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A RELATED FIXED POINT THEOREM FOR m MAPPINGS ON m COMPLETE QUASI-METRIC SPACES

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Abstract. We prove a related fixed point theorem for m mappings in m quasi-metric spaces. This result unifies, generalizes and extends several of well-known fixed point theorems for metric spaces to quasi-metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

Many authors, in [1], [2], [4], [5], [6], [10], [11] etc. proved several fixed point theorems for one, two and three metric spaces.

In this paper, we will prove a related fixed point theorem for m mappings on m quasi-metric spaces, $m-1$ of mappings must be continuous. This result generalizes the theorems of Rhoades [11], Banach [1], Kannan [6], Bianchini [2], Reich [10], Fisher [4], Jain et al. [5] etc. and in the same time extends them in quasi-metric spaces constituted to a more general setting than the metric ones in many applications (see [9]).

Now we give some standard definitions and notations.

Quasi-metric spaces concept is treated differently by many authors. In this paper our concept is in line with this treated on [3], [7], [8], [9], [12], [13], [14] etc.:

Definition 1. Let X be an arbitrary set and R^+ the set of nonnegative real numbers. A function $d : X \times X \to R^+$ is called quasi-distance on X if and only if there exists a constant $k \geq 1$, such that for all x, y and $z \in X$ the following conditions hold:

(1) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

Inequality (3) is often called quasi-triangular inequality and k is often called quasi-triangular constant of d. A pair (X, d) is called quasi-metric space if X is a set and d is a quasi-distance on X. Metric spaces are a special case of quasi-metric spaces (for $k = 1$).

The quasi-distance d is not always continuous. For this reason, in some articles on fixed point theory, authors require the continuity of the used quasi-distance, as complementary conditions.

The following example illustrates the existence of the quasi-distance for an arbitrary constant $k \geq 1$.

EXAMPLE 1. [7] Let $X = R \times R$ and $x = (x_1, x_2) \in X, y = (y_1, y_2) \in X$. The function $d : X \times X \to R^+$ such that

$$d(x,y) = \begin{cases} k\,|x_1 - y_1| + |x_2 - y_2|\,, & \text{for } |x_1 - y_1| \le |x_2 - y_2| \\ |x_1 - y_1| + k\,|x_2 - y_2|\,, & \text{for } |x_1 - y_1| > |x_2 - y_2| \end{cases}$$

is a quasi-distance.

DEFINITION 2. A sequence $\{x_n\}$ in a quasi-metric space (X, d) is called Cauchy sequence if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.

DEFINITION 3. Let (X, d) be a quasi-metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} d(x_n, x) = 0$.

(2) It is called compact if every sequence contains a convergent subsequence.

DEFINITION 4. A quasi-metric space (X, d) is called complete, if every Cauchy sequence is convergent.

Before stating the main theorem we define new classes implicit functions, whose role will be crucial in the main result of this paper.

DEFINITION 5. The set of all upper semi-continuous functions with k variables $\varphi : [0, +\infty)^k \to R$ satisfying the properties:

(a). φ is non decreasing in respect with each variable.

(b). $\varphi(t, t, \ldots, t) \le t, t \in [0, +\infty)$.

will be noted Φ_k and every such function will be called a Φ_k-function.

Some examples of Φ_4-function are as follows:

1. $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$
2. $\varphi(t_1, t_2, t_3, t_4) = [\max\{t_1t_2, t_2t_3, t_3t_4, t_4t_1\}]^{1/2}$
3. $\varphi(t_1, t_2, t_3, t_4) = [\max\{t_1^p, t_2^p, t_3^p, t_4^p\}]^{1/p}$
4. $\varphi(t_1, t_2, t_3, t_4) = \frac{at_1+bt_2+ct_3+dt_4}{a+b+c+d}$, where $a, b, c, d \ge 0$ etc.

2. MAIN RESULTS

We prove now the main theorem.

THEOREM 1. *Let (X_i, d_i) for $i = 1, 2, \ldots, m$ be m complete quasi-metric spaces with constants k_i and continuous quasi-distances d_i. Let $T_i : X_i \to X_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $T_m : X_m \to X_1$ be m mappings from which $(m-1)$ are continuous. If there exists $c \in \left[0, \frac{1}{k}\right)$, $k = \max\{k_i : i = 1, 2, \ldots, m\}$ and the following inequalities hold*

$$\begin{aligned} &d_1\left(T_m \ldots T_2T_1x_1, T_m \ldots T_2T_1x_1'\right) \le \\ (1)\qquad &c\varphi_1 \begin{pmatrix} d_1\left(x_1, x_1'\right), d_1\left(x_1, T_m \ldots T_1x_1\right), d_1\left(x_1', T_m \ldots T_1x_1'\right), \\ d_2\left(T_1x_1, T_1x_1'\right),\ d_3\left(T_2T_1x_1, T_2T_1x_1'\right), \ldots, \\ d_m\left(T_{m-1} \ldots T_1x_1, T_{m-1} \ldots T_1x_1'\right) \end{pmatrix}, \end{aligned}$$

