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ON FINITE GENERATION OF HOCHSCHILD COHOMOLOGY ALGEBRAS OF SOME FULLY GROUP GRADED ALGEBRAS

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Abstract. In this paper we study the finite generation problem for the Hochschild cohomology algebras of some fully group graded algebras using Grothendieck spectral sequences. Some minimal examples for which we can apply our results are given.

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Key words. Spectral sequence, group graded algebra, Hochschild cohomology.

1. PRELIMINARIES

Let k be a commutative ring, G be a finite group and let R be a fully G-graded k-algebra. By definition, R has a decomposition $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h = R_{gh}$ for all $g, h \in G$. These algebras are also called in literature "strongly", but we will use "fully". We denote by $\mathrm{HH}^*(R, M)$ the Hochschild cohomology of R with coefficients in the $R-R$-bimodule M and by $\mathrm{H}^*(G, A)$ the group cohomology of G with coefficients in a kG-module A. The finite generation problem of a cohomology ring is intensively studied for different types of cohomologies. Finite generation questions are of interest in their own right, but there are also important applications. If the cohomology ring is finitely generated, one may define algebraic varieties associated to modules, called support varieties, that contain useful information.

This paper is organized as follows. In Section 3 we recall a Grothendieck spectral sequence for Hochschild cohomology of a fully group graded algebra from [4] (Remark 4) and we also give a slightly modified variant of this spectral sequence (Remark 3). The main result of Section 4 is Proposition 2. We prove that for a fully group graded algebra R belonging to a class of rings, which we call HIF-rings (see Definition 1), there is a spectral sequence which is a module over a spectral sequence of rings. We warn the reader that it is not our intention to analyze this class of HIF-rings in this paper, but some examples are provided. In Section 5 we define a class of fully G-graded algebras and give some examples. In the main result of this paper (Theorem 1) we give

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conditions for the finite generation of the Hochschild cohomology algebra of a fully group graded algebra.

2. GROTHENDIECK SPECTRAL SEQUENCE FOR FULLY GRADED ALGEBRAS

In this section we give the Grothendieck spectral sequence for fully G-graded algebras, from [4], in a slightly more general setup. We recall from there some basic notations and results. If H is a subgroup of G then $R_H = \bigoplus_{h\in H} R_h$ is a fully H-graded subalgebra of R. If M, N are R-modules then $\mathrm{Hom}_{R_1}(M,N)$ is a kG-module with

$$^{g}f(m) = \sum_{i=1}^{\lambda} r_i f(r_i' m),$$

where $f \in \mathrm{Hom}_{R_1}(M,N), g \in G, m \in M$ and $r_i \in R_g, r_i' \in R_{g^{-1}}$ such that $\sum_{i=1}^{\lambda} r_i r_i' = 1$. Also $\mathrm{Hom}_{R_1}(M,N)^H = \mathrm{Hom}_{R_H}(M,N)$ for every subgroup H of G.

With R^{op} we denote the opposite algebra, which is also a G-graded algebra with components $R_g^{op} = R_{g^{-1}}$. Then $R^e = R\otimes_k R^{op}$ is a $G\times G$-graded algebra which has $\Delta(R) = \bigoplus_{g\in G} R_g \otimes_k R_{g^{-1}}$ as a k-subalgebra. Moreover $\Delta(R)$ is fully G-graded with $\Delta(R)_g = R_g \otimes R_g^{op}$. The first component is $\Delta(R)_1 = R_1^e$.

For simplicity we assume that R_1 is projective as k-module, hence R is projective as k-module. Let M be an $\Delta(R)$-module. Then it is also an R_1^e-module, i.e an $R_1 - R_1$-bimodule. Since M is a $\Delta(R)$-module we have that $\mathrm{Hom}_{R_1^e}(R_1, M) = \mathrm{Hom}_{\Delta(R)_1}(R_1, M)$ is a kG-module and moreover

$$\mathrm{Hom}_{kG}(k, \mathrm{Hom}_{R_1^e}(R_1, M) \cong \mathrm{Hom}_{\Delta(R)_1}(R_1, M)^G$$
$$= \mathrm{Hom}_{\Delta(R)_G}(R_1, M) = \mathrm{Hom}_{\Delta(R)}(R_1, M).$$

