tensor product over Z of R and S is no longer an algebra. More precisely, we have:

Remark 1.1. 1) R and S are F-algebras and R_1 and S_1 are actually Z-algebras.

2) Many arguments in [3] where based on the fact that $R \otimes_F S^{\mathrm{op}}$ is a $G \times G$-graded F-algebra, and then the subalgebra

$$\Delta(R \otimes_F S^{\mathrm{op}}) := \bigoplus_{g \in G} (R_g \otimes_F S_g^{\mathrm{op}})$$

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is a strongly G-graded F-algebra.

3) Here we may also construct $R\otimes_Z S$, but this is not a ring in general, nor is $\Delta(R\otimes_Z S)$.

Nevertheless, we show in Section 2 that we can still consider G-graded Morita equivalences over Z (not only over F) between R and S. For this, we introduce the following notion.

Definition 1.2. We say that M is a *G-graded (R,S)-bimodule over Z* if

- M is a (R,S)-bimodule,
- M has a decomposition $M=\bigoplus_{x\in G} M_x$ such that $R_gM_xS_h=M_{gxh}$,
- $m_gz={}^g zm_g$ for all $g\in G$, $z\in Z$, and $m_g\in M_g$.

Remark 1.3. If M is a G-graded (R,S)-bimodule then $R_gM_1R_{g^{-1}}=M_1$. But obviously, we can not say M is a $R\otimes_Z S^{\mathrm{op}}$-module because $R\otimes_Z S^{\mathrm{op}}$ is not a ring in general.

In Section 3 we discuss the particular case of skew-group algebras and the relation between G-graded Morita equivalences and G-equivariant Morita equivalences.

Finally, in Section 4 we associate a central simple G-graded algebra over Z to an irreducible character of R, in a unique way up to graded Morita equivalence over Z.

2. G-graded F-algebras over Z and Morita equivalences

We keep the notations and assumptions of the previous section.

Lemma 2.1. 1) *$\Delta(R\otimes_Z S^{\mathrm{op}})$ is a ring. Moreover, it is an G-algebra over Z, that is, there exists a G-algebra homomorphism from Z to $Z(R_1\otimes_Z S_1^{\mathrm{op}})$.*

2) *$R\otimes_Z S^{\mathrm{op}}$ is a right $\Delta(R\otimes_Z S^{\mathrm{op}})$-module.*

Proof. 1) We first show that $\Delta(R\otimes_Z S^{\mathrm{op}})$ is a ring. Clearly, it is an abelian group, and we have

$$\Delta:=\Delta(R\otimes_Z S^{\mathrm{op}})=\bigoplus_{g\in G}(R_g\otimes S_g^{\mathrm{op}}).$$

By definition,

$$(r_1\otimes_Z s_1^{\mathrm{op}})(r_2\otimes_Z s_2^{\mathrm{op}})=r_1r_2\otimes_Z s_1^{\mathrm{op}}s_2^{\mathrm{op}}=r_1r_2\otimes_Z (s_2s_1)^{\mathrm{op}}.$$

We need to verify that this is well-defined. Indeed, on the one hand, we have

$$(r_gz\otimes s_g)(r'_h\otimes s'_h)=r_gzr'_h\otimes s_g * s'_h={}^g zr_gr'_h\otimes s'_hs_g={}^g zr_gr'_h\otimes s'_hs_g,$$

where we have denoted by $*$ the multiplication in S^{op}. On the other hand,

$$\begin{aligned}(r_g\otimes zs_g)(r'_h\otimes s'_h) &= r_gr'_h\otimes(zs_g)*s'_h=r_gr'_h\otimes s'_h(zs_g)\\ &= r_gr'_h\otimes {}^h zs'_hs_g=r_gr'_h{}^h z\otimes s'_hs_g=r_gzr_h\otimes s'_hs_g\\ &= {}^g zr_gr_h\otimes s'_hs_g.\end{aligned}$$

By similar calculations we have

$$(r_g \otimes s_g)(r'_h z \otimes s'_h) = r_g r'_h z \otimes s_g * s'_h = r_g{}^h z r'_h \otimes s'_h s_g =^{gh} z r_g r'_h \otimes s'_h s_g,$$

and

$$\begin{aligned}(r_g \otimes s_g)(r'_h \otimes z s'_h) &= r_g r'_h \otimes s_g * (z s'_h) = r_g r'_h \otimes z s'_h s_g \\ &= r_g r'_h z \otimes s'_h s_g = r_g{}^h z r'_h \otimes s'_h s_g =^{gh} z r_g r'_h \otimes s'_h s_g.\end{aligned}$$

Hence Δ is a ring, and it is clearly strongly G-graded, since R and S are strongly G-graded.

