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THE EXTENSION MONOID PRODUCT OF PREINJECTIVE KRONECKER MODULES

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Abstract. We explore the combinatorial properties of the extension monoid product of preinjective Kronecker modules. We state a theorem which characterizes the extension monoid product of preinjective (and dually preprojective) Kronecker modules in the most general case, over an arbitrary base field. As corollaries, we give another proof of an interesting theorem from [7] and restate the main theorem from [9] (also extending these results over arbitrary fields).

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1. INTRODUCTION

Let K be the **Kronecker quiver**, i.e. the quiver having two vertices and two parallel arrows:

$$K:\ 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$$

and κ an arbitrary field. The path algebra of the Kronecker quiver is the **Kronecker algebra** and we will denote it by κK. A finite dimensional right module over the Kronecker algebra is called a **Kronecker module**. We denote by mod-κK the category of finite dimensional right modules over the Kronecker algebra.

A (finite dimensional) κ-linear representation of the quiver K is a quadruple $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ where V_1, V_2 are finite dimensional κ-vector spaces (corresponding to the vertices) and $\varphi_\alpha, \varphi_\beta : V_2 \to V_1$ are κ-linear maps (corresponding to the arrows). Thus a κ-linear representation of K associates vector spaces to the vertices and compatible κ-linear functions (or equivalently, matrices) to the arrows. Let us denote by rep-κK the category of finite dimensional κ-representations of the Kronecker quiver. There is a well-known equivalence of categories between mod-κK and rep-κK, so that every Kronecker module can be identified with a representation of K.

The **simple Kronecker module**s (up to isomorphism) are

$$S_1 : \kappa \leftleftarrows 0 \quad \text{and} \quad S_2 : 0 \leftleftarrows \kappa.$$

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For a Kronecker module M we denote by $\underline{\dim}M$ its **dimension**. The dimension of M is a vector $\underline{\dim}M = ((\dim M)_1, (\dim M)_2) = (m_{S_1}(M), m_{S_2}(M))$, where $m_{S_i}(M)$ is the number of factors isomorphic with the simple module S_i in a composition series of M, $i = \overline{1,2}$. Regarded as a representation, $M : V_1 \overset{\varphi_\alpha}{\underset{\varphi_\beta}{\leftleftarrows}} V_2$, we have that $\underline{\dim}M = (\dim_\kappa V_1, \dim_\kappa V_2)$.

The **defect** of $M \in$ mod-κK with $\underline{\dim}M = (a, b)$ is defined in the Kronecker case as $\partial M = b - a$.

An indecomposable module $M \in$ mod-κK is a member in one of the following three families: preprojectives, regulars and preinjectives. In what follows we give some details on these families.

The **preprojective indecomposable Kronecker module**s are determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by P_n the indecomposable preprojective module of dimension $(n+1, n)$. So P_0 and P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple). It is known that (up to isomorphism) $P_n = (\kappa^{n+1}, \kappa^n; f, g)$, where choosing the canonical basis in κ^n and κ^{n+1}, the matrix of $f : \kappa^n \to \kappa^{n+1}$ (respectively of $g : \kappa^n \to \kappa^{n+1}$) is $\begin{pmatrix} \mathbb{I}_n \\ 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 \\ \mathbb{I}_n \end{pmatrix}$). Thus in this case

$$P_n : \ \kappa^{n+1} \overset{\begin{pmatrix} 0 \\ \mathbb{I}_n \end{pmatrix}}{\underset{\begin{pmatrix} \mathbb{I}_n \\ 0 \end{pmatrix}}{\leftleftarrows}} \kappa^n \ ,$$

where $\mathbb{I}_n$ is the identity matrix. We have for the defect $\partial P_n = -1$.

We define a **preprojective Kronecker module** P as being a direct sum of indecomposable preprojective modules: $P = P_{a_1} \oplus P_{a_2} \oplus \cdots \oplus P_{a_l}$, where we use the convention that $a_1 \leq a_2 \leq \cdots \leq a_l$.

The **preinjective indecomposable Kronecker module**s are also determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by I_n the indecomposable preinjective module of dimension $(n, n+1)$. So I_0 and I_1 are the injective indecomposable modules ($P_0 = S_2$ being simple). It is known that (up to isomorphism) $I_n = (\kappa^n, \kappa^{n+1}; f, g)$, where choosing the canonical basis in κ^{n+1} and κ^n, the matrix of $f : \kappa^{n+1} \to \kappa^n$ (respectively of $g : \kappa^{n+1} \to \kappa^n$) is $\begin{pmatrix} \mathbb{I}_n & 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 & \mathbb{I}_n \end{pmatrix}$). Thus in this case

$$I_n : \ \kappa^n \overset{(\mathbb{I}_n\, 0)}{\underset{(0\, \mathbb{I}_n)}{\leftleftarrows}} \kappa^{n+1} \ ,$$

where $\mathbb{I}_n$ is the identity matrix. We have for the defect $\partial I_n = 1$.

We define a **preinjective Kronecker module** I as being a direct sum of indecomposable preinjective modules: $I = I_{a_1} \oplus I_{a_2} \oplus \cdots \oplus I_{a_l}$, where we use the convention that $a_1 \geq a_2 \geq \cdots \geq a_l$.

The category of Kronecker modules has been extensively studied because the Kronecker algebra is a very important example of a tame hereditary algebra. Moreover, the category has also a geometric interpretation, since it is derived equivalent with the category of coherent sheaves on the projective line. In addition, Kronecker modules correspond to matrix pencils in linear algebra, so the Kronecker algebra relates representation theory with numerical linear algebra and matrix theory. We refer the reader to standard textbooks on representation theory of algebras, such as [1, 5, 2, 4] for further details on definitions, calculations, justifications and proofs leading to these results.

In the sequel we denote integer sequences as $(a_1, \ldots, a_n)$, we take empty sums to be zero and denote the set of permutations of $\{1, 2, \ldots, n\}$ by S_n. For giving a permutation $\sigma \in S_n$ we may use either the tabular notation as in

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

or the cyclic notation (i.e. σ given as a product of cycles) as in $\sigma = \zeta_1 \zeta_2 \ldots \zeta_k$, where every $\zeta_j = (i \; \sigma(i) \; \sigma(\sigma(i)) \; \ldots \; \sigma^{\ell-1}(i))$ with $\sigma^{\ell}(i) = i$ is a cyclic permutation (or a cycle). A product of permutations is to be computed from right to left (as usual for the composition of functions).