$$
(2)\qquad
\begin{aligned}
&d_2\left(T_1T_m\dots T_3T_2x_2, T_1T_m\dots T_3T_2x_2'\right)\le\\
&c\varphi_2\begin{pmatrix}
d_2\left(x_2,x_2'\right), d_2\left(x_2,T_1T_m\dots T_2x_2\right), d_2\left(x_2',T_1T_m\dots T_2x_2'\right),\\
d_3\left(T_2x_2,T_2x_2'\right), d_4\left(T_3T_2x_2,T_3T_2x_2'\right),\dots,\\
d_m\left(T_{m-1}\dots T_2x_2, T_{m-1}\dots T_2x_2'\right),\\
d_1\left(T_m\dots T_2x_2, T_m\dots T_2x_2'\right)
\end{pmatrix},
\end{aligned}
$$

$$\dots$$

$$
(i)\qquad
\begin{aligned}
&d_i\left(T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_i, T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_i'\right)\le\\
&c\varphi_i\begin{pmatrix}
d_i\left(x_i,x_i'\right), d_i\left(x_i,T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_i\right),\\
d_i\left(x_i',T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_i'\right), d_{i+1}\left(T_ix_i,T_ix_i'\right),\\
d_{i+2}\left(T_{i+1}T_ix_i,T_{i+1}T_ix_i'\right),\dots,\\
d_m\left(T_{m-1}T_{m-2}\dots T_ix_i, T_{m-1}T_{m-2}\dots T_ix_i'\right),\\
d_1\left(T_mT_{m-1}\dots T_ix_i, T_mT_{m-1}\dots T_ix_i'\right),\\
d_2\left(T_1T_mT_{m-1}\dots T_ix_i, T_1T_mT_{m-1}\dots T_ix_i'\right),\dots,\\
d_{i-1}\left(T_{i-2}\dots T_1T_m\dots T_ix_i, T_{i-2}\dots T_1T_m\dots T_ix_i'\right)
\end{pmatrix},
\end{aligned}
$$

$$\dots$$

$$
(m)\qquad
\begin{aligned}
&d_m\left(T_{m-1}T_{m-2}\dots T_1T_mx_m, T_{m-1}T_{m-2}\dots T_1T_mx_m'\right)\le\\
&\varphi_m\begin{pmatrix}
d_m\left(x_m,x_m'\right), d_m\left(x_m,T_{m-1}T_{m-2}\dots T_1T_mx_m\right),\\
d_m\left(x_m',T_{m-1}T_{m-2}\dots T_1T_mx_m'\right), d_1\left(T_mx_m,T_mx_m'\right),\\
d_2\left(T_1T_mx_m,T_1T_mx_m'\right),\dots,\\
d_{m-1}\left(T_{m-2}T_{m-3}\dots T_1T_mx_m, T_{m-2}T_{m-3}\dots T_1T_mx_m'\right)
\end{pmatrix}
\end{aligned}
$$

for all $x_i, x_i' \in X_i$ and $\varphi_i \in \Phi_{m+2}$ *for* $i = 1, 2, \dots, m$, *then the maps:* $T_mT_{m-1}\dots T_1$, $T_1T_m\dots T_2,\dots$, $T_{i-1}T_{i-2}\dots T_1T_m\dots T_i,\dots$, $T_{m-1}\dots T_1T_m$ *have unique fixed point* $\alpha_1 \in X_1$, $\alpha_2 \in X_2,\dots$, $\alpha_i \in X_i,\dots$, $\alpha_m \in X_m$, *respectively. Further,* $T_i\alpha_i = \alpha_{i+1}$ *for* $i = 1,\dots,m-1$ *and* $T_m\alpha_m = \alpha_1$.

Proof. Let $x_0^{(1)}$ be an arbitrary point in X_1. We define the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\},\dots,\{x_n^{(i)}\},\dots,\{x_n^{(m)}\}$ in $X_1, X_2,\dots,X_i,\dots,X_m$ respectively, as follows: $x_n^{(1)} = (T_mT_{m-1}\dots T_1)^n x_0^{(1)}$, $x_n^{(2)} = T_1x_{n-1}^{(1)}$, $\dots$, $x_n^{(i)} = T_{i-1}x_n^{(i-1)}$, $\dots$, $x_n^{(m)} = T_{m-1}x_n^{(m-1)}$, $n \in N$.