The G-ring homomorphism $\varphi : Z \to Z(\Delta_1)$, where $\Delta_1 = R_1 \otimes_Z S_1^{\mathrm{op}}$ is given by

$$\varphi(z) = \varphi_1(z) \otimes 1 = 1 \otimes \varphi_2(z),$$

where $\varphi_1 : Z \to Z(R_1)$ and $\varphi_2 : Z \to Z(S_1)$ are G-ring homomorphisms.

2) As above, we must verify that the scalar multiplication is well-defined. On the one hand, we have

$$(r_x z \otimes s_y^{\mathrm{op}})(r_g \otimes s_g^{\mathrm{op}}) = r_x z r_g \otimes s_y^{\mathrm{op}} * s_g^{\mathrm{op}} = {}^x z r_x r_g \otimes s_g^{\mathrm{op}} s_y^{\mathrm{op}} = {}^x z r_x r_g \otimes s_g^{\mathrm{op}} s_y^{\mathrm{op}},$$

and on the other hand,

$$\begin{aligned}(r_x \otimes z s_y^{\mathrm{op}})(r_g \otimes s_g^{\mathrm{op}}) &= r_x r_g \otimes (z s_y^{\mathrm{op}}) * s_g^{\mathrm{op}} = r_x r_g \otimes s_g^{\mathrm{op}}(z s_y^{\mathrm{op}}) \\ &= r_x r_g \otimes {}^g z s_g^{\mathrm{op}} s_y = r_x r_g{}^g z \otimes s_g^{\mathrm{op}} s_y^{\mathrm{op}} = r_x z r_g \otimes s_g^{\mathrm{op}} s_y \\ &= {}^x z r_x r_g \otimes s_g^{\mathrm{op}} s_y^{\mathrm{op}},\end{aligned}$$

hence $R \otimes_Z S^{\mathrm{op}}$ is indeed a right Δ-module $\square$

Let N be a Δ-module, where $\Delta = \Delta(R \otimes_Z S^{\mathrm{op}})$. Then N it is also a Δ_1-module, where $\Delta_1 := R_1 \otimes_Z S_1^{\mathrm{op}}$, so we may consider the induced modules $R \otimes_{R_1} N$ and $N \otimes_{S_1} S$. The above lemma implies that, similarly to [3, Lemma 2.6], the following holds.

Lemma 2.2. *If N be a Δ-module, there exists a isomorphism*

$$R \otimes_{R_1} N \simeq N \otimes_{S_1} S \simeq (R \otimes_Z S^{\mathrm{op}}) \otimes_\Delta N$$

of G-graded (R, S)-bimodules over Z. We shall denote these isomorphic bimodules by $\widetilde{N}$.

Recall that since S is fully graded, for all $g \in G$ we have $S_{g^{-1}} S_g = S_1$. Hence there exist elements $s'_i \in S_{g^{-1}}$, $s_i \in S_g$, $1 \le i \le l$, such that

$$\sum_{i=1}^{l} s'_i s_i = 1. \tag{1}$$

Lemma 2.3. 1) *Assume that* N *is a left* $\Delta(R \otimes_Z S^{\mathrm{op}})$*-module and* N' *is a left* $\Delta(S \otimes_Z T^{\mathrm{op}})$*-module. Then* $N \otimes_{S_1} N'$ *is a* $\Delta(R \otimes_Z T^{\mathrm{op}})$*-module with multiplication*

$$(r_g \otimes_Z t_g^{\mathrm{op}})(n \otimes_{S_1} n') = \sum_{i=1}^{l} r_g n s_i' \otimes_{S_1} s_i n t_{g^{-1}}.$$

Moreover we have the isomorphism

$$\widetilde{N \otimes_{S_1} N'} \simeq \widetilde{N} \otimes_S \widetilde{N'}$$

of G*-graded* (R,T)*-bimodules over* Z.

2) *Assume that* N *is a* $\Delta(S \otimes_Z R^{\mathrm{op}})$*-module and* N *a* $\Delta(S \otimes_Z T^{\mathrm{op}})$*-module. Then* $\mathrm{Hom}_{S_1}(N, N')$ *is a* $\Delta(R \otimes_Z T^{\mathrm{op}})$*-module with multiplication:*

$$(r_{g^{-1}} f t_g)(n) = \sum_{i=1}^{l} s_i' f(s_i n r_{g^{-1}}) t_g$$

for $n \in N$ *and* $f \in \mathrm{Hom}_{S_1}(N, N')$*. Moreover we have the isomorphism*

$$\mathrm{Hom}_S(\widetilde{N}, \widetilde{N'}) \simeq \widetilde{\mathrm{Hom}_{S_1}(N, N')}$$

of G*-graded* (R,T)*-bimodules over* Z.

