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JACK'S LEMMA AND A CLASS OF POLYNOMIAL INEQUALITIES

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Abstract. We study Jack's lemma from the point of view of a class of polynomial inequalities involving bound-preserving operators.

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1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disc$\{z \in \mathbb{C} \mid |z| < 1\}$ of the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ the set of functions analytic on $\mathbb{D}$. We define for $f \in \mathcal{H}(\mathbb{D})$

$$|f|_{\mathbb{D}} := \sup_{z\in\mathbb{D}} |f(z)|.$$

Let also $\mathcal{P}_n$ denote the set of polynomials of degree at most n with coefficients in $\mathbb{C}$. The inequality (valid for any $p \in \mathcal{P}_n$)

$$(1) \qquad |zp'(z)/n - p(z)| + |zp'(z)/n| \leq |p|_{\mathbb{D}}, \quad |z| \leq 1,$$

is a well-known refinement of the classical Bernstein inequality for polynomials on the unit disc. The paper [1] contains references concerning various proofs of (1).

It has been observed by Sheil–Small [7, p. 152] that equality holds in (1) for any $p \in \mathcal{P}_n$ and any u in the closed unit disc $\overline{\mathbb{D}}$ such that $|p(u)| = |p|_{\mathbb{D}}$. (The only other case of equality, as proved in [1], occurs when $p(z) = Az^n + B$ at any point u with $|u| = 1$). This leads to a painless proof of Jack's lemma for polynomials: indeed if $p \in \mathcal{P}_n$ and $|p(u)| = |p|_{\mathbb{D}}$ for some $|u|$, ($|u| = 1$), then

$$(2) \qquad |up'(u)/n - p(u)| + |up'(u)/n| = |p|_{\mathbb{D}} = |p(u)|$$

and (2) is easily seen to amount to

$$(3) \qquad 0 \leq \tfrac{up'(u)}{np(u)} \leq 1,$$

which is a version of Jack's lemma for polynomials. It has been established in [4] and [3] that $0 < \frac{up'(u)}{p(u)}$ unless the polynomial p is constant or equivalently that $\frac{up'(u)}{p(u)} < n$ unless p is a monomial of degree n. We refer to the book of Miller and Mocanu [5] concerning Jack's lemma and its applications in geometric function theory.

Let $P_{1/2}$ denote the class of functions F in $\mathcal{H}(\mathbb{D})$ with $F(0) = 1$ and $\operatorname{Re} F(z) > \frac{1}{2}$ if $z \in \mathbb{D}$; let also $\star$ denote the usual Hadamard product of functions in $\mathcal{H}(\mathbb{D})$. Ruscheweyh [6, p. 128] proved that for $p \in \mathcal{P}_n$

$$|W \star p(z)| + |\widetilde{W} \star p(z)| \le |p|_{\mathbb{D}}, \quad |z| \le 1, \tag{4}$$

where $W \in \mathcal{P}_{n-1} \cap P_{1/2}$ and $\widetilde{W}(z) := z^n \overline{W(1/\bar{z})} \in \mathcal{P}_n$. Given the fact that $W(z) := \sum_{k=0}^{n-1} \bigl(1 - \frac{k}{n}\bigr) z^k \in \mathcal{P}_{n-1} \cap P_{1/2}$, we see that (4) is a striking generalization of (1). It was proved in [4] that a corresponding generalization of Jack's lemma follows from (4).

Let $F(z) := 1 + \sum_{k=1}^{\infty} A_k z^k \in P_{1/2}$ where, for a given $n \ge 1$, the associated Toeplitz $(n+1) \times (n+1)$ determinant $D_n(F)$ with first row $(1, A_1, \dots, A_n)$ is strictly positive. We recently established in [2] the existence of a constant $d_n = d_n(F)$ such that $0 < d_n(F) \le 1$ and for any $p \in \mathcal{P}_n$

$$|p \star F(z)| + d_n |p(z) - p \star F(z)| \le |p|_{\mathbb{D}}, \quad |z| \le 1. \tag{5}$$

