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FRACTIONAL q-CALCULUS AND CERTAIN SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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Abstract. By applying the concept (and theory) of fractional q-calculus, we first define and introduce new classes of univalent functions analytic in the open unit disk involving a q-differeintegral operator. Among the results investigated for these function classes are the coefficient inequalities and distortion theorems. Special cases are briefly pointed out.

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1. INTRODUCTION, PRELIMINARIES AND DEFINITIONS

The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering and also in the geometric function theory of complex variables (see, for example [8] and [13]). The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis. One may refer to the books [5]-[6], and the recent papers [1], [2], [4], [7], [11] and [12] on the subject.

Recently, authors in [10] have used the fractional q-calculus operators and investigated some new classes of functions which are analytic in the open disk. Purohit [9] also studied similar work and considered new classes of multivalently analytic functions in the open unit disk. In the present paper, our purpose is to introduce further new subclasses of functions defined by applying the fractional q-calculus operators which are univalent and analytic in the open unit disk. Among the results derived include, the coefficient inequalities and distortion theorems for the subclasses defined and introduced below. Special cases of the results are also pointed out briefly.

For the convenience of the reader, we deem it proper to give here the basic definitions and related details of the q-calculus:

The q-shifted factorial (see [5]) is defined for $\alpha,\, q \in \mathbb{C}$ as a product of n factors by

$$(1)\qquad (\alpha\,;q)_n = \begin{cases} 1 & , \quad n=0; \\ (1-\alpha)\,(1-\alpha\,q)\,\cdots\,(1-\alpha\,q^{n-1}) & , \quad n\in\mathbb{N}\,, \end{cases}$$

and in terms of the basic analogue of the gamma function by

$$(q^{\alpha}\,;q)_n = \frac{\Gamma_q(\alpha+n)\,(1-q)^n}{\Gamma_q(\alpha)} \qquad (n>0), \tag{2}$$

where the q-gamma function is defined by ([5, p. 16, eqn. (1.10.1)])

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}\ (0<q<1). \tag{3}$$

If $|q|<1$, the definition (1) remains meaningful for $n=\infty$, as a convergent infinite product given by

$$(\alpha\,;q)_\infty = \prod_{j=0}^{\infty}(1-\alpha\, q^j)\ .$$

We recall here the following q-analogue definitions given by Gasper and Rahman [5].
The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = \frac{(1-q^x)\,\Gamma_q(x)}{1-q} \tag{4}$$

and the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{5}$$

Also, the Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [5, pp. 19, 22])

$$D_{q,\,z}f(z) = \frac{f(z)-f(zq)}{z(1-q)}\ (z\neq 0,\ q\neq 1) \tag{6}$$

and

$$\int_0^z f(t)\,\mathrm{d}_q t = z\ (1-q)\sum_{k=0}^{\infty} q^k f(zq^k)\ . \tag{7}$$

In view of the relation that

$$\underset{q\to 1^-}{Lim}\,\frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n\ , \tag{8}$$

we observe that the q-shifted factorial (1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ $(n\in\mathbb{N})$.

We now mention below the fractional q-calculus operators of a complex-valued function $f(z)$ (which were recently studied by Purohit and Raina [10]).

Definition 1. (Fractional q-Integral Operator) The fractional q-integral operator $I^{\alpha}_{q,\,z}f(z)$ of a function $f(z)$ of order α is defined by

$$I^{\alpha}_{q,\,z}f(z) \equiv D^{-\alpha}_{q,\,z}f(z) = \frac{1}{\Gamma_q(\alpha)}\int_0^z (z-tq)_{\alpha-1}\,f(t)\ \mathrm{d}_q t \qquad (\alpha>0)\ , \tag{9}$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin. In view of relation (5), the q-binomial function $(z-tq)_{\alpha-1}$ can be expressed as

$$(z-tq)_{\alpha-1} = z^{\alpha-1}\ {}_1\Phi_0[q^{-\alpha+1};\ -;\ q,\ tq^{\alpha}/z]. \tag{10}$$