Proof. 1) In order to prove that $N \otimes_{S_1} N'$ is a left $\Delta(R \otimes_Z T^{op})$-module, the only difference from [3, Lemma 2.9] is that we have to verify that the scalar multiplication is well-defined. Indeed, we have

$$\begin{aligned}
(r_g \otimes_Z z t_g^{op})(n \otimes_{S_1} n') &= (r_g \otimes_Z z t_{g^{-1}})(n \otimes_{S_1} n') \\
&= \sum_{i=1}^{l} r_g n s_i' \otimes_{S_1} s_i n' z t_{g^{-1}} \quad \text{(by definition)} \\
&= \sum_{i=1}^{l} r_g n s_i' \otimes_{S_1} s_i z n' t_{g^{-1}} = \sum_{i=1}^{l} r_g n s_i' \otimes_{S_1} {}^{g}z s_i n' t_{g^{-1}} \\
&= \sum_{i=1}^{l} r_g n s_i'{}^{g}z \otimes_{S_1} s_i n' t_{g^{-1}} = \sum_{i=1}^{l} r_g n^{g^{-1}g} z s_i' \otimes_{S_1} s_i n' t_{g^{-1}} \\
&= \sum_{i=1}^{l} r_g n z s_i' \otimes_{S_1} s_i n' t_{g^{-1}} = \sum_{i=1}^{l} r_g z n s_i' \otimes_{S_1} s_i n' t_{g^{-1}} \\
&= \sum_{i=1}^{l} \left(r_g z n s_i' \otimes_{S_1} s_i n' t_{g^{-1}} \right) \quad \text{(by definition)} \\
&= (r_g z \otimes_Z t_g^{\mathrm{op}})(n \otimes_{S_1} n').
\end{aligned}$$

2) As above, one verifies that for all $f \in \mathrm{Hom}_{S_1}(N, N')$, $r_{g^{-1}} \in R_{g^{-1}} R$, $t_g \in T_g$, $n \in N$ and $z \in Z$ we have that $(r_{g^{-1}} f z t_g)(n)$ equals $(r_{g^{-1}} z f t_g)(n)$. $\square$

THEOREM 2.4. *Let M_1 be a (R_1, S_1)-bimodule and N_1 a (S_1, R_1)-bimodule such that the bimodules M_1 and N_1 induce a Morita equivalence between R_1 and S_1. Moreover, if M_1 is a Δ_1-module, then N_1 extends to a Δ-module, and $\widetilde{M_1}$, $\widetilde{N_1}$ induce G-graded Morita equivalences between R and S over Z.*

Proof. It is straightforward to verify that the map

$$\alpha =: M \otimes_S N \to R$$

of the proof of [3, Theorem 3.4] is an isomorphisms of G-graded (R, R)-bimodules over Z, while the map

$$\beta = \ N \otimes_R M \to S$$

is an isomorphisms of G-graded S-bimodules over Z. □

3. SKEW GROUP ALGEBRAS

DEFINITION 3.1. Let G be a group and A a G-algebra. The *skew group algebra*, denoted by $A * G$, has as underlying A-module the free left A-module having as basis all the elements in G, with the multiplication defined by

$$(ag)(bh) = a^g bgh$$

for all $a, b \in A$ and $g, h \in G$.

Let G be a finite group, Z a commutative G-algebra, and let $F = Z^G$. Throughout this section we assume that A and B are G-algebras over Z, that is, there exist the G-ring homomorphisms $Z \to Z(A)$ and $Z \to Z(B)$. In this case, the skew group algebras $A * G$ and $B * G$ are G-graded F-algebras over Z. We denote $R = A * G$ and $S = B * G$.

Note that the tensor products $A \otimes_Z B$ and $A \otimes_Z B^{\mathrm{op}}$ are both G-algebras with diagonal action.

DEFINITION 3.2. A Morita equivalence over Z between A and B induced by two bimodules M and N is said to be *G-equivariant* if

(1) M is a $(A \otimes_Z B^{\mathrm{op}}) * G$ module,
(2) N is a $(B \otimes_Z A^{\mathrm{op}}) * G$ module,

and all homomorphisms involved in the Morita equivalence are ZG-linear.