In some sense, (5) is an extension of (1) which corresponds to the case where $F(z) = \sum_{k=0}^{n} \bigl(1 - \frac{k}{n}\bigr) z^k \in P_{1/2}$ with the associated strictly positive Toeplitz determinant and of course $d_n(F) = 1$. We define

$$P^{\star}_{1/2} := \{F \in P_{1/2} \mid D_n(F) > 0 \text{ and } d_n(F) = 1\} \ne \emptyset.$$

For $F \in P^{\star}_{1/2}$ we have

$$|p \star F(z)| + |p(z) - p \star F(z)| \le |p|_{\mathbb{D}}, \quad |z| \le 1, \tag{6}$$

and if for some $p \in \mathcal{P}_n$ and $u \in \partial\mathbb{D}$ we have $|p(u)| = |p|_{\mathbb{D}}$, we obtain by (6)

$$|p \star F(u)| + |p(u) - p \star F(u)| = |p(u)|$$

and clearly

$$0 \le \tfrac{p \star F(u)}{p(u)} \le 1 \quad \text{if} \quad F \in P^{\star}_{1/2},\ p \in \mathcal{P}_n \quad \text{and} \quad |p(u)| = |p|_{\mathbb{D}}. \tag{7}$$

In particular $0 \le A_k \le 1$ if $F(z) = 1 + \sum_{k=1}^{n} A_k z^k + o(z^n)$. At first sight, (7) looks like an exciting extension of Jack's lemma [1]. The main result of this note shows that this is not indeed the case. We shall prove

THEOREM 1. *The members of $P^{\star}_{1/2}$ are of the type*

$$F_t(z) = \sum_{k=0}^{n} \bigl(1 - t\tfrac{k}{n}\bigr) z^k + o(z^n)$$

with $0 \le t \le 1$.

According to (7), we obtain for any $p \in \mathcal{P}_n$ with $|p(u)| = |p|_{\mathbb{D}}$

$$0 \le \frac{p \star F_t(u)}{p(u)} = 1 - t\tfrac{up'(u)}{np(u)} \le 1$$

i.e., nothing more than (3)!

2. PROOF OF THE THEOREM

We shall rely on the following

LEMMA 1. *For any $n \geq 2$, there exists a polynomial $p \in \mathcal{P}_n$ and $u \in \partial\mathbb{D}$ such that $|p(u)| = |p|_{\mathbb{D}}$ and $\frac{p(0)}{p(u)}$ is not real.*

Proof. Let $p(z) = 1 - z - z^2$. Then $|p(\mathrm{e}^{\mathrm{i}\theta})| = |1 - \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{2\mathrm{i}\theta}| = |-1 - 2\mathrm{i}\sin(\theta)|$ and clearly $|p(\mathrm{i})| = |p|_{\mathbb{D}}$ but $\frac{p(0)}{p(\mathrm{i})} = \frac{1}{2-\mathrm{i}}$ is not real.

When $n > 2$, we set $p(z) = (1+z)(1 - z^{n-1})$. We have

$$p(\mathrm{e}^{\mathrm{i}\theta}) = (1 + \mathrm{e}^{\mathrm{i}\theta})\left(1 - \mathrm{e}^{\mathrm{i}(n-1)\theta}\right) = 4\mathrm{e}^{\mathrm{i}\left(\frac{n}{2}\theta + \frac{n}{2}\right)} \cos\left(\tfrac{\theta}{2}\right) \sin\left(\tfrac{n-1}{2}\theta\right).$$

Clearly if $p(\mathrm{e}^{\mathrm{i}\theta}) = \pm |p|_{\mathbb{D}}$, then

$$\sin\left(\tfrac{n}{2}\theta + \tfrac{\pi}{2}\right) = \cos\left(\tfrac{n}{2}\theta\right) = 0$$

and $\frac{\mathrm{d}}{\mathrm{d}\,\theta} \cos\frac{\theta}{2} \sin\frac{n-1}{2}\theta = \pm\frac{(n-2)}{2}\sin(\frac{\theta}{2})\cos\left(\frac{\theta}{2}\right) = \pm\frac{(n-2)}{4}\sin\theta = 0$. This is impossible, because $p(1) = p(-1) = 0$, and again in this case $\frac{p(0)}{p(u)}$ is not real. □