The series ${}_1\Phi_0[\alpha; -; q, z]$ (which is a special case of the series ${}_2\Phi_1[\alpha, \beta; \gamma; q, z]$ for $\gamma = \beta$) is obviously single-valued when $|\arg(z)| < \pi$ and $|z| < 1$, (see for details [5, pp. 104-106]), therefore, in view of the representation of the integral defined by (7), it may be noted that the function $(z-tq)_{\alpha-1}$ in (9) is single-valued when $|\arg(-tq^{\alpha}/z)| < \pi,\ |tq^{\alpha}/z)| < 1$ and $|\arg z| < \pi$. Thus, for suitably selected function $f(z)$ (which ensures its convergence), the operator (9) is well defined.

Definition 2. (Fractional q-Derivative Operator) The fractional q-derivative operator $D^{\alpha}_{q,z} f(z)$ of a function $f(z)$ of order α is defined by

$$D^{\alpha}_{q,z} f(z) = D_{q,z}\ I^{1-\alpha}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\alpha)} D_{q,z} \int_0^z (z-tq)_{-\alpha} f(t)\ \mathrm{d}_q t, \tag{11}$$
$$(0 \leq \alpha < 1),$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above.

Definition 3. (Extended Fractional q-Derivative Operator) Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order α is defined by

$$D^{\alpha}_{q,z} f(z) = D^{m}_{q,z}\, I^{m-\alpha}_{q,z} f(z) \qquad (m-1 \leq \alpha < m;\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{12}$$

where $\mathbb{N}$ denotes the set of natural numbers.

Also for $\alpha = 1$, we have

$$D^{1}_{q,z} f(z) = D_{q,z} f(z). \tag{13}$$

In the sequel, we shall be using the following image formulas which are easy consequences of the operators (9) and (12) ([10, pp. 58-59]):

$$I^{\alpha}_{q,z} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda+\alpha)}\ z^{\lambda+\alpha}\ (\alpha > 0,\ \lambda > -1) \tag{14}$$

and

$$D^{\alpha}_{q,z} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-\alpha)}\ z^{\lambda-\alpha}\ (\alpha \geq 0,\ \lambda > -1). \tag{15}$$

2. NEW CLASSES OF FUNCTIONS

By $\mathcal{A}_n$, we denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k\, z^k \qquad (n \in \mathbb{N}), \tag{16}$$

which are analytic and univalent in the open unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}.$$

Also, let $\mathcal{A}_n^-$ denote the subclass of $\mathcal{A}_n$ consisting of analytic and univalent functions expressed in the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ (a_k \geq 0, n \in \mathbb{N}). \tag{17}$$

For the purpose of this paper, we define a fractional q-differintegral operator $\Omega_{q,z}^{\alpha}$ for a function $f(z)$ by

$$\Omega_{q,z}^{\alpha} f(z) = \frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)} z^{\alpha} D_{q,z}^{\alpha} f(z) \tag{18}$$

$$(\alpha < 2;\ 0 < q < 1;\ z \in \mathbb{U}),$$

where $D_{q,z}^{\alpha} f(z)$ in (18) represents, respectively, a fractional q-integral of $f(z)$ of order α when $-\infty < \alpha < 0$, and a fractional q-derivative of $f(z)$ of order α when $0 \leq \alpha < 2$. Thus, in view of (17), the operator (18) has the form:

$$\Omega_{q,z}^{\alpha} f(z) = z - \sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k z^k \ (-\infty < \alpha < 2), \tag{19}$$

where

$$A(\alpha, k, q) = \frac{\Gamma_q(2-\alpha)\Gamma_q(1+k)}{\Gamma_q(2)\Gamma_q(1+k-\alpha)}. \tag{20}$$

Remark 1. In view of (15), the above operator (19) has the following form:

$$\Omega_{q,z}^{\alpha} f(z) = z - \sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k z^k \ (0 \leq \alpha < 2), \tag{21}$$

where $A(\alpha, k, q)$ is given by (20).