We prove that $\{x_n^{(i)}\}$ are Cauchy sequences for $i = 1, 2,\dots,m$. Denote

$$d_n^{(i)} = d_i(x_n^{(i)}, x_{n+1}^{(i)}), i = 1, 2,\dots,m.$$

We will assume that $x_n^{(i)} \neq x_{n+1}^{(i)}$ for all n, otherwise if $x_n^{(i)} = x_{n+1}^{(i)}$ for some n, we could put $\alpha_i = x_n^{(i)}$.

Applying the inequality (2) for $x_2 = x_{n-1}^{(2)}$ and $x_2' = x_n^{(2)}$, we have:

$$\begin{aligned} d_n^{(2)} &= d_2(x_n^{(2)}, x_{n+1}^{(2)}) = d_2\left(T_1T_m\dots T_3T_2x_{n-1}^{(2)}, T_1T_m\dots T_2x_n^{(2)}\right) \\ &\le c\phi_2 \begin{pmatrix} d_2\left(x_{n-1}^{(2)}, x_n^{(2)}\right), d_2\left(x_{n-1}^{(2)}, T_1T_m\dots T_2x_{n-1}^{(2)}\right), \\ d_2\left(x_n^{(2)}, T_1T_m\dots T_2x_n^{(2)}\right), \\ d_3\left(T_2x_{n-1}^{(2)}, T_2x_n^{(2)}\right), d_4\left(T_3T_2x_{n-1}^{(2)}, T_3T_2x_n^{(2)}\right), \dots, \\ d_m\left(T_{m-1}T_{m-2}\dots T_2x_{n-1}^{(2)}, T_{m-1}T_{m-2}\dots T_2x_n^{(2)}\right), \\ d_1\left(T_mT_{m-1}\dots T_2x_{n-1}^{(2)}, T_mT_{m-1}\dots T_2x_n^{(2)}\right) \end{pmatrix} \\ &= c\phi_2\left(d_{n-1}^{(2)}, d_{n-1}^{(2)}, d_n^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, \dots, d_{n-1}^{(m)}, d_{n-1}^{(1)}\right). \end{aligned}$$

Between the coordinates in the left side of the inequality, $d_n^{(2)}$ can not be the greatest, because in contrary, applying the properties of φ we will have the inequality $d_n^{(2)} \le cd_n^{(2)}$ from which it follows $x_n^{(2)} = x_{n+1}^{(2)}$, which contradicts the assumption. Then applying the properties of φ replacing the coordinates with the greatest we have:

(2′) $\quad d_n^{(2)} \le c\max\left\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, \dots, d_{n-1}^{(m)}\right\}$.

Applying inequality (i) for $x_i = x_{n-1}^{(i)}$ and $x_i' = x_n^{(i)}$, we obtain:

$$\begin{aligned} d_n^{(i)} &= d_i(x_n^{(i)}, x_{n+1}^{(i)}) \\ &= d_i\left(T_{i-1}T_{i-2}\dots T_1T_m\dots T_ix_{n-1}^{(i)}, T_{i-1}T_{i-2}\dots T_1T_m\dots T_ix_n^{(i)}\right) \\ &\le c\phi_i \begin{pmatrix} d_i\left(x_{n-1}^{(i)}, x_n^{(i)}\right), d_i\left(x_{n-1}^{(i)}, T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_{n-1}^{(i)}\right), \\ d_i\left(x_n^{(i)}, T_{i-1}T_{i-2}\dots T_1T_mT_{m-1}\dots T_ix_n^{(i)}\right), \\ d_{i+1}\left(T_ix_{n-1}^{(i)}, T_ix_n^{(i)}\right), d_{i+2}\left(T_{i+1}T_ix_{n-1}^{(i)}, T_{i+1}T_ix_n^{(i)}\right), \dots, \\ d_m\left(T_{m-1}T_{m-2}\dots T_ix_{n-1}^{(i)}, T_{m-1}T_{m-2}\dots T_ix_n^{(i)}\right), \\ d_1\left(T_mT_{m-1}\dots T_ix_{n-1}^{(i)}, T_mT_{m-1}\dots T_ix_n^{(i)}\right), \\ d_2\left(T_1T_mT_{m-1}\dots T_ix_{n-1}^{(i)}, T_1T_mT_{m-1}\dots T_ix_n^{(i)}\right), \dots, \\ d_{i-1}\left(T_{i-2}T_{i-3}\dots T_1T_m\dots T_ix_{n-1}^{(i)}, T_{i-2}\dots T_1T_m\dots T_ix_n^{(i)}\right) \end{pmatrix} \\ &= c\varphi_i(d_{n-1}^{(i)}, d_{n-1}^{(i)}, d_n^{(i)}, d_{n-1}^{(i+1)}, d_{n-1}^{(i+2)}, \dots, d_{n-1}^{(m)}, d_{n-1}^{(1)}, d_n^{(2)}, d_n^{(3)}, \dots, d_n^{(i-1)}). \end{aligned}$$