Proof of Theorem 1. We shall prove our Theorem by induction on $n \geq 1$. When $n = 1$, let $F(z) = 1 + A_1 z + o(z)$ and $p(z) = a_0 + a_1 z \in \mathcal{P}_1$ satisfy (6). This is easily seen to amount to

$$|a_0| + |A_1|\,|a_1| + |a_1|\,|1 - A_1| \leq |a_0| + |a_1|$$

and since this must hold for an arbitrary polynomial in $\mathcal{P}_1$, we obtain $|A_1| + |1 - A_1| \leq 1$, i.e., $0 \leq A_1 \leq 1$. We then have

$$1 + A_1 z + o(z^n) = \sum_{k=0}^{n} \left(1 - t\tfrac{k}{n}\right) z^k + o(z^n) \text{ for } \; n = 1 \quad \text{and} \quad t = 1 - A_1.$$

Let us now assume our result valid for $n-1$ and consider $Q(z) = 1 + \sum_{k=1}^{n} A_k z^k + o(z^n) \in P^{\star}_{1/2}$. We then have, by the induction hypothesis, for any $p \in \mathcal{P}_{n-1} \subset \mathcal{P}_n$

$$|(Q(z) - A_n z^n) \star p(z)| + |p(z) - (Q(z) - A_n z^n) \star p(z)|$$
$$= |Q \star p(z)| + |p(z) - Q(z) \star p(z)| \leq |p|_{\mathbb{D}}$$

and therefore, for some $t \in [0,1]$,

$$Q(z) = \sum_{k=0}^{n-1} \left(1 - t\tfrac{k}{n-1}\right) z^k + A_n z^n. \tag{8}$$

It follows that for any $p(z) = a_n z^n + \cdots$ in $\mathcal{P}_n$,

$$\frac{Q\star p(z)}{p(z)} = \frac{p(z)-a_nz^n-\frac{t}{n-1}(zp'(z)-na_nz^n)+A_na_nz^n}{p(z)} \tag{9}$$

$$= 1 - \frac{1}{n-1}\frac{zp'(z)}{p(z)} + \frac{a_nz^n}{p(z)}\big[-1+\frac{nt}{n-1}+A_n\big].$$

Assuming now that $|u| = 1$ and $|p(u)| = |p|_{\mathbb{D}}$, it follows from (3) and (7) that $0 \le \frac{up'(u)}{p(u)}$ and $0 \le \frac{Q\star p(u)}{p(u)}$. We therefore obtain from (9) that $\frac{a_nu^n}{p(u)}\big[-1+\frac{nt}{n-1}+A_n\big]$ is real and $A_n = 1-t\frac{n}{n-1}$, because otherwise $\frac{a_nu^n}{p(u)}$ would be real; this is a violation of Lemma 1, because p is arbitrary. We therefore have $A_n = 1-t\frac{n}{n-1}$ and, by (8),

$$Q(z) = \sum_{k=0}^{n-1}\big(1-t\tfrac{k}{n-1}\big)z^k + \big(1-t\tfrac{n}{n-1}\big)z^n + o(z^n)$$

$$= \sum_{k=0}^{n}\big(1-\tau\tfrac{k}{n}\big)z^k + o(z^n),$$

where $0 \le \tau = \frac{nt}{n-1} \le 1$, because, by (7), $1-t\frac{n}{n-1} \ge 0$. This concludes the proof of our Theorem. □

3. CONCLUSION

We first remark that cases of equality in (6) for $F = F_t$ are not difficult to establish. Indeed, if for some $0 \le t \le 1$, $u \in \partial\mathbb{D}$ and $p \in \mathcal{P}_n$ we have

$$|p|_{\mathbb{D}} = \big|p(u) - t\tfrac{up'(u)}{n}\big| + \big|t\tfrac{up'(u)}{n}\big|,$$

then

$$|p|_{\mathbb{D}} = \big|t\big(p(u)-\tfrac{up'(u)}{n}\big)+(1-t)p(u)\big| + t\big|\tfrac{up'(u)}{n}\big| \tag{10}$$

$$\le t\big(\big|p(u)-\tfrac{up'(u)}{n}\big| + \big|\tfrac{up'(u)}{n}\big|\big) + (1-t)|p(u)|$$

$$\le t|p|_{\mathbb{D}} + (1-t)|p|_{\mathbb{D}}$$

and equality holds everywhere in (10). It then follows from our introduction that either $p \in \mathcal{P}_n$ and $|p(u)| = |p|_{\mathbb{D}}$ if $0 \le t \le 1$ or else $p(z) = Az^n + B$ with $u \in \partial\mathbb{D}$ if $t = 1$.