Remark 2. Again, on making use of (14), we get

$$\Omega_{q,z}^{\alpha} f(z) = z - \sum_{k=n+1}^{\infty} A(-\alpha, k, q)\, a_k z^k \ (\alpha > 0), \tag{22}$$

where $A(\alpha, k, q)$ is given by (20) (with α replaced by $-\alpha$, therein).

Suppose $\mathcal{S}_n^{\alpha}(\lambda, \delta, q)$ denotes the subclass of $\mathcal{A}_n^-$ consisting of functions $f(z)$ which satisfy the inequality that

$$\left| \frac{1}{\delta} \left\{ \frac{z D_{q,z} F_q^{(\alpha,\lambda)}(z)}{F_q^{(\alpha,\lambda)}(z)} - 1 \right\} \right| < 1, \tag{23}$$

where

$$F_q^{(\alpha,\lambda)}(z) = \lambda\, z\, D_{q,z}\left(\Omega_{q,z}^{\alpha} f(z)\right) + (1-\lambda)\, \Omega_{q,z}^{\alpha} f(z) \tag{24}$$

$$(\alpha < 2;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}).$$

Also, let $\mathcal{R}_n^{\alpha}(\lambda, \delta, q)$ denote the subclass of $\mathcal{A}_n^-$ consisting of functions $f(z)$ which satisfy the inequality that

$$\left| \frac{1}{\delta} \left\{ D_{q,z} \left(\Omega_{q,\,z}^{\alpha} f(z) \right) + \lambda\ z\ D_{q,z}^2 \left(\Omega_{q,\,z}^{\alpha} f(z) \right) - 1 \right\} \right| < 1 \tag{25}$$

$$(\alpha < 2;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}).$$

The following give the characterization properties for functions of the form (17) to belong to the classes defined above.

THEOREM 1. *Let the function f defined by* (17) *be in the class $\mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, then*

$$\sum_{k=n+1}^{\infty} \frac{A(\alpha, k, q)}{B(\lambda, \delta, k, q)}\ a_k \leq |\delta|\,, \tag{26}$$

where $A(\alpha, k, q)$ is given by (20) *and $B(\lambda, \delta, k, q)$ is given by*

$$B(\lambda, \delta, k, q) = \frac{(1-q)^2}{[\lambda\ q(1-q^{k-1}) + 1 - q]\,[q(1-q^{k-1}) + |\delta|\,(1-q)]}. \tag{27}$$

The result is sharp.

Proof. Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, then on using (23), we get

$$\Re \left\{ \frac{z D_{q,z} F_q^{(\alpha,\lambda)}(z) - F_q^{(\alpha,\lambda)}(z)}{F_q^{(\alpha,\lambda)}(z)} \right\} > -\,|\delta|\,.$$

In view of (19), (24) and the q-derivative formula, namely:

$$D_{q,z}\ z^k = \left(\frac{1-q^k}{1-q} \right)\ z^{k-1},$$

we obtain

$$F_q^{(\alpha,\lambda)}(z) = z - \sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1}) + 1 - q}{1-q} \right]\ z^k \tag{28}$$

and

$$z\ D_{q,z}\ F_q^{(\alpha,\lambda)}(z) = z - \sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1}) + 1 - q}{(1-q)^2} \right]\ (1-q^k)\, z^k. \tag{29}$$

Now, on making use of the above relations, we get

$$\Re \left\{ \frac{-\sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{(1-q)^2} \right]\ q\ (1-q^{k-1})\, z^{k-1}}{1 - \sum_{k=n+1}^{\infty} A(\alpha, k, q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{1-q} \right]\ z^{k-1}} \right\} > -\,|\delta|\,.$$

By putting $z = r$, and noting that the denominator is positive for $r = 0$, and also remains positive for $0 < r < 1$, so that on letting $r \to 1^-$, we get

$$\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{(1-q)^2}\right]\ q\ (1-q^{k-1})\ < \tag{30}$$
$$|\delta| \left(1-\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{1-q}\right]\right),$$

which yields the desired coefficient bound inequality (26).