Since the function φ_i satisfies the properties (a) and (b), we will have:

$d_n^i \le c\max(d_{n-1}^{(i)}, d_{n-1}^{(i+1)}, d_{n-1}^{(i+2)}, \dots, d_{n-1}^{(m)}, d_{n-1}^{(1)}, d_n^{(2)}, d_n^{(3)}, \dots, d_n^{(i-1)}).$ $\qquad$ (*)

By (*), for $i=3$ we have $d_n^3 \le c\max(d_{n-1}^{(3)}, d_{n-1}^{(4)}, \dots, d_{n-1}^{(m)}, d_{n-1}^{(1)}, d_n^{(2)})$.
By this inequality and (2') it follows:
$(3')\quad d_n^{(3)} \le c\max\left\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, \dots, d_{n-1}^{(m)}\right\}$.
In similar way, for $i=4,5,\dots,m-1$ and by the inequalities $(2')$, $(3')$,..., $((i-1)')$ we get:
(i) $\quad d_n^{(i)} \le c\max\left\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, \dots, d_{n-1}^{(m)}\right\}\ \ i=2,3,\dots,m-1.$
Applying inequality (m) for $x_m = x_{n-1}^{(m)}$ and $x'_m = x_n^{(m)}$, we have:

$$\begin{aligned}
d_n^{(m)} &= d_m(x_n^{(m)}, x_{n+1}^{(m)})\\
&= d_m\left(T_{m-1}T_{m-2}\dots T_1T_mx_{n-1}^{(m)}, T_{m-1}T_{m-2}\dots T_1T_mx_n^{(m)}\right)\\
&\le c\phi_m\left(\begin{array}{l}
d_m(x_{n-1}^{(m)}, x_n^{(m)}), d_m\left(x_{n-1}^{(m)}, T_{m-1}T_{m-2}\dots T_1T_mx_{n-1}^{(m)}\right),\\
d_m\left(x_n^{(m)}, T_{m-1}T_{m-2}\dots T_1T_mx_n^{(m)}\right),\\
d_1\left(T_mx_{n-1}^{(m)}, T_mx_n^{(m)}\right), d_2\left(T_1T_mx_{n-1}^{(m)}, T_1T_mx_n^{(m)}\right), \dots,\\
d_{m-1}\left(T_{m-2}T_{m-3}\dots T_1T_mx_{n-1}^{(m)}, T_{m-2}T_{m-3}\dots T_1T_mx_n^{(m)}\right)
\end{array}\right)\\
&= c\varphi_m(d_{n-1}^{(m)}, d_{n-1}^{(m)}, d_n^{(m)}, d_{n-1}^{(1)}, d_n^{(2)}, d_n^{(3)}, \dots, d_n^{(m-1)}).
\end{aligned}$$

By the properties of φ_m we have

$$d_n^m \le c\max(d_{n-1}^{(m)}, d_{n-1}^{(1)}, d_n^{(2)}, d_n^{(3)}, \dots, d_n^{(m-1)}).$$

By this inequality and by $(2')$, $(3')$, ..., $((m-1)')$ it follows:
$(m')\quad d_n^{(m)} \le c\max(d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, \dots, d_{n-1}^{(m)})$.
Applying inequality (1) for $x_1 = x_{n-1}^{(1)}$ and $x'_1 = x_n^{(1)}$, we have:

$$\begin{aligned}
d_n^{(1)} &= d_1(x_n^{(1)}, x_{n+1}^{(1)}) = d_1\left(T_mT_{m-1}\dots T_2T_1x_{n-1}^{(1)}, T_mT_{m-1}\dots T_2T_1x_n^{(1)}\right)\\
&\le c\phi_1\left(\begin{array}{l}
d_1\left(x_{n-1}^{(1)}, x_n^{(1)}\right), d_1\left(x_{n-1}^{(1)}, T_mT_{m-1}\dots T_1x_{n-1}^{(1)}\right),\\
d_1\left(x_n^{(1)}, T_mT_{m-1}\dots T_1x_n^{(1)}\right),\\
d_2\left(T_1x_{n-1}^{(1)}, T_1x_n^{(1)}\right),\ \mathrm{d}_3\left(T_2T_1x_{n-1}^{(1)}, T_2T_1x_n^{(1)}\right), \dots,\\
d_m\left(T_{m-1}T_{m-2}\dots T_1x_n^{(1)}, T_{m-1}T_{m-2}\dots T_1x_n^{(1)}\right)
\end{array}\right) =\\
&= c\phi_1(d_{n-1}^{(1)}, d_{n-1}^{(1)}, d_n^{(1)}, d_n^{(2)}, d_n^{(3)}, \dots, d_n^{(m)}).
\end{aligned}$$

By the properties of φ_1 we have:

$$d_n^{(1)} \le c\max\{d_{n-1}^{(1)}, d_n^{(1)}, d_n^{(1)}, d_n^{(1)}, \dots, d_n^{(1)}\}.$$

By this inequality and by (2'),(3'),...,(m') it follows:
$(1')\quad d_n^{(1)} \le c\max\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, \dots, d_{n-1}^{(m)}\}$.