In the case that M is an R^e-module considered by restriction as $\Delta(R)$-module, we use that $\mathrm{Ind}_{\Delta(R)}^{R^e} R_1 \cong R$ as R^e-modules (see [4, Section 2]) to obtain as above, that

$$\mathrm{Hom}_{kG}(k, \mathrm{Hom}_{R_1^e}(R_1, M) \cong \mathrm{Hom}_{R^e}(R, M).$$

We consider the functors

$$\mathcal{F}_1 = \mathrm{Hom}_{kG}(k, -) : kG\text{-Mod} \to k\text{-Mod},$$
$$\mathcal{F}_2 = \mathrm{Hom}_{R_1^e}(R_1, -) : \Delta(R)\text{-Mod} \to kG\text{-Mod},$$
$$\mathcal{F} = \mathrm{Hom}_{\Delta(R)}(R_1, -) : \Delta(R)\text{-Mod} \to k\text{-Mod}.$$

LEMMA 1. *Under the above assumptions we have that $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ and that $\mathcal{F}_2$ sends $\Delta(R)$-injective modules to $\mathcal{F}_1$-acyclic modules.*

Proof. The first statement follows from the above results. The second statement follows from a more general proposition applied to Hopf-Galois extension, see [7, Proposition 3.2]. □

Remark 1. Since $\mathcal{F}_1, \mathcal{F}_2$ are left exact, by Lemma 2 it follows that there is a Grothendieck spectral sequence

$$\mathrm{H}^p(G, \mathrm{HH}^q(R_1, M) \Rightarrow \mathrm{Ext}^{p+q}_{\Delta(R)}(R_1, M)),$$

for any $\Delta(R)$-module M and p, q nonnegative integers.

We notice that if R is a crossed product then $\Delta(R)$ is actually $\Delta(G)$ considered by S. J. Witherspoon in [9, Section 3]. When we take $M = R$ then $\mathrm{HH}^*(R) \cong \mathrm{Ext}^*_{\Delta(R)}(R_1, R)$ as graded k-modules and there is a cup product on $\mathrm{Ext}^*_{\Delta(R)}(R_1, R)$ such that we obtain an algebra isomorphism. See the results before [9, Lemma 3.8].

Remark 2. If M is a R^e-module we make the similar constructions with similar functors as above, and since

$$\mathrm{Ext}^{p+q}_{\Delta(R)}(R_1, M) \cong \mathrm{HH}^{p+q}(R, M)$$

we obtain the Grothendieck spectral sequence for Hochschild cohomology

$$\mathrm{H}^p(G, \mathrm{HH}^q(R_1, M)) \Rightarrow \mathrm{HH}^{p+q}(R, M).$$

This is the spectral sequence from [4, 3.1] and it will be used in the following sections.

3. GROTHENDIECK SPECTRAL SEQUENCE FOR FULLY GRADED ALGEBRAS

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In the case that M is an R^e-module considered by restriction as $\Delta(R)$-module, we use that $\mathrm{Ind}_{\Delta(R)}^{R^e} R_1 \cong R$ as R^e-modules (see [4, Section 2]) to obtain as above, that

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We consider the functors

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LEMMA 2. *Under the above assumptions we have that $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ and that $\mathcal{F}_2$ sends $\Delta(R)$-injective modules to $\mathcal{F}_1$-acyclic modules.*

Proof. The first statement follows from the above results. The second statement follows from a more general proposition applied to Hopf-Galois extension, see [7, Proposition 3.2]. □

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REMARK 4. If M is a R^e-module we make the similar constructions with similar functors as above, and since

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This is the spectral sequence from [4, 3.1] and it will be used in the following sections.