We have so far identified all functions F satisfying (6) for all $p \in \mathcal{P}_n$: these are the functions F_t, $0 \le t \le 1$, introduced in Theorem 1. These functions also satisfy

$$0 \le \frac{F\star p(u)}{p(u)} \le 1, \quad p \in \mathcal{P}_n,\ |p(u)| = |p|_{\mathbb{D}}. \tag{11}$$

It is a natural question to ask if other functions $F \in \mathcal{H}(\mathbb{D})$ with $F(0) = 1$ may also satisfy (11), since a negative answer would in some sense assert some sort of unicity in the statement of Jack's lemma for polynomials of fixed degree n (which is equivalent to (7) with $F = F_t$ and $0 \le t \le 1$). As a matter of fact, the induction argument used in the proof of Theorem 1 can also be used to prove that only functions of the type F_t can satisfy (11). We supply a sketch of the induction argument.

Let us assume that any $F \in \mathcal{H}(\mathbb{D})$ satisfying (11) and $F(0) = 1$ is of the type

$$F(z) = \sum_{k=0}^{n} \left(1 - \tfrac{tk}{n}\right) z^k + o(z^n)$$

for some $0 \le t \le 1$. Let now

$$0 \le \tfrac{G \star p(u)}{p(u)} \le 1, \quad p \in \mathcal{P}_{n+1}, \quad |p(u)| = |p|_{\mathbb{D}} \tag{12}$$

for some G in $\mathcal{H}(\mathbb{D})$ with $G(0) = 1 + \cdots + A_{n+1} z^{n+1} + o(z^{n+1})$.

Then of course

$$0 \le \left. \frac{(G(z) - A_{n+1} z^{n+1}) \star p(z)}{p(u)} \right|_{z=u} \le 1$$

for all $p \in \mathcal{P}_n$ with $|p(u)| = |p|_{\mathbb{D}}$. By the induction hypothesis, for some $0 \le t \le 1$

$$G(z) = \sum_{k=0}^{n} \left(1 - t\frac{k}{n}\right) z^k + A_{n+1} z^{n+1} + o(z^{n+1})$$

with

$$\begin{aligned} G \star g(z) &= g(z) - a_{n+1} z^{n+1} - \frac{t}{n}\left(zg'(z) - (n+1) a_{n+1} z^{n+1}\right) + A_{n+1} a_{n+1} z^{n+1} \\ &= g(z) - \frac{t}{n} z g'(z) + a_{n+1} z^{n+1} \left(-1 + \frac{(n+1)t}{n} + A_{n+1} \right) \end{aligned} \tag{13}$$

for any $g \in \mathcal{P}_{n+1}$ with leading coefficient $a_{n+1} \ne 0$. If we also assume that $|g(u)| = |g|_{\mathbb{D}}$ for some $u \in \partial\mathbb{D}$, it shall follow from (12), (13) and the standard Jack's lemma that

$$\tfrac{a_{n+1} u^{n+1}}{g(u)} \left(-1 + \tfrac{(n+1)t}{n} + A_{n+1} \right) \quad \text{is real.}$$

Because g is arbitrary, this shall contradict our lemma if $A_{n+1} \ne 1 - \frac{(n+1)t}{n}$. We therefore obtain

$$\begin{aligned} G(z) &= \sum_{k=0}^{n} \left(1 - t\tfrac{k}{n}\right) z^k + \left(1 - t\tfrac{(n+1)}{n}\right) z^{n+1} + o(z^{n+1}) \\ &= \sum_{k=0}^{n+1} \left(1 - t\tfrac{k}{n}\right) z^k + o(z^{n+1}) \\ &= \sum_{k=0}^{n+1} \left(1 - \tau\tfrac{k}{n+1}\right) z^k + o(z^{n+1}) \end{aligned}$$

with $\tau = \frac{t(n+1)}{n} \in [0,1]$, because, as above,

$$0 \leq A_{n+1} = 1 - \frac{(n+1)}{n} t \leq 1.$$

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