It now follows from (1'),(2'),...,(m') that for large enough n

$$\begin{aligned} d_n^{(i)} &= d_i(x_n^{(i)}, x_{n+1}^{(i)}) \leq c\max\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, \ldots, d_{n-1}^{(m)}\} \\ &\leq c^2 \max\{d_{n-2}^{(1)}, d_{n-2}^{(2)}, d_{n-2}^{(3)}, \ldots, d_{n-2}^{(m)}\} \\ &\leq c^3 \max\{d_{n-3}^{(1)}, d_{n-3}^{(2)}, d_{n-3}^{(3)}, \ldots, d_{n-3}^{(m)}\} \\ &\ldots \\ &\leq c^{n-1} \max\{d_1^{(1)}, d_1^{(2)}, d_1^{(3)}, \ldots, d_1^{(m)}\} = c^{n-1} \cdot l, \end{aligned}$$

where $l = \max\{d_1^{(1)}, d_1^{(2)}, d_1^{(3)}, \ldots, d_1^{(m)}\}$.

Let us prove that the sequences $\{x_n^{(i)}\}$ are Cauchy sequences:

$$\begin{aligned} d_i(x_n^{(i)}, x_{n+p}^{(i)}) &\leq k_i[d_i(x_n^{(i)}, x_{n+1}^{(i)}) + d_i(x_{n+1}^{(i)}, x_{n+p}^{(i)})] \\ &\leq k_i d_i(x_n^{(i)}, x_{n+1}^{(i)}) + k_i^2[d_i(x_{n+1}^{(i)}, x_{n+2}^{(i)}) + d_i(x_{n+2}^{(i)}, x_{n+p}^{(i)}))] \\ &\leq k_i d_i(x_n^{(i)}, x_{n+1}^{(i)}) + k_i^2 d_i(x_{n+1}^{(i)}, x_{n+2}^{(i)}) + k_i^3 d_i(x_{n+2}^{(i)}, x_{n+3}^{(i)}) \\ &\quad + \cdots + k_i^{p-1}[d_i(x_{n+p-2}^{(i)}, x_{n+p-1}^{(i)}) + d_i(x_{n+p-1}^{(i)}, x_{n+p}^{(i)})] \\ &\leq k_i c^{n-1} l + k_i^2 c^n l + \cdots + k_i^{p-1} c^{n+p-3} l + k_i^{p-1} c^{n+p-2} l \\ &\leq k_i c^{n-1} l[1 + (k_i c) + (k_i c)^2 + \cdots + (k_i c)^{p-1}] = \\ &= k_i c^{n-1} l \frac{1-(k_i c)^p}{1-k_i c} < k_i c^{n-1} \frac{l}{1-k_i c}. \end{aligned}$$

Since $k_i c < 1$, letting n tend to infinity, we have: $\lim\limits_{n\to\infty} d_i(x_n^{(i)}, x_{n+p}^{(i)}) = 0$. So, $\{x_n^{(i)}\}$ is a Cauchy sequence for $c < \frac{1}{k}$, where $k = \max\{k_i : i = 1, 2, \ldots, m\}$. Since the quasi metric spaces are completes we have: $\lim\limits_{n\to\infty} x_n^{(i)} = \alpha_i \in X_i$, for $i = 1, 2, \ldots, m$.

Now suppose that T_i for $i = 1, 2, \ldots, m-1$ are continuous. We have $\lim\limits_{n\to\infty} x_n^{(2)} = \lim\limits_{n\to\infty} T_1 x_{n-1}^{(1)} \Rightarrow T_1\alpha_1 = \alpha_2$ and $\lim\limits_{n\to\infty} x_{n+1}^{(i+1)} = \lim\limits_{n\to\infty} T_i x_n^{(i)} \Rightarrow T_i\alpha_i = \alpha_{i+1}$, for $i = 2, 3, \ldots, m-1$.