Recall that Herman and Mitra [2, Proposition 9] says that two central separable G-algebras are equivalent (in the sense of Turull [7]) if and only if they are equivariantly Morita equivalent. They also proved that the group of equivalence classes of central separable G-algebras over a commutative G-algebra is isomorphic to the Brauer-Clifford group $\mathrm{BrClif}(Z, G)$. We wish to relate this to strongly graded central simple algebras.

LEMMA 3.3. *Let A and B to be G-algebras over Z and let $R = A * G$ and $S = B * G$. Then there is an isomorphism*

$$\Delta(R \otimes_Z S^{\mathrm{op}}) \simeq (A \otimes_Z B^{\mathrm{op}}) * G \tag{2}$$

of G-graded F-algebras over Z.

Proof. We show that there is a G-graded F-algebra isomomorphism

$$f:(A\otimes_Z B^{\mathrm{op}})*G\to\Delta(R\otimes S^{\mathrm{op}}),$$

where G acts on $A\otimes_Z B^{\mathrm{op}}$ diagonally. We define f as follows:

$$f((a\otimes_Z b^{\mathrm{op}})g)=ag\otimes g^{-1}b^{\mathrm{op}}.$$

We first need to show that this map is well-defined:

$$\begin{aligned}f((az\otimes b^{\mathrm{op}})g)&=azg\otimes g^{-1}b^{\mathrm{op}}=agg^{-1}zg\otimes g^{-1}b^{\mathrm{op}}\\&=ag({}^{g^{-1}}z)g^{-1}g\otimes g^{-1}b^{\mathrm{op}}=ag({}^{g^{-1}}z)\otimes g^{-1}b^{\mathrm{op}}\\&=ag\otimes({}^{g^{-1}}z)g^{-1}b^{\mathrm{op}}=ag\otimes g^{-1}zb^{\mathrm{op}}\\&=f((a\otimes zb^{\mathrm{op}})g).\end{aligned}$$

To show that f is a homomorphism, notice that for all $a,c\in A$, $b^{\mathrm{op}},d^{\mathrm{op}}\in B^{\mathrm{op}}$ and $g,h\in G$, we have

$$\begin{aligned}f((a\otimes b^{\mathrm{op}})g\cdot(c\otimes d^{\mathrm{op}})h)&=f((a\otimes b^{\mathrm{op}})^{g}(c\otimes d^{\mathrm{op}})gh)\\&=f((a\otimes b^{\mathrm{op}})({}^{g}c\otimes({}^{g^{-1}}d^{\mathrm{op}}))gh)\\&=f((a^{g}c\otimes({}^{g^{-1}}d^{\mathrm{op}})b^{\mathrm{op}})gh)\\&=a({}^{g}c)gh\otimes h^{-1}g^{-1}({}^{g^{-1}}d^{\mathrm{op}})b^{\mathrm{op}},\end{aligned}$$

while

$$\begin{aligned}f((a\otimes b^{\mathrm{op}})g)f((c\otimes d^{\mathrm{op}})h)&=(ag\otimes g^{-1}b^{\mathrm{op}})(ch\otimes h^{-1}d^{\mathrm{op}})\\&=agch\otimes h^{-1}d^{\mathrm{op}}g^{-1}b^{\mathrm{op}}\\&=a^{g}cgh\otimes h^{-1}g^{-1}gd^{\mathrm{op}}g^{-1}b^{\mathrm{op}}\\&=a^{g}cgh\otimes h^{-1}g^{-1}({}^{g}(d^{\mathrm{op}}))gg^{-1}b^{\mathrm{op}}\\&=a^{g}cgh\otimes h^{-1}g^{-1}({}^{g^{-1}}d^{\mathrm{op}})b^{\mathrm{op}}.\end{aligned}$$

The last equality is true because the action of g on $d^{\mathrm{op}}\in B^{\mathrm{op}}$ is equal to ${}^{g^{-1}}d^{\mathrm{op}}$, where the operation is now the one in B.

Now let $z\in F=Z^G$ and $a\in A$ and $b^{\mathrm{op}}\in B^{\mathrm{op}}$. Then

$$\begin{aligned}f(z(a\otimes_Z b^{\mathrm{op}})g)=f((za\otimes_Z b^{\mathrm{op}})g)&=(za)g\otimes_Z g^{-1}b^{\mathrm{op}}\\&=z(ag\otimes_Z g^{-1}b^{\mathrm{op}})=zf((a\otimes_Z b^{\mathrm{op}})g).\end{aligned}$$

We used that z is G-invariant in the second row of the equation above.