4. MULTIPLICATIVE STRUCTURE OF THE ABOVE SPECTRAL SEQUENCES

In the following sections we assume that k is a field. Any Grothendieck spectral sequence is actually obtained as the spectral sequence associated to a specific cochain double complex (see [3, VIII. Theorem 9.3]). Let M be an R^e-module. In this section we give the construction of the double complex, denoted $E_0^{*,*}(M)$ (used to obtain the spectral sequence from Remark 4), and we assure the reader that under some assumptions there is a natural pairing. First we take an injective R^e-resolution of M

$$0 \to M \to I_0 \to I_1 \to I_2 \to \dots$$

Then we apply $\mathrm{Hom}_{R_1^e}(R_1, -)$ to get a complex of kG-modules

$$0 \to \mathrm{Hom}_{R_1^e}(R_1, M) \to \mathrm{Hom}_{R_1^e}(R_1, I_0) \to \mathrm{Hom}_{R_1^e}(R_1, I_1) \to \dots,$$

and consequently a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R_1^e}(R_1, I_0) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I_1) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I_2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J_{0,0} & \longrightarrow & J_{1,0} & \longrightarrow & J_{2,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J_{0,1} & \longrightarrow & J_{1,1} & \longrightarrow & J_{2,1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & ,
\end{array}$$

in which every column is an injective resolution of the top kG-module. We apply the functor $\mathrm{Hom}_{kG}(k, -)$ and obtain the double cochain complex denoted by $E_0^{*,*}(M)$

$$\begin{array}{ccccccc}
\mathrm{Hom}_{kG}(k, J_{0,0}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{1,0}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{2,0}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{kG}(k, J_{0,1}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{1,1}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{2,1}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{kG}(k, J_{0,2}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{1,2}) & \longrightarrow & \mathrm{Hom}_{kG}(k, J_{2,2}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & ,
\end{array}$$

The above diagram give us the spectral sequence from Remark 4, now denoted

$$E_0^{*,*}(M) \Rightarrow E_\infty^{*,*}(M).$$

To prove the next results we introduce the following class of rings.

Definition 1. A ring R is called *right HIF-ring* if any injective right R-module is flat and any homomorphic image of an injective right R-module is also flat.

This class of rings is nonempty since includes the Von Neumann regular rings, and also it is included in the class of right IF-rings, a class of rings first considered by S. Jain in [6].

Proposition 1. Let R be a fully G-graded algebra such that R is a right HIF-ring. Then there is a natural pairing of double complexes

$$E_0^{*,*}(M) \otimes E_0^{*,*}(N) \to E_0^{*,*}(M \otimes_R N),$$

for any $M, N \in R^e$-Mod.

Proof. Let $M, N \in R^e$-Mod and $E_0^{i,j}(M) = \mathrm{Hom}_{kG}(k, J_{i,j})$ defined as in the beginning of this section, for i, j two nonnegative integers. Similarly we can obtain the double complex $E_0^{*,*}(N) = \mathrm{Hom}_{kG}(k, J'_{s,t})$, for any s, t two nonnegative integers, using an injective R^e-resolution of N

$$0 \to N \to I'_0 \to I'_1 \to I'_2 \to \ldots,$$

and the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R_1^e}(R_1, I'_0) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I'_1) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I'_2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J'_{0,0} & \longrightarrow & J'_{1,0} & \longrightarrow & J'_{2,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J'_{0,1} & \longrightarrow & J'_{1,1} & \longrightarrow & J'_{2,1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & .
\end{array}$$

Next we have that $M \otimes_R N$ is an R^e-module, since it is well known that R^e-Mod is a monoidal category with tensor product $- \otimes_R -$ such that R is the tensor identity. As above we obtain the double complex $E_0^{*,*}(M \otimes_R N) = \mathrm{Hom}_{kG}(k, J''_{*,*})$ by considering the injective R^e-resolution of $M \otimes_R N$

$$0 \to M \otimes_R N \to I''_0 \to I''_1 \to I''_2 \to \ldots,$$

and the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R_1^e}(R_1, I''_0) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I''_1) & \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I''_2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J''_{0,0} & \longrightarrow & J''_{1,0} & \longrightarrow & J''_{2,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
J''_{0,1} & \longrightarrow & J''_{1,1} & \longrightarrow & J''_{2,1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & .
\end{array}$$

We want to establish a natural map

$$E_0^{i,j}(M) \otimes E_0^{s,t}(N) \to E_0^{i+s,j+t}(M \otimes_R N),$$

compatible with the differentials (pairing of double complexes), see [2, Definition 3.9.1].