Later we will show that $T_m\alpha_m = \alpha_1$. To prove that α_1 is a fixed point of $T_m T_{m-1} \ldots T_1$. Using the inequality (1) for $x_1 = \alpha_1$ and $x_1^{'} = x_{n-1}^{(1)}$, we obtain:

$$d_1\left(T_m T_{m-1}\ldots T_1\alpha_1, x_n^{(1)}\right) = d_1\left(T_m T_{m-1}\ldots T_1\alpha_1, T_m T_{m-1}\ldots T_1 x_{n-1}^{(1)}\right)$$

$$\leq c\varphi_1 \begin{pmatrix} d_1\left(\alpha_1, x_{n-1}^{(1)}\right), d_1\left(\alpha_1, T_m T_{m-1}\ldots T_1\alpha_1\right), d_1\left(x_{n-1}^{(1)}, x_n^{(1)}\right), \\ d_2\left(T_1\alpha_1, T_1 x_{n-1}^{(1)}\right),\ \mathrm{d}_3\left(T_2 T_1\alpha_1, T_2 T_1 x_{n-1}^{(1)}\right), \ldots, \\ d_m\left(T_{m-1}T_{m-2}\ldots T_1\alpha_1, T_{m-1}T_{m-2}\ldots T_1 x_{n-1}^{(1)}\right) \end{pmatrix}.$$

Since d_i, T_i for $i = 1, 2, \ldots, m-1$ are continuous and φ_1 is upper semi-continuous, letting n tend to infinity, we have

$$d_1\left(T_m \ldots T_2T_1\alpha_1, \alpha_1\right) \leq c\varphi_1 \begin{pmatrix} d_1\left(\alpha_1, \alpha_1\right), d_1\left(\alpha_1, T_mT_{m-1} \ldots T_1\alpha_1\right), \\ d_1\left(\alpha_1, \alpha_1\right), d_2\left(T_1\alpha_1, T_1\alpha_1\right), \\ \mathrm{d}_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1\right), \ldots, \\ d_m\left(T_{m-1} \ldots T_1\alpha_1, T_{m-1} \ldots T_1\alpha_1\right) \end{pmatrix}$$
$$= c\varphi_1\left(0, d_1\left(\alpha_1, T_m \ldots T_1\alpha_1\right), 0, 0, \ldots, 0\ \right) \leq cd_1\left(\alpha_1, T_m \ldots T_1\alpha_1\right).$$

Thus $T_mT_{m-1} \ldots T_1\alpha_1 = \alpha_1$, since $0 \leq$ c<1 and so α_1 is a fixed point of $T_mT_{m-1} \ldots T_1$.

We now have $T_1T_mT_{m-1} \ldots T_2\alpha_2 = T_1T_mT_{m-1} \ldots T_2T_1\alpha_1 = T_1\alpha_1 = \alpha_2$ and in general

$$\begin{aligned} T_{i-1}T_{i-2} \ldots T_1T_mT_{m-1} \ldots T_i\alpha_i &= T_{i-1}(T_{i-2} \ldots T_1T_mT_{m-1} \ldots T_iT_{i-1}\alpha_{i-1}) \\ &= T_{i-1}\alpha_{i-1} = \alpha_i, i = 2, 3, \ldots, m. \end{aligned}$$

Hence α_i are fixed points of $T_{i-1}T_{i-2} \ldots T_1T_mT_{m-1} \ldots T_i$, $i = 2, 3, \ldots, m$.

We now prove the uniqueness of the fixed point α_i. Let us prove for α_1. Suppose that $T_mT_{m-1} \ldots T_1$ has a second fixed point $\alpha_1^{'} \neq \alpha_1$. Using the inequality (1) for $x_1 = \alpha_1$ and $x_1^{'} = \alpha_1^{'}$ we have:

$$d_1\left(\alpha_1, \alpha_1^{'}\right) = d_1\left(T_m \ldots T_2T_1\alpha_1, T_m \ldots T_2T_1\alpha_1^{'}\right) \leq$$
$$\leq c\varphi_1 \begin{pmatrix} d_1(\alpha_1, \alpha_1^{'}), d_1\left(\alpha_1, T_m \ldots T_1\alpha_1\right), d_1\left(\alpha_1^{'}, T_m \ldots T_1\alpha_1^{'}\right), \\ d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right),\ \mathrm{d}_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), \ldots, \\ d_m\left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'}\right) \end{pmatrix}$$
$$= c\varphi_1 \begin{pmatrix} d_1(\alpha_1, \alpha_1^{'}), 0, 0, d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right), \mathrm{d}_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), \ldots, \\ d_m\left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'}\right) \end{pmatrix}.$$

Between the coordinates in the left side of inequality, $d_1\left(\alpha_1, \alpha_1^{'}\right)$ can not be the greatest. In contrary, applying the properties of φ we will get the inequality $d_1\left(\alpha_1, \alpha_1^{'}\right) \leq cd_1\left(\alpha_1, \alpha_1^{'}\right)$ from which it follows $\alpha_1 = \alpha_1^{'}$, which contradicts the assumption.