2. THE EXTENSION MONOID PRODUCT OF PREINJECTIVE KRONECKER MODULES

For $d \in \mathbb{N}^2$ let $M_d = \{[M] | M \in \text{mod-}\kappa K, \underline{\dim} M = d\}$ be the set of isomorphism classes of Kronecker modules of dimension d. Following Reineke in [3] for subsets $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$ we define $\mathcal{A} * \mathcal{B} = \{[X] \in M_{d+e} \,|\, \exists\, 0 \to N \to X \to M \to 0$ exact for some $[M] \in \mathcal{A}, [N] \in \mathcal{B}\}$. So the product $\mathcal{A} * \mathcal{B}$ is the set of isoclasses of all extensions of modules M with $[M] \in \mathcal{A}$ by modules N with $[N] \in \mathcal{B}$. This is in fact Reineke's extension monoid product using isomorphism classes of modules instead of modules. It is important to know (see [3]) that the product above is associative, i.e. for $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$, $\mathcal{C} \subset M_f$, we have $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$. Also $\{[0]\} * \mathcal{A} = \mathcal{A} * \{[0]\} = \mathcal{A}$. We will call the operation "$*$" simply the **extension monoid product**.

REMARK 1. For $M, N \in \text{mod-}\kappa K$ and κ finite, the product $\{[M]\} * \{[N]\}$ coincides with the set $\{[M][N]\}$ of terms in the Ringel-Hall product $[M][N]$ (see Section 4 from [9]).

From now on we deal only with the extension monoid product of preinjective Kronecker modules. It is very important to mention that all results can be dualized in natural way to preprojective Kronecker modules as well.

According to the main result from [8] (Theorem 3.3), the possible middle terms in preprojective (and dually preinjective) short exact sequences do not depend on the base field. This allows us to describe the combinatorial rules governing the extension monoid product of preinjective Kronecker modules in a field independent way. Specifically, this allows us to restate some theorems involving the Ringel-Hall product (valid only over finite fields) in terms of the extension monoid product (in a field independent manner).

Let us begin with a step-by-step presentation of our results so far, going from the most specific towards the more general.

The first step is to state the combinatorial rule describing the extension monoid product of two preinjective indecomposable Kronecker modules. The following theorem appears in [6] (and also in [9]) written in terms of the Ringel-Hall product, so we may give its equivalent, field independent version:

THEOREM 1. *Let $I_i, I_j \in \text{mod-}\kappa K$ be preinjective indecomposable Kronecker modules. Then we have the following rule describing their extension monoid product:*

$$\{[I_i]\} * \{[I_j]\} = \begin{cases} \{[I_i \oplus I_j]\} & i \geq j \\ \{[I_j \oplus I_i], [I_{j-1} \oplus I_{i+1}], \dots, [I_{j-\lfloor \frac{j-i}{2} \rfloor} \oplus I_{i+\lfloor \frac{j-i}{2} \rfloor}]\} & i < j \end{cases}.$$

REMARK 2. In the theorem above $\lfloor x \rfloor$ denotes the integer part of x, therefore $\{[I_j \oplus I_i], [I_{j-1} \oplus I_{i+1}], \dots, [I_{j-\lfloor \frac{j-i}{2} \rfloor} \oplus I_{i+\lfloor \frac{j-i}{2} \rfloor}]\} = \{[I_{j-m} \oplus I_{i+m}] | m \in \mathbb{N},\ j-m \geq i+m\}$.

The next step is to consider the extension monoid product of a preinjective with a preinjective indecomposable. Again, we can just rewrite the corresponding result from [9]:

LEMMA 1. *For $a_1 \geq \dots \geq a_{n-1} \geq 0$, $c_1 \geq \dots \geq c_p \geq 0$ and $a_n \geq 0$ nonnegative integers ($n \geq 2$), we have that*

$$[I_{c_1} \oplus \dots \oplus I_{c_p}] \in \{[I_{a_1} \oplus \dots \oplus I_{a_{n-1}}]\} * \{[I_{a_n}]\}$$

if and only if $p = n$, $c_1 = a_1, \dots, c_{k-1} = a_{k-1}$, $c_k = a_n - \sum_{j=k}^{n-1} m_j$, $c_{k+1} = a_k + m_k, \dots, c_n = a_{n-1} + m_{n-1}$ for some $k \in \{1, \dots, n\}$ and $m_j \in \mathbb{N}$, $k \leq j < n$.

In an analogue way we have (see [10]):

LEMMA 2. *For $b_1 \geq \dots \geq b_n \geq 0$, $c_1 \geq \dots \geq c_p \geq 0$ and $a \geq 0$ nonnegative integers, we have that*

$$[I_{c_1} \oplus \dots \oplus I_{c_p}] \in \{[I_a]\} * \{[I_{b_1} \oplus \dots \oplus I_{b_n}]\}$$

if and only if $p = n+1$, $c_1 = b_1 - m_1, \dots, c_{k-1} = b_{k-1} - m_{k-1}$, $c_k = a + \sum_{i=1}^{k-1} m_i$, $c_{k+1} = b_k, \dots, c_{n+1} = b_n$ for some $k \in \{1, \dots, n+1\}$ and $m_i \in \mathbb{N}$, $i = \{1, \dots, k-1\}$.

The main result from [9] gives an implicit description of the Ringel-Hall product of two arbitrary preinjective Kronecker modules over finite fields. The field independent version is our next step towards the most general case:

THEOREM 2. *If $a_1 \geq \dots a_p \geq 0$, $b_1 \geq \dots \geq b_n \geq 0$ and $c_1 \geq \dots \geq c_r \geq 0$ are nonnegative integers, then $[I_{c_1} \oplus \dots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \dots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \dots \oplus I_{b_n}]\}$*

if and only if $r = n + p$, $\exists\beta : \{1, \dots, n\} \to \{1, \dots, n+p\}$, $\exists\alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *and* $\exists m_j^i \geq 0$, $1 \leq i \leq n$, $1 \leq j \leq p$, *such that* $\forall \ell \in \{1, \dots, n+p\}$

$$(1) \qquad c_\ell = \begin{cases} b_i - \sum_{\substack{\beta(i)<\alpha(j) \\ 1\leq j\leq p}} m_j^i, \ \text{where } i = \beta^{-1}(\ell) & \ell \in \mathrm{Im}\beta \\ a_j + \sum_{\substack{\beta(i)<\alpha(j) \\ 1\leq i\leq n}} m_j^i, \ \text{where } j = \alpha^{-1}(\ell) & \ell \in \mathrm{Im}\alpha \end{cases}.$$

Having surveyed what we have so far, let us deal with the most general case, that is with an extension monoid product of the following form:

$$\{[I_{a_1}]\} * \{[I_{a_2}]\} * \cdots * \{[I_{a_n}]\},$$

where the sequence $(a_1, a_2, \dots, a_n)$ parameterizing the preinjective indecomposables is arbitrary.