Conversely, by applying hypothesis (26) and letting $|z| = 1$, we find that

$$\left|\frac{zD_{q,z}F_q^{(\alpha,\lambda)}(z)-F_q^{(\alpha,\lambda)}(z)}{F_q^{(\alpha,\lambda)}(z)}\right| =$$

$$\left|\frac{-\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{(1-q)^2}\right]\ q\ (1-q^{k-1})z^{k-1}}{1-\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{1-q}\right] z^{k-1}}\right|$$

$$\leq \frac{|\delta| \left(1-\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{1-q}\right]\right)}{1-\sum_{k=n+1}^{\infty} A(\alpha,k,q)\, a_k \left[\frac{\lambda\ q\ (1-q^{k-1})+1-q}{1-q}\right]} = |\delta|\,.$$

Hence, by the maximum modulus principle and the condition (23), we infer that

$$f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\delta,q).$$

It is easy to verify that the equality in (26) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{|\delta|\, B(\lambda,\delta,n+1,q)}{A(\alpha,n+1,q)}\ z^{n+1} \quad (n \in \mathbb{N}), \tag{31}$$

where $A(\alpha,k,q)$ and $B(\lambda,\delta,k,q)$ are given by (20) and (27), respectively. $\square$

Similarly, we can prove the following.

Theorem 2. *Let the function f defined by* (17) *be in the class $\mathcal{R}_n^{\alpha}(\lambda,\delta,q)$, then*

$$\sum_{k=n+1}^{\infty} \frac{A(\alpha,k,q)}{C(\lambda,k,q)}\ a_k \leq |\delta|\ , \tag{32}$$

where $A(\alpha,k,q)$ is given by (20) *and $C(\lambda,k,q)$ is given by*

$$C(\lambda,k,q) = \frac{(1-q)^2}{(1-q^k)\,[(1-q)+\lambda(1-q^{k-1})]}. \tag{33}$$

The result is sharp with the extremal function given by

$$f(z) = z - \frac{|\delta|\, C(\lambda, n+1, q)}{A(\alpha, n+1, q)}\; z^{n+1} \quad (n \in \mathbb{N}). \tag{34}$$

We next prove a simple inclusion property which is given as follows:

THEOREM 3. *Let* $\alpha < 2,\ 0 \leq \lambda \leq 1,\ 0 < q < 1,\ \delta_1, \delta_2 \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *and* $|\delta_1| < |\delta_2|$, *then*

$$\mathcal{S}_n^{\alpha}(\lambda, \delta_1, q) \ \subset\ \mathcal{S}_n^{\alpha}(\lambda, \delta_2, q). \tag{35}$$

Proof. Suppose that $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \delta_1, q)$, then in view of Theorem 1, we have

$$\sum_{k=n+1}^{\infty} \frac{A(\alpha, k, q)}{B(\lambda, \delta_1, k, q)}\; a_k \leq |\delta_1| \ ,$$

where $A(\alpha, k, q)$ and $B(\lambda, \delta_1, k, q)$ are given by (20) and (27), respectively. Now $|\delta_1| < |\delta_2|$ implies that

$$\sum_{k=n+1}^{\infty} \frac{A(\alpha, k, q)}{B(\lambda, \delta_1, k, q)}\; a_k \leq |\delta_1| \ \leq |\delta_2| \ ,$$

which in view of Theorem 1 immediately leads to $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \delta_2, q)$, and the result (33) follows. $\square$

Similarly, we can prove the following inclusion property:

THEOREM 4. *Let* $\alpha < 2,\ 0 \leq \lambda \leq 1,\ 0 < q < 1,\ \delta_1, \delta_2 \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *and* $|\delta_1| < |\delta_2|$, *then*

$$\mathcal{R}_n^{\alpha}(\lambda, \delta_1, q) \ \subset\ \mathcal{R}_n^{\alpha}(\lambda, \delta_2, q). \tag{36}$$

3. DISTORTION THEOREMS

In this section, we establish distortion theorems for classes of functions defined above involving the fractional q-calculus operators.