Since R is a right HIF-ring, by the Künneth theorem for cochain complexes (the similar variant of [8, Theorem 3.6.1]), we obtain that $0 \to M \otimes_R N \to I_* \otimes_R I'_*$ is an acyclic complex. From [8, Theorem 2.3.7], we can build the following commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & M \otimes_R N & \longrightarrow & I_* \otimes_R I'_* \\
 & & \| & & \downarrow \\
0 & \longrightarrow & M \otimes_R N & \longrightarrow & I''_*.
\end{array}$$

Then there is a cochain map $I_* \otimes_R I'_* \to I''_*$. By applying $\mathrm{Hom}_{R_1^e}(R_1, -)$ we obtain a map of kG-modules

$$\mathrm{Hom}_{R_1^e}(R_1, I_i \otimes_R I'_s) \to \mathrm{Hom}_{R_1^e}(R_1, I''_{i+s}),$$

for i, s nonnegative integers. From the cup product in Hochschild cohomology we know that there are maps

$$\mathrm{Hom}_{R_1^e}(R_1, I_i) \otimes \mathrm{Hom}_{R_1^e}(R_1, I'_s) \to \mathrm{Hom}_{R_1^e}(R_1, I_i \otimes_{R_1} I'_s) \to \mathrm{Hom}_{R_1^e}(R_1, I_i \otimes_R I'_s),$$

where the right map is induced by $a \otimes_{R_1} b \mapsto a \otimes_R b$, for any $a \in I_i, b \in I'_s$. We compose the above two maps and obtain a natural map

$$\mathrm{Hom}_{R_1^e}(R_1, I_i) \otimes \mathrm{Hom}_{R_1^e}(R_1, I'_s) \to \mathrm{Hom}_{R_1^e}(R_1, I''_{i+s}). \tag{1}$$

Since $0 \to \mathrm{Hom}_{R_1^e}(R_1, I_i) \to J_{i,*}$ and $0 \to \mathrm{Hom}_{R_1^e}(R_1, I'_s) \to J'_{s,*}$ are injective resolutions of kG-modules, we repeat the above construction (made with R^e-resolutions) to obtain the commutative diagram (k is a field and we can apply

Künneth theorem again)

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I_i) \otimes \mathrm{Hom}_{R_1^e}(R_1, I'_s) & \longrightarrow & J_{i,*} \otimes J'_{s,*} \\ & \downarrow{\scriptstyle (1)} & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_{R_1^e}(R_1, I''_{i+s}) & \longrightarrow & J''_{i+s,*+*}. \end{array}$$

In particular we obtain a map $J_{i,j} \otimes J'_{s,t} \to J''_{i+s,j+t}$ and consequently $\mathrm{Hom}_{kG}(k, J_{i,j} \otimes J'_{s,t}) \to \mathrm{Hom}_{kG}(k, J''_{i+s,j+t})$. Since kG is a Hopf algebra there is a map compatible with the differentials

$$\mathrm{Hom}_{kG}(k, J_{i,j}) \otimes \mathrm{Hom}_{kG}(k, J'_{s,t}) \to \mathrm{Hom}_{kG}(k, J_{i,j} \otimes J'_{s,t}).$$

Consequently we get the desired map

$$\mathrm{Hom}_{kG}(k, J_{i,j}) \otimes \mathrm{Hom}_{kG}(k, J'_{s,t}) \to \mathrm{Hom}_{kG}(k, J''_{i+s,j+t}).$$

The above maps respect the differentials in $E_0^{*,*}$ since $\mathrm{Hom}_{kG}(k, -)$ and tensor product are applied on cochain complexes, and we don't check it. □

By [2, Proposition 3.9.3], Proposition 1 induces a product on each page of the Grothendieck spectral sequence:

$$E_n^{i,j}(M) \otimes E_n^{s,t}(N) \to E_n^{i+s,j+t}(M \otimes_R N),$$

and hence on limit objects

$$E_\infty^{i,j}(M) \otimes E_\infty^{s,t}(N) \to E_\infty^{i+s,j+t}(M \otimes_R N).$$

If we take $M = N = R$ from Remark 4 we have the following proposition.