Since the extension monoid product is associative, one can compute it in many ways. One strategy could be to always put parentheses around adjacent terms and use only Theorem 1 along the process. Formally:

$$\begin{aligned} \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\} &\supseteq \{[I_{\alpha_1^1}]\} * \cdots * \left(\{[I_{\alpha_{k_1}^1}]\} * \{[I_{\alpha_{k_1+1}^1}]\}\right) * \cdots * \{[I_{\alpha_n^1}]\} \\ &\supseteq \{[I_{\alpha_1^2}]\} * \cdots * \left(\{[I_{\alpha_{k_2}^2}]\} * \{[I_{\alpha_{k_2+1}^2}]\}\right) * \cdots * \{[I_{\alpha_n^2}]\} \\ &\ \vdots \\ &\supseteq \{[I_{\alpha_1^p}]\} * \cdots * \left(\{[I_{\alpha_{k_p}^p}]\} * \{[I_{\alpha_{k_p+1}^p}]\}\right) * \cdots * \{[I_{\alpha_n^p}]\} \\ &\supseteq \{[I_{\alpha_1^{p+1}}]\} * \{[I_{\alpha_2^{p+1}}]\} * \cdots * \{[I_{\alpha_n^{p+1}}]\} \\ &= \{[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_n}]\}, \end{aligned}$$

where $(a_1, \dots, a_n) = (\alpha_1^1, \dots, \alpha_n^1)$, $k_i \in \{1, \dots, n-1\}$ for all $i \in \{1, \dots, p\}$ and $(\alpha_1^{p+1}, \dots, \alpha_n^{p+1}) = (c_1, \dots, c_n)$ is a decreasing sequence, i.e. $c_1 \geq \cdots \geq c_n \geq 0$. At any step $i \in \{1, \dots, p\}$, when computing $\left(\{[I_{\alpha_{k_i}^i}]\} * \{[I_{\alpha_{k_i+1}^i}]\}\right)$ we have two cases according to Theorem 1, hence we may define the sequences $(\alpha_1^i, \dots, \alpha_n^i)$ in the following way: if $\alpha_{k_i}^i \geq \alpha_{k_i+1}^i$ then

$$(\alpha_1^{i+1}, \dots, \alpha_{k_i}^{i+1}, \alpha_{k_i+1}^{i+1}, \dots, \alpha_n^{i+1}) = (\alpha_1^i, \dots, \alpha_{k_i}^i, \alpha_{k_i+1}^i, \dots, \alpha_n^i),$$

otherwise (if $\alpha_{k_i}^i < \alpha_{k_i+1}^i$)

$$(\alpha_1^{i+1}, \dots, \alpha_{k_i}^{i+1}, \alpha_{k_i+1}^{i+1}, \dots, \alpha_n^{i+1}) = (\alpha_1^i, \dots, \alpha_{k_i+1}^i - m, \alpha_{k_i}^i + m, , \dots, \alpha_n^i),$$

for some $m \in \mathbb{N}$ such that $\alpha_{k_i+1}^i - m \geq \alpha_{k_i}^i + m$. So either the sequence remains unchanged, or there is a “swap” between adjacent elements with an optional increasing of the element left behind at the expense of the element going in front. Either case we will have $\alpha_{k_i}^{i+1} \geq \alpha_{k_i+1}^{i+1}$ in the new sequence. Therefore we can disregard the steps where $\alpha_{k_i}^i \geq \alpha_{k_i+1}^i$ and suppose we compute by putting parentheses only in the case when $\alpha_{k_i}^i < \alpha_{k_i+1}^i$, stopping the process when the sequence is decreasing. It also is easy to see that the process stops after a finite number $p \in \mathbb{N}$ of steps since for every $i \in \{1, \dots, p\}$ the sequence $(\alpha_1^{i+1}, \dots, \alpha_n^{i+1})$ is strictly greater lexicographically than $(\alpha_1^i, \dots, \alpha_n^i)$ and the number of sequences having a fixed lenght and a fixed sum of their elements if finite. We may also keep track of the elements of the "original" sequence $(a_1, \dots, a_n)$. Clearly, every swap defines a transposition of the form $(k_i\ k_i+1)$. Let us denote their product by $\sigma = (k_1\ k_1+1)\dots(k_p\ k_p+1)$. At the end we will have $(c_1, \dots, c_n) = (a_{\sigma(1)}+m_1, \dots, a_{\sigma(n)}+m_n)$, for some $m_i \in \mathbb{Z}$ satisfying $\sum_{i=1}^n m_i = 0$. We summarize all this reasoning in the following proposition:

Proposition 1. *For $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{N}$, $c_1 \geq \dots \geq c_n \geq 0$, $n \geq 2$ we have that $[I_{c_1} \oplus \dots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \dots * \{[I_{a_n}]\} \iff (c_1, \dots, c_n) = (a_{\sigma(1)} + m_1, \dots, a_{\sigma(n)} + m_n)$, with $\sigma = \sigma_{p+1} \in S_n$ and $m_i = \alpha_i^{p+1} - a_{\sigma(i)}$ for some $p \in \mathbb{N}$, where for every $i \in \{1, \dots, p\}$ and $k_i \in \{1, \dots, n-1\}$ we have $\alpha_{k_i}^i < \alpha_{k_i+1}^i$ using the following recursive definitions:*

- *$(\alpha_1^1, \dots, \alpha_n^1) = (a_1, \dots, a_n)$ and $(\alpha_1^{i+1}, \dots, \alpha_{k_i}^{i+1}, \alpha_{k_i+1}^{i+1}, \dots, \alpha_n^{i+1}) = (\alpha_1^i, \dots, \alpha_{k_i+1}^i - m^{(i)}, \alpha_{k_i}^i + m^{(i)}, \dots, \alpha_n^i)$ for some $m^{(i)} \in \mathbb{N}$ such that $\alpha_{k_i+1}^i - m^{(i)} \geq \alpha_{k_i}^i + m^{(i)}$;*
- *$\sigma_1 = ()$ is the identity permutation and $\sigma_{i+1} = \sigma_i \cdot (k_i\ k_i+1) \in S_n$.*

Let us state now a lemma, which solves most of the technical difficulties in our way towards the main theorem.

Lemma 3. *For $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{N}$, $c_1 \geq \dots \geq c_n \geq 0$, $n \geq 2$ we have that*

$$[I_{c_1} \oplus \dots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \dots * \{[I_{a_n}]\}$$

if and only if $\exists \sigma \in \mathcal{S}_n$ a permutation and $\exists m_j^i \geq 0$ nonnegative integers, $1 \leq j < i \leq n$, such that $\forall \ell \in \{1, \dots, n\}$,

$$c_\ell = a_{\sigma(\ell)} + \sum_{i=\sigma(\ell)+1}^{n} m_{\sigma(\ell)}^i - \sum_{j=1}^{\sigma(\ell)-1} m_j^{\sigma(\ell)}, \tag{2}$$

and the following condition holds: $m_j^i > 0 \Longrightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$.