THEOREM 5. *Let* $\alpha, \lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities*

$$\alpha < 2, n \in \mathbb{N};\ 0 \leq \lambda \leq 1,\ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (17) *be in the class* $\mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, *then*

$$\begin{aligned} |z| - |\delta|\ \ D(\alpha, \lambda, \delta, n, q)\, |z|^{n+1} \leq\ \ |f(z)| \leq |z| + \\ |\delta|\ \ D(\alpha, \lambda, \delta, n, q)\ \ |z|^{n+1}\ \ (z \in \mathbb{U}), \end{aligned} \tag{37}$$

where

$$D(\alpha, \lambda, \delta, n, q) = \frac{B(\lambda, \delta, n+1, q)}{A(\alpha, n+1, q)}, \tag{38}$$

$A(\alpha,n+1,q)$ and $B(\lambda,\delta,n+1,q)$ are given by (20) *and* (27)*, respectively.*

Proof. Since $f(z)\in\mathcal{S}_n^{\alpha}(\lambda,\delta,q)$, then under the hypotheses of Theorem 1, we have

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{|\delta|\,B(\lambda,\delta,n+1,q)}{A(\alpha,n+1,q)}\,, \tag{39}$$

which in view of (38), implies that

$$\sum_{k=n+1}^{\infty} a_k \leq |\delta|\;D(\alpha,\lambda,\delta,n,q).$$

This last inequality in conjunction with the following inequality (easily obtainable from (16)):

$$|z|-|z|^{n+1}\sum_{k=n+1}^{\infty} a_k \leq |f(z)| \leq |z|+|z|^{n+1}\sum_{k=n+1}^{\infty} a_k \tag{40}$$

yields the assertion (37) of Theorem 5. □

A further distortion theorem involving the generalized fractional q-differintegral operator $\Omega_{q,\,z}^{\alpha}f(z)$ is given by the following result.

Theorem 6. *Let $\alpha<2, 0\leq\lambda\leq 1,\ \delta\in\mathbb{C}\setminus\{0\}\in\mathbb{N},\ n\in\mathbb{N},\ 0<q<1$ and let the function $f(z)$ defined by* (17) *be in the class $\mathcal{S}_n^{\alpha}(\lambda,\delta,q)$. Then*

$$|z|-|\delta|\;B(\lambda,\delta,n+1,q)\,|z|^{n+1} \leq \left|\Omega_{q,\,z}^{\alpha}f(z)\right| \leq \tag{41}$$
$$|z|+|\delta|\;B(\lambda,\delta,n+1,q)\;|z|^{n+1}\quad(z\in\mathbb{U}),$$

where $B(\lambda,\delta,n+1,q)$ is given by (27).

Proof. In view of (19), we first show that the function $A(\alpha,k,q)$ $(-\infty<\alpha<2, k\geq n+1;\ n\in\mathbb{N})$ is a decreasing function of k for $\alpha<2, 0<q<1$. It follows that

$$\frac{A(\alpha,k+1,q)}{A(\alpha,k,q)}=\frac{\Gamma_q(2+k)\Gamma_q(1+k-\alpha)}{\Gamma_q(1+k)\Gamma_q(2+k-\alpha)}\quad(k\geq n+1;n\in\mathbb{N}),$$

and it is sufficient to consider here the value $k=n+1$, so that on using (4), we get $\frac{A(\alpha,k+1,q)}{A(\alpha,k,q)}=\frac{1-q^{2+n}}{1-q^{2+n-\alpha}}$ $(0<q<1)$. The function $A(\alpha,k,q)$ is a decreasing function of k if $\frac{A(\alpha,n+2,q)}{A(\alpha,n+1,q)}\leq 1\,(n\in\mathbb{N})$, and this gives $\frac{1-q^{2+n}}{1-q^{2+n-\alpha}}\leq 1\;(0<q<1)$. Multiplying the above inequality both sides by $1-q^{2+n-\alpha}$ (provided that $\alpha<2$), we are at once lead to the inequality $\alpha\leq 0$. Thus, $A(\alpha,k,q)$ $(k\geq n+1;n\in\mathbb{N})$ is a decreasing function of k for $-\infty<\alpha<2, 0<q<1$.
Now, on using (19), we observe that

$$\left|\Omega_{q,\,z}^{\alpha}f(z)\right|\geq|z|-\sum_{k=n+1}^{\infty}A(\alpha,k,q)\,a_k\;|z|^k$$

$$\geq |z| - A(\alpha, n+1, q) \ \ |z|^{n+1} \sum_{k=n+1}^{\infty} a_k.$$

Since $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, then under the hypotheses of Theorem 1 on using inequality (39), we obtain

$$\left|\Omega_{q,\,z}^{\alpha} f(z)\right| \geq |z| - |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^{n+1} \ . \tag{42}$$