Applying the properties of φ replacing the coordinates with the greatest we have:

$$(1'') \quad d_1(\alpha_1, \alpha_1^{'}) \leq c \max \left\{ \begin{array}{l} d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right), \mathrm{d}_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), \ldots, \\ d_m\left(T_{m-1} \ldots T_1\alpha_1, T_{m-1} \ldots T_1\alpha_1^{'}\right) \end{array} \right\}.$$

Further, using inequality (2) for $x_2 = T_1\alpha_1$ and $x_2^{'} = T_1\alpha_1^{'}$, we have

$$d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right) = d_2\left(T_1T_m\dots T_3T_2T_1\alpha_1, T_1T_m\dots T_3T_2T_1\alpha_1^{'}\right)$$

$$\leq c\varphi_2\left(\begin{array}{l} d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right), d_2\left(T_1\alpha_1, T_1T_m\dots T_2T_1\alpha_1\right), \\ d_2\left(T_1\alpha_1^{'}, T_1T_m\dots T_2T_1\alpha_1^{'}\right), d_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), \\ d_4\left(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1^{'}\right), \dots, \\ d_m\left(T_{m-1}T_{m-2}\dots T_2T_1\alpha_1, T_{m-1}T_{m-2}\dots T_2T_1\alpha_1^{'}\right), \\ d_1\left(T_mT_{m-1}\dots T_2T_1\alpha_1, T_mT_{m-1}\dots T_2T_1\alpha_1^{'}\right) \end{array}\right) =$$

$$= c\varphi_2\left\{\begin{array}{l} d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right), 0, 0, d_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), \\ d_4\left(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1^{'}\right), \dots, \\ d_m\left(T_{m-1}\dots T_2T_1\alpha_1, T_{m-1}\dots T_2T_1\alpha_1^{'}\right), d_1\left(\alpha_1, \alpha_1^{'}\right) \end{array}\right\}.$$

In similar way and taking in consideration $(1'')$ we obtain:

$$(2'') \qquad d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right)$$
$$\leq c\max\left\{\begin{array}{l} d_3\left(T_2T_1\alpha_1, T_2T_1\alpha_1^{'}\right), d_4\left(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1^{'}\right), \\ \dots \\ d_m\left(T_{m-1}\dots T_2T_1\alpha_1, T_{m-1}\dots T_2T_1\alpha_1^{'}\right), \end{array}\right\}.$$

Similarly, applying the inequality (i) for $x_i = T_{i-1}T_{i-2}\dots T_2T_1\alpha_1$ and $x_i^{'} = T_{i-1}T_{i-2}\dots T_2T_1\alpha_1^{'}$ and using these inequalities $(1''), (2''), \dots, ((i-1)'')$, we have

$$d_i\left(T_{i-1}T_{i-2}\dots T_1\alpha_1, T_{i-1}T_{i-2}\dots T_1\alpha_1^{'}\right) \leq$$

$$(i'') \qquad \leq c\max\left(\begin{array}{l} d_{i+1}\left(T_iT_{i-1}\dots T_1\alpha_1, T_iT_{i-1}\dots T_1\alpha_1^{'}\right), \\ d_{i+2}\left(T_{i+1}T_i\dots T_1\alpha_1, T_{i+1}T_i\dots T_1\alpha_1^{'}\right), \dots, \\ d_m\left(T_{m-1}\dots T_1\alpha_1, T_{m-1}T_{m-2}\dots T_i\alpha_1^{'}\right), d_1\left(\alpha_1, \alpha_1^{'}\right), \\ d_2\left(T_1\alpha_1, T_1\alpha_1^{'}\right), \dots, d_{i-1}\left(T_{i-2}\dots T_1\alpha_1, T_{i-2}\dots T_1\alpha_1^{'}\right) \end{array}\right)$$

$$= c\max\left(\begin{array}{l} d_{i+1}\left(T_iT_{i-1}\dots T_1\alpha_1, T_iT_{i-1}\dots T_1\alpha_1^{'}\right), \\ d_{i+2}\left(T_{i+1}T_i\dots T_1\alpha_1, T_{i+1}T_i\dots T_1\alpha_1^{'}\right), \dots, \\ d_m\left(T_{m-1}\dots T_1\alpha_1, T_{m-1}\dots T_i\alpha_1^{'}\right) \end{array}\right).$$