Proof. "$\Longleftarrow$" Suppose we have a permutation $\sigma \in S_n$ and the values $m_j^i \in \mathbb{N}$, $1 \leq j < i \leq n$ such that equation 2 holds. Taking into account the condition $m_j^i > 0 \Longrightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$, we can rewrite the equation in the following equivalent form $\forall \ell \in \{1, \dots, n\}$:

$$c_\ell = a_{\sigma(\ell)} + \sum_{\substack{i>\sigma(\ell) \\ \sigma^{-1}(i)<\ell}} m_{\sigma(\ell)}^i - \sum_{\substack{j<\sigma(\ell) \\ \sigma^{-1}(j)>\ell}} m_j^{\sigma(\ell)}.$$

Next, we divide the indices $\ell \in \{1, \dots, n\}$ into two different sets. Let $D = \{\ell \mid c_\ell \leq a_{\sigma(\ell)}\}$ and $A = \{\ell \mid c_\ell > a_{\sigma(\ell)}\}$. If $A = \emptyset$, then $D = \{1, \dots, n\}$ and $(c_1, \dots, c_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$ is only a decreasing ordering of the elements of the sequence $(a_1, \dots, a_n)$, and using Proposition 1 we are done. So, let us suppose that $A \neq \emptyset$. We are going to show that the following inequality is true for every $k \in A$:

$$\sum_{\substack{\ell \in A \\ \ell \leq k}} (c_\ell - a_{\sigma(\ell)}) \leq \sum_{\substack{\ell \in D \\ \ell < k}} (a_{\sigma(\ell)} - c_\ell). \tag{3}$$

Since

$$\sum_{\substack{\ell \in A \\ \ell \leq k}} (c_\ell - a_{\sigma(\ell)}) = \sum_{\substack{\ell \in A \\ \ell \leq k \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} m^i_{\sigma(\ell)} - \sum_{\substack{\ell \in A \\ \ell \leq k \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} m^{\sigma(\ell)}_j$$

and

$$\sum_{\substack{\ell \in D \\ \ell < k}} (a_{\sigma(\ell)} - c_\ell) = - \sum_{\substack{\ell \in D \\ l < k \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} m^i_{\sigma(\ell)} + \sum_{\substack{\ell \in D \\ \ell < k \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} m^{\sigma(\ell)}_j,$$

we have that inequality 3 is equivalent with

$$\sum_{\substack{\ell \in A \\ \ell \leq k \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} m^i_{\sigma(\ell)} + \sum_{\substack{\ell \in D \\ l < k \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} m^i_{\sigma(\ell)} \leq \sum_{\substack{\ell \in A \\ \ell \leq k \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} m^{\sigma(\ell)}_j + \sum_{\substack{\ell \in D \\ \ell < k \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} m^{\sigma(\ell)}_j,$$

which in turn is equivalent with

$$\sum_{\substack{\ell \leq k \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} m^i_j \leq \sum_{\substack{\ell \leq k \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} m^{\sigma(\ell)}_j.$$

Using the change of variables $\ell = \sigma^{-1}(j)$ and $\ell = \sigma^{-1}(i)$ respectively, this last inequality can be rewritten in the following equivalent form:

$$\sum_{\substack{\sigma^{-1}(j) \leq k \\ i > j \\ \sigma^{-1}(i) < \sigma^{-1}(j)}} m^i_j \leq \sum_{\substack{\sigma^{-1}(i) \leq k \\ j < i \\ \sigma^{-1}(j) > \sigma^{-1}(i)}} m^i_j,$$

which is true, because the terms in the sum from the left hand side also appear as terms in the sum on the right hand side. Thus inequality 3 holds. As a consequence it is clear that $\exists \bar{m}^i_j \in \mathbb{N}$ such that

$$c_\ell = \begin{cases} a_{\sigma(\ell)} + \sum_{\substack{\sigma^{-1}(i) \in D \\ i > \sigma(\ell) \\ \sigma^{-1}(i) < \ell}} \bar{m}^i_{\sigma(\ell)} = a_{\sigma(\ell)} + \sum_{i=\sigma(\ell)+1}^{n} \bar{m}^i_{\sigma(\ell)} & \ell \in A \\ a_{\sigma(\ell)} - \sum_{\substack{\sigma^{-1}(j) \in A \\ j < \sigma(\ell) \\ \sigma^{-1}(j) > \ell}} \bar{m}^{\sigma(\ell)}_j = a_{\sigma(\ell)} - \sum_{j=1}^{\sigma(\ell)-1} \bar{m}^{\sigma(\ell)}_j & \ell \in D \end{cases}, \tag{4}$$

this also implying that $\bar{m}^i_j = 0$ for all $1 \le j < i$ if $\sigma^{-1}(i) \in A$ and $\bar{m}^i_j = 0$ for all $j < i \le n$ if $\sigma^{-1}(j) \in D$.

We use Proposition 1 to show that $[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$. We set $(\alpha^1_1, \ldots, \alpha^1_n) = (a_1, \ldots, a_n)$ and perform a finite number of "controlled" swaps to reach the final decreasing sequence $(c_1, \ldots, c_n) = (\alpha^{p+1}_1, \ldots, \alpha^{p+1}_n)$, where $p \in \mathbb{N}$ is the number of swaps performed. Using the notation introduced in the statement of Proposition 1 and initializing the permutation $\sigma_1 = ()$ and a set with $W_1 = \emptyset$ we could describe formally the process in the following way: for every $i \in \{1, \ldots, p\}$ let $(\alpha^{i+1}_1, \ldots, \alpha^{i+1}_{k_i}, \alpha^{i+1}_{k_i+1}, \ldots, \alpha^{i+1}_n) = (\alpha^i_1, \ldots, \alpha^i_{k_i+1} - m^{(i)}, \alpha^i_{k_i} + m^{(i)}, \ldots, \alpha^i_n)$, $W_{i+1} = W_i \cup \{(\sigma_i(k_i), \sigma_i(k_i+1))\}$ and $\sigma_{i+1} = \sigma_i \cdot (k_i \ k_i+1) \in S_n$ for some $k_i \in \{1, \ldots, n-1\}$ if $\alpha^i_{k_i} < \alpha^i_{k_i+1}$, where $m^{(i)} = \bar{m}^{\sigma_i(k_i+1)}_{\sigma_i(k_i)}$ if $\sigma_i(k_i+1) > \sigma_i(k_i)$, $\sigma^{-1}(\sigma_i(k_i)) \in A$ and $\sigma^{-1}(\sigma_i(k_i+1)) \in D$ are satisfied, otherwise $m^{(i)} = 0$.