Similarly, it follows that

$$\left|\Omega_{q,\,z}^{\alpha} f(z)\right| \leq |z| + |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^{n+1} \,, \tag{43}$$

which establishes the assertion (41) of Theorem 6. □

In view of (18) and (21), Theorem 6 gives the following distortion inequality for the function $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \delta, q)$ involving the fractional q-derivative operator $D_{q,\,z}^{\alpha}$:

Corollary 1. *Let the function $f(z)$ defined by* (17) *be in the class $\mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} \, |z| \, \{1 - |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^n\} \leq \left|D_{q,\,z}^{\alpha} f(z)\right| \leq \tag{44}$$

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} \, |z| \, \{1 + |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^n\} \ \ (z \in \mathbb{U}),$$

where $0 \leq \alpha < 2, \ n \in \mathbb{N}; \ 0 \leq \lambda \leq 1; \ 0 < q < 1$ *and* $B(\lambda, \delta, n+1, q)$ *is given by* (27)*.*

Also, in view of (18) and (22), Theorem 6 gives the following inequality involving fractional q-integral operator $I_{q,\,z}^{\alpha}$:

Corollary 2. *Let the function $f(z)$ be in the class $\mathcal{S}_n^{\alpha}(\lambda, \delta, q)$, then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)} \, |z| \, \{1 - |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^n\} \leq \left|I_{q,\,z}^{\alpha} f(z)\right| \leq \tag{45}$$

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)} \, |z| \, \{1 + |\delta| \, B(\lambda, \delta, n+1, q) \ \ |z|^n\} \ \ (z \in \mathbb{U}),$$

where $\alpha > 0, \ n \in \mathbb{N}; \ 0 \leq \lambda \leq 1; \ 0 < q < 1$ *and* $B(\lambda, \delta, n+1, q)$ *is given by* (27)*.*

Similarly, one can easily prove the following distortion inequalities for the function $f(z) \in \mathcal{R}_n^{\alpha}(\lambda, \delta, q)$:

Theorem 7. *Let* $\alpha, \lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities*

$$\alpha < 2, n \in \mathbb{N}; \ 0 \leq \lambda \leq 1, \ 0 < q < 1.$$

Also, let the function $f(z)$ defined by (17) *be in the class $\mathcal{R}_n^{\alpha}(\lambda, \delta, q)$, then*

$$|z| - |\delta| \ \ E(\alpha, \lambda, n, q) \, |z|^{n+1} \leq \ |f(z)| \leq |z| + |\delta| \ \ E(\alpha, \lambda, n, q) \ \ |z|^{n+1} \tag{46}$$

$$(z \in \mathbb{U}),$$

where

$$E(\alpha, \lambda, n, q) = \frac{C(\lambda, n+1, q)}{A(\alpha, n+1, q)}, \tag{47}$$

$A(\alpha, n+1, q)$ and $C(\lambda, n+1, q)$ are given by (20) and (33), respectively.

Theorem 8. *Let $\alpha < 2, 0 \leq \lambda \leq 1,\ \delta \in \mathbb{C} \setminus \{0\} \in \mathbb{N},\ n \in \mathbb{N},\ 0 < q < 1$ and let the function $f(z)$ defined by* (17) *be in the class $\mathcal{R}_n^\alpha(\lambda, \delta, q)$. Then*

$$|z| - |\delta|\ \ C(\lambda, n+1, q)\, |z|^{n+1} \leq\ \left|\Omega_{q,\,z}^\alpha f(z)\right| \leq |z| + \tag{48}$$

$$|\delta|\ \ C(\lambda, n+1, q)\ \ |z|^{n+1}\ \ (z \in \mathbb{U}),$$

where $C(\lambda, n+1, q)$ is given by (33).