Proposition 2. Let R be a fully G-graded algebra such that R is a right HIF-ring. Then there is a ring structure on the second page

$$\mathrm{H}^*(G, \mathrm{HH}^*(R_1, R)) \Rightarrow \mathrm{HH}^*(R, R),$$

over which the following is a module

$$\mathrm{H}^*(G, \mathrm{HH}^*(R_1, M)) \Rightarrow \mathrm{HH}^*(R, M),$$

for any $M \in R^e$-Mod.

5. FINITE GENERATION

In this section we identify conditions for a particular class of fully G-graded algebras, which will give the finite generation of the Hochschild cohomology algebra.

Definition 2. We say that a fully G-graded algebra R is a *fg-G-graded algebra* if R_1 is of finite projective dimension as R_1^e-module and the spectral sequence

$$\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{R_1^e}(R_1, R)) \Longrightarrow \mathrm{Ext}^{i+j}_{R^e}(R, R)$$

is a module over the ring $\mathrm{H}^*(G, k)$.

The above definition seems cumbersome at a first view but it is obvious that $R = kG$, the group algebra, is a fg-G-graded algebra. Moreover we have the next remark.

Remark 5. Let R_1 be a separable k-algebra. Then the above spectral sequence collapses to give $\mathrm{HH}^*(R) = \mathrm{H}^*(G, \mathrm{Hom}_{R_1^e}(R_1, R))$, hence the spectral sequence is a module over $\mathrm{H}^*(G, k)$.

Theorem 1. *Let R be a fg-G-graded algebra such that R is a right HIF-ring. Then $\mathrm{HH}^*(R, M)$ is a finitely generated $\mathrm{H}^*(G, k)$-module for any $M \in R^e$-Mod. In particular the Hochschild cohomology $\mathrm{HH}^*(R)$ is a finitely generated algebra.*

Proof. By Proposition 2, since R is a fg-G-graded algebra we have that the spectral sequence

$$\mathrm{Ext}^i_{kG}(k, \mathrm{Ext}^j_{R_1^e}(R_1, M)) \Longrightarrow \mathrm{Ext}^{i+j}_{R^e}(R, M)$$

is a module over $\mathrm{H}^*(G, k)$. Since R_1 is of finite projective dimension as R_1^e-module $\mathrm{H}^j(R_1, M) = \mathrm{Ext}^j_{R_1^e}(R_1, M)$ vanishes for large j. The finite generation theorem of Evens and Venkov says that for each j, $\mathrm{Ext}^*_{kG}(k, \mathrm{Ext}^j_{R_1^e}(R_1, M))$ is a finitely generated $\mathrm{Ext}^*_{kG}(k, k)$-module. We know that E_∞ is a subquotient of page E_2 of the cohomology spectral sequence, then $\mathrm{HH}^*(R, M)$ is a finitely generated $\mathrm{H}^*(G, k)$-module.

In particular if $M = R$ it follows that $\mathrm{HH}^*(R)$ is a finitely generated $\mathrm{H}^*(G, k)$-module. Since $\mathrm{H}^*(G, k)$ is a finitely generated algebra we obtain that $\mathrm{HH}^*(R)$ is finitely generated algebra. □

We give an example of fully group graded algebras which satisfies the hypothesis of the above theorem.

Example 1. Let R be a fully group graded algebra with $\mid G \mid$ invertible in R. Let R_1 be a matrix algebra over the field k. Then R is fg-G-graded by Remark 5. Moreover from [5, 3.8.3], since R_1 is von Neumann regular ring, we have that R is von Neumann regular, hence a right HIF-ring.

But in the above case the finite generation of the Hochschild cohomology algebra with R_1 a matrix algebra, can be obtained alternatively from the finite generation theorem of Evens and Venkov by using the spectral sequence from Remark 4, which collapses. We end with the following remark.

Remark 6. Let R be a fully group graded algebra. Let R_1 be a k-algebra of finite projective dimension as R_1^e-module different from 0 and with the $\mathrm{Ext}^i_{R_1^e}(R_1, R) = 0$ for $i > 0$. If R is a right HIF-ring then the hypothesis of Theorem 1 are satisfied.

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