Note that $\alpha^i_{k_i+1} - m^{(i)} \ge \alpha^i_{k_i} + m^{(i)}$ is always true because if $m^{(i)} > 0$, then $\alpha^i_{k_i+1} - m^{(i)} \ge a_{\sigma_i(k_i+1)} - \sum_{j=1}^{\sigma_i(k_i+1)-1} \bar{m}^{\sigma_i(k_i+1)}_j \ge a_{\sigma_i(k_i)} + \sum_{i=\sigma_i(k_i)+1}^{n} \bar{m}^i_{\sigma_i(k_i)} \ge \alpha^i_{k_i} + m^{(i)}$. One can see that the permutations $\sigma_1, \ldots \sigma_{p+1} \in S_n$ are used for tracking the elements of the sequence $(a_1, \ldots, a_n)$ while performing the swaps. Intuitively, if for a pair of indices we have that $(j, j') \in W_i$, this means that elements corresponding to a_j and $a_{j'}$ have been swapped at least once in the first i iterations of the process. Hence for every $i \in \{1, \ldots, p+1\}$ and every $\ell \in \{1, \ldots, n\}$ the following is true for the elements of the the sequence $(\alpha^i_1, \ldots, \alpha^i_n)$:

$$\alpha^i_\ell = \begin{cases} a_{\sigma_i(\ell)} + \sum_{\substack{n \ge j > \sigma_i(\ell) \\ (\sigma_i(\ell), j) \in W_i}} \bar{m}^j_{\sigma_i(\ell)} & \sigma^{-1}(\sigma_i(\ell)) \in A \\ a_{\sigma_i(\ell)} - \sum_{\substack{1 \le j < \sigma_i(\ell) \\ (j, \sigma_i(\ell)) \in W_i}} \bar{m}^{\sigma_i(\ell)}_j & \sigma^{-1}(\sigma_i(\ell)) \in D \end{cases}.$$

Obviously, this process can only stop when $\alpha^{p+1}_1 \ge \cdots \ge \alpha^{p+1}_n$, meaning that $(c_1, \ldots, c_n) = (\alpha^{p+1}_1, \ldots, \alpha^{p+1}_n)$. If in addition $\sigma_{p+1} = \sigma$, then equation 4 is readily satisfied and we are done. If $\sigma_{p+1} \ne \sigma$, then one can perform a number of swaps between adjacent elements having the same value to get exactly the permutation σ.

"$\Longrightarrow$" Suppose now that we have $[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$. Using the associative law, $[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in (((\{[I_{a_1}]\} * \{[I_{a_2}]\}) * \{[I_{a_3}]\}) * \cdots * \{[I_{a_n}]\})$. This only reflects the obvious fact that any element of the set $\{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$ can be obtained by repeated application of Lemma 1 and conversely, by iterating the process described in Lemma 1 we can get nothing but an isoclass from the set $\{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$. So, in order to compute the element $[I_{c_1} \oplus \cdots \oplus I_{c_n}]$ of the product $\{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$ we can proceed by starting from the second term and apply $n-1$ times Lemma 1.

As it can be seen, the position $k \in \{1, \ldots, n\}$ in Lemma 1 determines a cyclic permutation of the form $\zeta = (k\ k+1\ \ldots\ n-1\ n)$ having length $n-k+1$. Therefore the isoclass $[I_{c_1} \oplus \cdots \oplus I_{c_p}]$ is completely determined by the cycle ζ, the values $m_j \in \mathbb{N}$, $1 \le j < n$ (where $j < k \Longrightarrow m_j = 0$) and the sequence $(a_1, \ldots, a_n)$.

We will number the $n-1$ iterations starting from 2, so let $i \in \{2, \ldots, n\}$ be the index used in counting iterations of Lemma 1. Correspondingly, $k_i \in \{1, \ldots, i\}$ will be the position where the element a_i is going to be inserted at the end of the i^{th} iteration, hence $\zeta_i = (k_i\ k_i+1\ \ldots\ i-1\ i)$ will be the cyclic permutation determined by k_i. Let us denote the product of these cycles by $\tau_i = \zeta_i \zeta_{i-1} \ldots \zeta_2$ and its inverse by $\sigma_i = \tau_i^{-1} = \zeta_2^{-1} \ldots \zeta_i^{-1}$. The nonnegative integer values by which a_i is decreased in favor of some of the other elements a_j, where $1 \le j < i$ will be denoted by m_j^i. Let us denote by $(\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i)$ the sequence obtained after the i^{th} iteration. For convenience we use the convention $(\alpha_1^1, \ldots, \alpha_n^1) = (a_1, \ldots, a_n)$ and $\sigma_1 = ()$ that enables us to write following recursive definition for all $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, n\}$:

$$\alpha_j^i = \begin{cases} \alpha_j^{i-1} & 1 \le j < k_i \\ a_i - \sum_{k=k_i}^{i-1} m_k^i & j = k_i \\ \alpha_j^{i-1} + m_{\sigma_{i-1}(j)}^i & k_i < j < i \\ a_j & i \le j \le n \end{cases}.$$

Finally we will have $c_1 = \alpha_1^n,\ c_2 = \alpha_2^n,\ \ldots,\ c_n = \alpha_n^n$ and

$$\begin{aligned} &(((\{[I_{a_1}]\} * \{[I_{a_2}]\}) * \{[I_{a_3}]\}) * \cdots * \{[I_{a_n}]\}) = \\ &\qquad = \left(\left(\left(\{[I_{\alpha_1^1}]\} * \{[I_{\alpha_2^1}]\}\right) * \{[I_{\alpha_3^1}]\}\right) * \cdots * \{[I_{\alpha_n^1}]\}\right) \\ &\qquad \supseteq \left(\left(\{[I_{\alpha_1^2} \oplus I_{\alpha_2^2}]\} * \{[I_{\alpha_3^2}]\}\right) * \cdots * \{[I_{\alpha_n^2}]\}\right) \\ &\qquad \supseteq \{[I_{\alpha_1^3} \oplus I_{\alpha_2^3} \oplus I_{\alpha_3^3}]\} * \cdots * \{[I_{\alpha_n^3}]\} \\ &\qquad \vdots \\ &\qquad \supseteq \{[I_{\alpha_1^n} \oplus I_{\alpha_2^n} \oplus \cdots \oplus I_{\alpha_n^n}]\} \\ &\qquad = \{[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_n}]\}. \end{aligned}$$

Iterating this process $n-1$ times it is clear that $\sigma = \sigma_n = (\zeta_n \zeta_{n-1} \ldots \zeta_2)^{-1}$ and equation 2 holds if we set the appropriate values m_j^i to zero so that the implication $m_j^i > 0 \Longrightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$ is satisfied. □

Being the main result of the present paper, we state the following theorem, characterizing the extension monoid product of preinjective Kronecker modules in the most general case:

Theorem 3. *For $a_1, \ldots, a_n, c_1, \ldots, c_n \in \mathbb{N}$, $c_1 \ge \cdots \ge c_n \ge 0$, $n \ge 2$ we have that*

$$[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$$

if and only if $\exists\sigma \in \mathcal{S}_n$ *a permutation and* $\exists m_j^i \geq 0$ *nonnegative integers,* $1 \leq j < i \leq n$*, such that* $\forall\ell \in \{1, \ldots, n\}$

$$c_\ell = a_{\sigma(\ell)} + \sum_{i=\sigma(\ell)+1}^{n} m_{\sigma(\ell)}^i - \sum_{j=1}^{\sigma(\ell)-1} m_j^{\sigma(\ell)},$$

and the following conditions hold:

(i) $m_j^i > 0 \Longrightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$

(ii) $a_j \geq a_i \Longrightarrow \sigma^{-1}(i) > \sigma^{-1}(j)$.