Now, again in view of (18), (21) and (22), Theorem 8 gives the following distortion inequalities for the function $f(z) \in \mathcal{R}_n^\alpha(\lambda, \delta, q)$ involving fractional q-derivative operator $D_{q,\,z}^\alpha$ and fractional q-integral operator $I_{q,\,z}^\alpha$.

Corollary 3. *Let the maof(z) defined by* (17) *be in the class $\mathcal{R}_n^\alpha(\lambda, \delta, q)$, then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)}\, |z| \left\{1 - |\delta|\, C(\lambda, n+1, q)\ \ |z|^n\right\} \leq \left|D_{q,\,z}^\alpha f(z)\right| \leq \tag{49}$$

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)}\, |z| \left\{1 + |\delta|\, C(\lambda, n+1, q)\ \ |z|^n\right\}\ \ (z \in \mathbb{U}),$$

where $0 \leq \alpha < 2,\ n \in \mathbb{N};\ 0 \leq \lambda \leq 1;\ 0 < q < 1$ and $C(\lambda, n+1, q)$ is given by (33).

Corollary 4. *Let the function $f(z)$ be in the class $\mathcal{R}_n^\alpha(\lambda, \delta, q)$, then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)}\, |z| \left\{1 - |\delta|\, C(\lambda, n+1, q)\ \ |z|^n\right\} \leq \left|I_{q,\,z}^\alpha f(z)\right| \leq \tag{50}$$

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\alpha)}\, |z| \left\{1 + |\delta|\, C(\lambda, n+1, q)\ \ |z|^n\right\}\ \ (z \in \mathbb{U}),$$

where $\alpha > 0,\ n \in \mathbb{N};\ 0 \leq \lambda \leq 1;\ 0 < q < 1$ and $C(\lambda, n+1, q)$ is given by (33).

4. SPECIAL CASES

In this section we briefly consider some special cases of the results derived in the preceding sections.
In view of the relationship (18), we find that

$$\Omega_{q,\,z}^0 f(z)\ \ =\ \ f(z). \tag{51}$$

When $\alpha = 0$ and $\delta = \gamma\beta$ ($\gamma \in \mathbb{C} \setminus \{0\}\,,\ 0 < \beta \leq 1$), the condition (23) reduces to the inequality

$$\left|\frac{1}{\gamma}\left\{\frac{z\ D_{q,z}f(z) + \lambda q z^2\ D_{q,z}^2 f(z)}{\lambda\ z\ D_{q,z}f(z) + (1-\lambda) f(z)} - 1\right\}\right| < \beta, \tag{52}$$

$$(0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\}\,;\ 0 < q < 1;\ z \in \mathbb{U}),$$

and we have

$$\mathcal{S}_n^0(\lambda, \gamma\beta, q) = \mathcal{H}_n(\lambda, \gamma, \beta, q), \tag{53}$$

where $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ represents a subclass of analytic functions, which satisfy the condition (52).

Similarly, the condition (25) when $\alpha = 0$ and $\delta = \gamma\beta$ reduces to the inequality

$$\left| \frac{1}{\gamma} \left\{ D_{q,z} f(z) + \lambda\ z\ D_{q,z}^2 f(z) - 1 \right\} \right| < \beta \tag{54}$$

$$(0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\}\,;\ 0 < q < 1;\ z \in \mathbb{U}),$$

and we write

$$\mathcal{R}_n^0(\lambda, \gamma\beta, q) = \mathcal{G}_n(\lambda, \gamma, \beta, q), \tag{55}$$

where $\mathcal{G}_n(\lambda, \gamma, \beta, q)$ is another subclass of analytic functions which satisfy the condition (54).

By setting $\alpha = 0$, $\delta = \gamma\beta$, and making use of the relations (53) and (55), Theorems 1 and 2 give the following coefficient inequalities for the classes $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ and $\mathcal{G}_n(\lambda, \gamma, \beta, q)$, respectively.