By definition, it is clear that the homomorphism is grading preserving. Hence (2) holds. $\square$

Corollary 3.4. *Let M_1 be a (A,B)-bimodule. If M_1 induces a G-equivariant Morita equivalence between A and B, then $R\otimes_{R_1}M_1$ induces a G-graded Morita equivalence between R and S.*

Conversely, let M be a G-graded (R,S)-bimodule. If M induces a G-graded Morita equivalence over Z between R and S, then M_1 induces a G-equvariant Morita equivalence over Z between A and B.

Proof. These statements follow immediately from Lemma 2.2, Theorem 2.4 and Lemma 3.3. □

4. THE CENTRAL SIMPLE GRADED ALGEBRA ASSOCIATED TO A CHARACTER

Assume that the field F has the characteristic zero and that the strongly G-graded F-algebra R is semisimple. Denote by $\overline{F}$ the algebraic closure of F and let $\overline{F}R := \overline{F} \otimes_F R$.

We take V to be a simple R-module, and let χ be the character of a simple submodule of the $\overline{F}R$-module $\overline{F} \otimes_F V$. Let θ_1 be an irreducible character contained in the restriction of χ to $\overline{F}A$ and let $\theta_1 \ldots, \theta_r \in \mathrm{Irr}(\overline{F}A)$ be the $G \times \mathrm{Gal}(\overline{F}/F)$-orbit of θ_1. Also let

$$\overline{\theta} := \theta_1 + \cdots + \theta_r,$$

and let $e_{\theta_1}, \ldots, e_{\theta_r}$ be the corresponding primitive idempotents of $Z(\overline{F}A)$. We denote by e the sum of these idempotents,

$$e := e_{\theta_1} + \cdots + e_{\theta_r} \in Z(A)$$

and $F_0 := e(A \cap Z(R))$. Let $\theta_1, \ldots, \theta_s$ be the representatives for the orbits of the action of $\mathrm{Gal}(\overline{F}/F)$ on $\{\theta_1 \ldots \theta_r\}$.

Then $\mathrm{End}_R(R \otimes_A V)^{\mathrm{op}}$ is a central simple G-graded F_0-algebra over Z where

$$Z := eZ(A) \simeq F(\theta_1) \oplus F(\theta_2) \oplus \cdots \oplus F(\theta_s)$$

as G-acted F_0-algebras.

Instead of V as above we may use another more general R-module.

Definition 4.1. An R-module W is *χ-quasihomogeneous* (*with respect to* R_1) if it is not 0 and the character of $\overline{F} \otimes_F W$ is of the form $m\overline{\theta}$, where m is a positive integer.

Proposition 4.2. *If W is a χ-quasihomogeneous R-module, then there is a G-graded Morita equivalence over Z between* $\mathrm{End}_R(R \otimes_A V)$ *and* $\mathrm{End}_R(R \otimes_A W)$.

Proof. By a well-known result of E. Dade we know that $\mathrm{End}_A(V)$ and $\mathrm{End}_A(W)$ are G-algebras in a natural way, and furthermore we have that

$$\mathrm{End}_R(R \otimes_A V) \simeq \mathrm{End}_A(V) * G \quad \text{and} \quad \mathrm{End}_R(R \otimes_A W) \simeq \mathrm{End}_A(W) * G.$$

Moreover, $\mathrm{End}_A(W)$ is also a central simple G-algebra over Z. As in the proof of Turull [7, Theorem 7.6], there is a G-equivariant Morita equivalence over Z between $\mathrm{End}_A(V)$ and $\mathrm{End}_A(W)$. Finally, we use Corollary 3.4.

Alternatively, let V_0 be a simple A-submodule of V, and observe that the fact that W is χ-quasihomogeneous mean that W belongs to the subcategory $(R \mid V_0)$-mod of R-modules lying over V_0. Consequently, both $R \otimes_A V$ and

$R \otimes_A W$ are G-graded bimodules over Z which generate $(R \mid V_0)$-mod, and hence the functors

$$\mathrm{Hom}_R(R \otimes_A V, -) : (R \mid V_0)\text{-mod} \to \mathrm{End}(R \otimes_A V)^{\mathrm{op}}\text{-mod}$$

and

$$\mathrm{Hom}_R(R \otimes_A W, -) : (R \mid V_0)\text{-mod} \to \mathrm{End}(R \otimes_A W)^{\mathrm{op}}\text{-mod}$$

are equivalences. □

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