Proof. Because of Lemma 3 we have to prove only the condition (ii) holds in the left to right implication.

We proceed by induction on n. When $n = 2$ we may have either $a_1 \geq a_2$ or $a_1 < a_2$.

If $a_1 \geq a_2$, then $\{[I_{a_1}]\} * \{[I_{a_2}]\} = \{[I_{a_1} \oplus I_{a_2}]\}$, the permutation $\sigma = ()$ is just the identity, $m_1^2 = 0$, so $c_1 = a_1 \geq a_2 = c_2$ and the conditions (i) and (ii) trivially hold.

If $a_1 < a_2$, then $\{[I_{a_1}]\} * \{[I_{a_2}]\} = \{[I_{c_1} \oplus I_{c_2}] \,|\, a_2 - m_1^2 = c_1 \geq c_2 = a_1 + m_1^2, m_1^2 \geq 0\}$, $\sigma = (2\ 1)$ and the conditions (i) and (ii) are also satisfied.

Let us assume that the implication holds for any product having $n-1$ or less terms ($n \geq 3$) and compute the product in the following way: $\{[I_{a_1}]\} * \ldots \{[I_{a_{n-1}}]\} * \{[I_{a_n}]\} \supseteq \left(\{[I_{a_1}]\} * \ldots \{[I_{a_{n-1}}]\}\right) * \{[I_{a_n}]\} \supseteq \{[I_{c'_1} \oplus \cdots \oplus I_{c'_{n-1}}]\} * \{[I_{a_n}]\}$, point at which we may use Lemma 1 to get the final element $[I_{c_1} \oplus \cdots \oplus I_{c_n}]$. Therefore $c_1 = c'_1, \ldots, c_{k-1} = c'_{k-1}$, $c_k = a_n - \sum_{j=k}^{n-1} m_j$, $c_{k+1} = c'_k + m_k$, $\ldots$, $c_n = c'_{n-1} + m_{n-1}$ for some $k \in \{1, \ldots, n\}$ and $m_j \in \mathbb{N}$, $k \leq j < n$.

Assuming the theorem holds for the product $\{[I_{a_1}]\} * \ldots \{[I_{a_{n-1}}]\}$, let us denote the corresponding permutation by $\bar\sigma \in S_{n-1}$ and apply again the trick of dividing the indices in two sets $D = \{\ell \,|\, c_\ell \leq a_{\bar\sigma(\ell)}\}$ and $A = \{\ell \,|\, c_\ell > a_{\bar\sigma(\ell)}\}$ as we did in the proof of Lemma 3. Following the same reasoning there exists values $\bar m_j^i \in \mathbb{N}$, such that $\forall\ell \in \{1, \ldots, n-1\}$

$$c'_\ell = \begin{cases} a_{\bar\sigma(\ell)} + \sum_{i=\bar\sigma(\ell)+1}^{n-1} \bar m_{\bar\sigma(\ell)}^i & \ell \in A \\ a_{\bar\sigma(\ell)} - \sum_{j=1}^{\bar\sigma(\ell)-1} \bar m_j^{\bar\sigma(\ell)} & \ell \in D \end{cases},$$

with $\bar m_j^i = 0$ for all $1 \leq j < i$ if $\bar\sigma^{-1}(i) \in A$ and $\bar m_j^i = 0$ for all $j < i \leq n-1$ if $\bar\sigma^{-1}(j) \in D$.

Moving the (decreased) element a_n into the k^{th} position is achieved via a sequence of swaps as described by Proposition 1. Set $(\alpha_1^1, \ldots, \alpha_{n-1}^1, \alpha_n^1) = (c'_1, \ldots, c'_{n-1}, a_n)$ and $(\alpha_1^{i+1}, \ldots, \alpha_{n-i}^{i+1}, \alpha_{n-i+1}^{i+1}, \ldots, \alpha_n^{i+1}) = (\alpha_1^i, \ldots, \alpha_{n-i+1}^i - m_{n-i}, \alpha_{n-i}^i + m_{n-i}, \ldots, \alpha_n^i)$ with $\alpha_{n-i+1}^i - m_{n-i} \geq \alpha_{n-i}^i + m_{n-i}$ for every

$i \in \{1, \dots, n-k\}$. The elements of the sequence $(\alpha_1^i, \dots, \alpha_n^i)$ are

$$\alpha_\ell^{i+1} = \begin{cases} c'_\ell & 1 \le \ell < n-i \\ a_n - \sum_{j=n-i}^{n-1} m_j & \ell = n-i \\ c'_{\ell-1} + m_{\ell-1} & n-i < \ell \le n \end{cases}.$$

Hence the final permutation would be $\bar{\sigma} \cdot (n-1\ n) \cdots (k\ k+1)$.

Suppose that we have just performed the i^{th} iteration and we have $a_n - \sum_{j=n-i}^{n-1} m_j = \alpha_{n-i}^{i+1} \ge \alpha_{n-i+1}^{i+1} = c'_{n-i} + m_{n-i}$, but $a_n \le a_{\bar{\sigma}(n-i)}$(if this is not the case, then condition (ii) is readily fulfilled and we are done). We must have $n-i \in D$, since $n-i \in A \Longrightarrow c'_{n-i} > a_{\bar{\sigma}(n-i)} \ge a_n$ and $c'_{n-i} \le a_n$ would lead to contradiction, so $c'_{n-i} = a_{\bar{\sigma}(n-i)} - \sum_{j=1}^{\bar{\sigma}(n-i)-1} \bar{m}_j^{\bar{\sigma}(n-i)}$. We can now replace the values $\bar{m}_j^{\bar{\sigma}(n-i)} \in \mathbb{N}$ with new ones (denoted by $m_j'^{\bar{\sigma}(n-i)} \in \mathbb{N}$) satisfying $m_j'^{\bar{\sigma}(n-i)} \le \bar{m}_j^{\bar{\sigma}(n-i)}$ for $1 \le j < \bar{\sigma}(n-i)$ such that

$$a_{\bar{\sigma}(n-i)} - \sum_{j=1}^{\bar{\sigma}(n-i)-1} m_j'^{\bar{\sigma}(n-i)} = \alpha_{n-i}^{i+1} = a_n - \sum_{j=n-i}^{n-1} m_j$$