Corollary 5. *Let f defined by* (17) *be in the class $\mathcal{H}_n(\lambda, \gamma, \beta, q)$, then*

$$\sum_{k=n+1}^{\infty} \Big[\lambda\ q(1 - q^{k-1}) + 1 - q \Big] \Big[q(1 - q^{k-1}) + \beta\, |\gamma|\, (1 - q) \Big]\ a_k \leq \beta\, |\gamma|\ (1 - q)^2. \tag{56}$$

The result is sharp with the extremal function given by

$$f(z) = z - \frac{\beta\, |\gamma|\ (1 - q)^2}{[\lambda\ q(1 - q^n) + 1 - q]\, [q(1 - q^n) + \beta\, |\gamma|\, (1 - q)]}\ z^{n+1} \quad (n \in \mathbb{N}). \tag{57}$$

Corollary 6. *Let f defined by* (17) *be in the class $\mathcal{G}_n(\lambda, \gamma, \beta, q)$, then*

$$\sum_{k=n+1}^{\infty} (1 - q^k) \Big[(1 - q) + \lambda(1 - q^{k-1}) \Big]\ a_k \leq \beta\, |\gamma|\ (1 - q)^2. \tag{58}$$

The result is sharp and the equality is attained for the function $f(z)$ is given by

$$f(z) = z - \frac{\beta\, |\gamma|\ (1 - q)^2}{(1 - q^{n+1})\, [(1 - q) + \lambda(1 - q^n)]}\ z^{n+1}\ (n \in \mathbb{N}). \tag{59}$$

If we put $\alpha = 0$, $\delta = \gamma\beta$, then Theorem 5 and Theorem 7, respectively, yield the following distortion theorems for the classes $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ and $\mathcal{G}_n(\lambda, \gamma, \beta, q)$.

Corollary 7. *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities*

$$n \in \mathbb{N};\ 0 \leq \lambda \leq 1, 0 < \beta \leq 1,\ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (17) *be in the class* $\mathcal{H}_n(\lambda, \gamma, \beta, q)$*, then*

$$|z| - \beta\,|\gamma|\ B(\lambda, \beta\gamma, n+1, q)\,|z|^{n+1} \leq\ |f(z)| \leq |z| + \beta\,|\gamma|\ B(\lambda, \beta\gamma, n+1, q)\ |z|^{n+1}\ \ (z \in \mathbb{U}), \tag{60}$$

where $B(\lambda, \beta\gamma, n+1, q)$ *(with* δ *and* k*) is given by* (27).

Corollary 8. *Let* $0 \leq \lambda \leq 1,\ 0 < \beta \leq 1,\ \gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N},\ n \in \mathbb{N},\ 0 < q < 1$ *and let the function* $f(z)$ *defined by* (17) *be in the class* $\mathcal{G}_n(\lambda, \gamma, \beta, q)$*. Then*

$$|z| - \beta\,|\gamma|\ C(\lambda, n+1, q)\,|z|^{n+1} \leq\ |f(z)| \leq |z| + \beta\,|\gamma|\ C(\lambda, n+1, q)\ |z|^{n+1}\ \ (z \in \mathbb{U}), \tag{61}$$

where $C(\lambda, n+1, q)$ *is given by* (33).

By letting $q \to 1^{-}$, and making use of the limit formula (8), we observe that the function class $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ and the inequality (56) of Corollary 5 provide, respectively, the q-extensions of the known class and the related inequality due to Altintaş, Özkan and Srivastava [3, p. 64, eqn. (16)]. Also, the function class $\mathcal{G}_n(\lambda, \gamma, \beta, q)$ defined by condition (54) and Corollary 6 are the q-extensions of the corresponding known function class and the related result due to Altintaş, Özkan and Srivastava [3, p. 65, Lemma 2, eqn. (20)].

Lastly, we conclude this paper by remarking that the fractional q-calculus operators defined in Section 2 can fruitfully be used in the investigation of several other multivalent (or meromorphic) analytic function classes and their various geometric properties like, the coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity etc. can be studied in the unit disk. These considerations can be pursued by using the theory of fractional q-calculus.

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