(this being possible because $a_n \le a_{\bar{\sigma}(n-i)}$) and we define for every element $\ell \in \{n-i+1, \dots, n-1\}$ the following:

$$c''_\ell = \begin{cases} a_{\bar{\sigma}(\ell)} + \sum_{\substack{n \ge i > \bar{\sigma}(\ell) \\ i \ne \bar{\sigma}(n-i)}} \bar{m}_{\bar{\sigma}(\ell)}^i + m_{\bar{\sigma}(\ell)}'^{\bar{\sigma}(n-i)} & \ell \in A \\ c'_\ell & \ell \in D \end{cases}.$$

Also set for every $j \in \{n-i+1, \dots, n-1\}$ the following: $m'_j = m_j + \bar{m}_{\bar{\sigma}(j)}^{\bar{\sigma}(n-i)} - m_{\bar{\sigma}(j)}'^{\bar{\sigma}(n-i)}$ if $\bar{\sigma}(j) < \bar{\sigma}(n-i)$ and $m'_j = m_j$ otherwise. One can see that

$$\begin{aligned} a_n - \sum_{j=n-i+1}^{n-1} m'_j &= a_n - \sum_{j=n-i+1}^{n-1} m_j - \sum_{j=1}^{\bar{\sigma}(n-i)-1} \bar{m}_j^{\bar{\sigma}(n-i)} + \sum_{j=1}^{\bar{\sigma}(n-i)-1} m_j'^{\bar{\sigma}(n-i)} \\ &= \alpha_{n-i}^{i+1} + m_{n-i} - \sum_{j=1}^{\bar{\sigma}(n-i)-1} \bar{m}_j^{\bar{\sigma}(n-i)} + \sum_{j=1}^{\bar{\sigma}(n-i)-1} m_j'^{\bar{\sigma}(n-i)} \\ &= a_{\bar{\sigma}(n-i)} - \sum_{j=1}^{\bar{\sigma}(n-i)-1} \bar{m}_j^{\bar{\sigma}(n-i)} + m_{n-i} \\ &= c'_{n-i} + m_{n-i} = \alpha_{n-i+1}^{i+1} \end{aligned}$$

(at the beginning of the previous chain of equalities we have used condition (i), namely $\bar{m}_j^{\bar{\sigma}(n-i)} > 0 \Longrightarrow n-i < \bar{\sigma}^{-1}(j)$ and the fact that $m_j'^{\bar{\sigma}(n-i)} \le \bar{m}_j^{\bar{\sigma}(n-i)}$). Additionally, for every $\ell \in \{n-i+2, \dots, n\}$ it can be shown that $c''_{\ell-1} + m'_{\ell-1} = \alpha_\ell^{i+1}$. If $\ell - 1 \in D$ with $\bar{\sigma}(n-i) > \bar{\sigma}(\ell-1)$ then

${m'}^{\bar\sigma(n-i)}_{\bar\sigma(\ell-1)} = \bar m^{\bar\sigma(n-i)}_{\bar\sigma(\ell-1)} = 0$ and $m'_{\ell-1} = m_{\ell-1}$ while if $\ell-1 \in A$, then $c''_{\ell-1} + m'_{\ell-1} = a_{\bar\sigma(\ell-1)} + \sum_{\substack{n\geq i>\bar\sigma(\ell-1)\\ i\neq\bar\sigma(n-i)}} \bar m^i_{\bar\sigma(\ell-1)} + {m'}^{\bar\sigma(n-i)}_{\bar\sigma(\ell-1)} + m_{\ell-1} + \bar m^{\bar\sigma(n-i)}_{\bar\sigma(\ell-1)} - {m'}^{\bar\sigma(n-i)}_{\bar\sigma(\ell-1)} = c'_{\ell-1} + m_{\ell-1} = \alpha^{i+1}_\ell$.

We can conclude that the sequence $(\alpha^{i+1}_1, \dots, \alpha^{i+1}_n)$ remained unchanged with a new underlying permutation, such that the (decreased) element a_n is not positioned in front of the greater or equal $a_{\bar\sigma(n-i)}$ anymore. So we can continue the swapping process with the same values to get the desired sequence $(\alpha^{n-k+1}_1, \dots, \alpha^{n-k+1}_n) = (c_1, \dots, c_n)$. If along the go a smaller element would be placed in front of a greater one, repeat the analogous process to the one just described and continue. At the end we will get a permutation $\sigma \in S_n$ and the values $m^i_j \in \mathbb{N}$ (using the appropriate change of variables) such that every condition imposed by the the theorem is satisfied. □

Remark 3. Note that in Theorem 3 for a given $\ell \in \{1, \dots, n\}$, $\sigma(\ell)$ indicates which element of $a_1, \dots, a_n$ got the ℓ^{th} place within the sequence $(c_1, \dots, c_n)$. Conversely, $\sigma^{-1}(\ell)$ will show the final position of a_ℓ in the sequence $(c_1, \dots, c_n)$. For $1 \leq j < i \leq n$, $m^i_j \geq 0$ is the amount by which a_i is decreased in favor of a_j, i.e. an amount of m^i_j is subtracted from a_i and added to a_j.

Example 1. Over an arbitrary field κ the following is true: $[I_9 \oplus I_8 \oplus I_8 \oplus I_7 \oplus I_5 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1] \in \{[I_5]\} * \{[I_2]\} * \{[I_6]\} * \{[I_9]\} * \{[I_1]\} * \{[I_2]\} * \{[I_{13}]\} * \{[I_0]\} * \{[I_{10}]\} * \{[I_3]\} * \{[I_1]\}$. To see this, take the sequences $(a_1, \dots, a_{11}) = (5, 2, 6, 9, 1, 2, 13, 0, 10, 3, 1)$ and $(c_1, \dots, c_{11}) = (9, 8, 8, 7, 5, 4, 4, 3, 2, 1, 1)$, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 3 & 4 & 1 & 9 & 2 & 6 & 5 & 10 & 8 & 11 \end{pmatrix}$$

and the values $m^7_3 = 3$, $m^4_1 = m^3_1 = m^7_2 = m^{10}_8 = m^9_2 = 1$, $m^9_6 = m^9_5 = 2$ with all other $m^i_j = 0$ and use Theorem 3 to verify the conditions and the equality.

Using the main theorem (or better, using Lemma 3 which imposes weaker condition on the permutation σ) one can give another proof for an interesting corollary, which was observed in [7]. In the following statement (Corollary 1) μ and λ denote partitions (if $\mu = (c_1, c_2, \dots, c_n)$, then $c_1 \geq \dots \geq c_n \geq 0$). The length of a partition $\mu = (c_1, c_2, \dots, c_n)$ is $|\mu| = \sum_{i=1}^n c_i$. We say that a partition $(p_1, p_2, \dots, p_n)$ is dominated by the partition $(p'_1, p'_2, \dots, p'_n)$ and we write $(p_1, p_2, \dots, p_n) \trianglelefteq (p'_1, p'_2, \dots, p'_n)$ if $\sum_{i=1}^k p_i \leq \sum_{i=1}^k p'_i$ for all $k \in \{1, \dots, n\}$. The domination relation is a partial order.

Corollary 1. *Suppose that* $0 \leq a_1 \leq \dots \leq a_n$. *Then* $[I_{c_1} \oplus \dots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \dots * \{[I_{a_n}]\}$ *if and only if* $c_1 \geq \dots \geq c_n \geq 0$, $|\mu| = |\lambda|$ *and* $\mu \trianglelefteq \lambda$, *where* $\mu = (c_1, c_2, \dots, c_n)$ *and* $\lambda = (a_n, a_{n-1}, \dots, a_1)$.

Proof. "$\Longrightarrow$" If $[I_{c_1} \oplus \dots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \dots * \{[I_{a_n}]\}$, then the conditions from Lemma 3 hold, so $c_1 \geq \dots \geq c_n \geq 0$ and $|\mu| = |\lambda|$ follow immediately.

We have to show only that $\mu \trianglelefteq \lambda$, i.e. $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} a_{n-i+1}$ for all $k \in \{1,\dots,n\}$. Take $k \in \{1,\dots,n\}$ and consider the following inequality:

$$\begin{aligned}
\sum_{\ell=1}^{k} c_\ell &= \sum_{\ell=1}^{k} a_{\sigma(\ell)} + \sum_{\ell=1}^{k} \sum_{i=\sigma(\ell)+1}^{n} m^i_{\sigma(\ell)} - \sum_{\ell=1}^{k} \sum_{j=1}^{\sigma(\ell)-1} m_j^{\sigma(\ell)} \\
&= \sum_{\substack{1\leq i\leq n \\ \sigma^{-1}(i)\leq k}} a_i + \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(j)\leq k}} m^i_j - \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k}} m^i_j \\
&= \sum_{\substack{1\leq i\leq n \\ \sigma^{-1}(i)\leq k}} a_i + \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k \\ \sigma^{-1}(j)\leq k}} m^i_j - \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k \\ \sigma^{-1}(j)\leq k}} m^i_j - \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k \\ \sigma^{-1}(j)> k}} m^i_j \\
&= \sum_{\substack{1\leq i\leq n \\ \sigma^{-1}(i)\leq k}} a_i - \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k \\ \sigma^{-1}(j)> k}} m^i_j \leq \sum_{i=1}^{k} a_{n-i+1}.
\end{aligned}$$

We have used that $m^i_{\sigma(\ell)} > 0 \Longrightarrow \sigma^{-1}(i) < \ell \leq k$, hence

$$\sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(j)\leq k}} m^i_j = \sum_{\substack{1\leq j<i\leq n \\ \sigma^{-1}(i)\leq k \\ \sigma^{-1}(j)\leq k}} m^i_j.$$

"$\Longleftarrow$" Let us take $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$. Note that because $a_1 \leq a_2 \leq \cdots \leq a_n$ and our choice for $\sigma \in \mathcal{S}_n$, the condition $m^i_{\sigma(\ell)} > 0 \Longrightarrow \sigma^{-1}(i) < \ell \leq k$ from Lemma 3 is readily satisfied. So, knowing that $c_1 \geq \cdots \geq c_n \geq 0$, $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} a_{n-i+1}$ for all $k \in \{1,\dots,n\}$, all we have to show is that $\exists m^i_j \geq 0$, $1 \leq j < i \leq n$ such that

$$c_\ell = a_{n-\ell+1} + \sum_{i=n-\ell+2}^{n} m^i_{n-\ell+1} - \sum_{j=1}^{n-\ell} m_j^{n-\ell+1} \tag{5}$$

for all $\ell \in \{1,\dots,n\}$.

Let $m^i = \max\{0, a_i - c_{n-i+1}\}$ and $m_j = \max\{0, c_{n-j+1} - a_j\}$ for all $i,j \in \{1,\dots,n\}$. Since $|\mu| = |\lambda|$ it is clear that $\sum_{i=1}^{n} m^i = \sum_{j=1}^{n} m_j$. We define the sets $M^i = \left\{\sum_{k=1}^{i-1} m^k + 1, \dots, \sum_{k=1}^{i} m^k\right\}$ and $M_j = \left\{\sum_{k=1}^{j-1} m_j + 1, \dots, \sum_{k=1}^{j} m_k\right\}$ for all $i,j \in \{1,\dots,n\}$. Obviously $m^i = \left|M^i\right|$, $m_j = \left|M_j\right|$, , $M^{i_1} \cap M^{i_2} = M_{j_1} \cap M_{j_2} = \emptyset$ for any $i_1, i_2 \in \{1,\dots,n\}$, $i_1 \neq i_2$ and $j_1, j_2 \in \{1,\dots,p\}$, $j_1 \neq j_2$, hence $\bigcup_{j=1}^{n} M_j = \bigcup_{i=1}^{n} M^i$. Remark that $M^i \neq \emptyset \Rightarrow M_i = \emptyset$ and $M_j \neq \emptyset \Rightarrow M^j = \emptyset$.

We set $m^i_j = \left|M^i \cap M_j\right|$ for $1 \le j < i \le n$ and show that this is a valid choice. In equation (5) we have three cases.

If $c_\ell = a_{n-\ell+1}$, then $m^{n-\ell+1} = m_{n-\ell+1} = 0 \Rightarrow M^i = M_j = \emptyset \Rightarrow m^i_{n-\ell+1} = m^{n-\ell+1}_j = 0$ for all $1 \le j < n-\ell+1 < i \le n$, hence the equation is true in this case.

If $c_\ell < a_{n-\ell+1}$ then $m_{n-\ell+1} = 0 \Rightarrow M_j = \emptyset \Rightarrow m^i_{n-\ell+1} = 0$ for all $n-\ell+1 < i \le n$ so we have to show only that $m^{n-\ell+1} = \sum_{j=1}^{n-\ell} m^{n-\ell+1}_j$. This last equality is true if and only if $M^{n-\ell+1} \cap \left(\bigcup_{j=1}^{n-\ell} M_j\right) = M^{n-\ell+1} \iff M^{n-\ell+1} \subseteq \bigcup_{j=1}^{n-\ell} M_j$. The last inclusion is true because $\sum_{i=1}^{k} c_i \le \sum_{j=1}^{k} a_{n-j+1} \Rightarrow \bigcup_{i=1}^{k} M^{m-i+1} \supseteq \bigcup_{j=1}^{k} M_{m-j+1} \Longrightarrow \bigcup_{i=1}^{k} M^i \subseteq \bigcup_{j=1}^{k} M_j$ for all $k \in \{1, \ldots, n\}$ and $M^{n-\ell+1} \ne \emptyset \Longrightarrow M_{n-\ell+1} = \emptyset$.

In case $c_\ell > a_{n-\ell+1}$ we can make an analogue reasoning. □

Remark 4. Using Theorem 3 one can also give an easy proof for Theorem 2.

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