By (i") for $i = m-1$ we get:

$$((m-1)'') \quad d_{m-1}\left(T_{m-2}T_{m-3}\dots T_1\alpha_1, T_{m-2}T_{m-3}\dots T_1\alpha_1^{'}\right)$$
$$\leq cd_m\left(T_{m-1}T_{m-2}\dots T_1\alpha_1, T_{m-1}T_{m-2}\dots T_1\alpha_1^{'}\right).$$

Applying the inequality (m), for $x_m = T_{m-1} \ldots T_1\alpha_1$, $x_m^{'} = T_{m-1} \ldots T_1\alpha_1^{'}$ and using the properties (a) and (b) of φ_m, we have

$$d_m \left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'} \right)$$
$$\leq c \max \left(\begin{array}{l} d_1 \left(\alpha_1, \alpha_1^{'} \right), d_2 \left(T_1\alpha_1, T_1\alpha_1^{'} \right), \ldots, \\ d_{m-1} \left(T_{m-2}T_{m-3} \ldots T_1\alpha_1, T_{m-2}T_{m-3} \ldots T_1\alpha_1^{'} \right) \end{array} \right).$$

In the above inequality, on using these inequalities $((m-1)'')$,$((m-2)'')$, $\ldots$,$(1'')$ we now have

(m'') $$d_m \left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'} \right)$$
$$\leq c d_m \left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'} \right)$$

and so

$$d_m \left(T_{m-1}T_{m-2} \ldots T_1\alpha_1, T_{m-1}T_{m-2} \ldots T_1\alpha_1^{'} \right) = 0.$$

Returning back we get:

$$d_{m-1} \left(T_{m-2}T_{m-3} \ldots T_1\alpha_1, T_{m-2}T_{m-3} \ldots T_1\alpha_1^{'} \right) = 0,$$
$$d_{m-2} \left(T_{m-3}T_{m-4} \ldots T_1\alpha_1, T_{m-3}T_{m-4} \ldots T_1\alpha_1^{'} \right) = 0,$$
$$\ldots$$
$$d_2 \left(T_1\alpha_1, T_1\alpha_1^{'} \right) = 0,$$
$$d_1 \left(\alpha_1, \alpha_1^{'} \right) = 0 \Leftrightarrow \alpha_1 = \alpha_1^{'}.$$

The fixed point α_1 of $T_mT_{m-1} \ldots T_1$ must therefore be unique.

Similarly, it can be proved that α_i is the unique fixed point of $T_{i-1}T_{i-2} \ldots T_1T_mT_{m-1} \ldots T_i$, for $i = 2, 3, \ldots, m$.

We finally prove that also have $T_m\alpha_m = \alpha_1$. To do this, note that

$$T_m\alpha_m = T_m(T_{m-1}T_{m-2} \ldots T_1T_m\alpha_m) = T_mT_{m-1}T_{m-2} \ldots T_1(T_m\alpha_m)$$

and so $T_m\alpha_m$ is a fixed point of $T_mT_{m-1}T_{m-2} \ldots T_1$. Since α_m is the unique fixed point, it follows that $T_m\alpha_m = \alpha_1$. This completes the proof of the theorem. □

3. COROLLARIES

The next corollary follows from Theorem 1 in the case $m = 1$, $T_1 = T$ and $k_1 = k$.

Corollary 1. *Let (X, d) be a complete quasi-metric space with constant k and continuous quasi-distance d. Let $T : X \to X$ a self map of X. If for some $c \in [0, \frac{1}{k})$ the following inequality holds*

$$d(Tx, Ty) \leq c\varphi\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$ *and* $\varphi \in \Phi_3$*, then* T *has a unique fixed point* α *in* X*.*

For different expressions of φ in first corollary we get different theorems:

In the cases: $\varphi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$, $\varphi(t_1, t_2, t_3) = t_1$, $\varphi(t_1, t_2, t_3) = \frac{t_2+t_3}{2}$, $\varphi(t_1, t_2, t_3) = \min\{t_2, t_3\}$ and $\varphi(t_1, t_2, t_3) = \frac{at_1+bt_2+ct_3}{a+b+c}$ we obtain, respectively, the extensions of Rhoades's Theorem [11], Banach's Theorem [1], Kannan's Theorem [6], Bianchini's Theorem [2] and Reich's Theorem [10] to quasi-metric spaces.

The next corollary follows from Theorem 1 in the case $m = 2$, $T_1 = S$ and $T_2 = R$.

Corollary 2. *Let* (X, d) *and* (Y, ρ) *are complete quasi-metric spaces with constants* k_1, k_2 *and continuous quasi-distances* d, ρ *respectively. Let* $S : X \to Y$, $R : Y \to X$ *be two maps, at least one of them being continuous. If for some* $c \in [0, \frac{1}{k})$, $k = \max\{k_1, k_2\}$*, the following inequalities are satisfied:*

$$\begin{aligned} d\left(RSx, RSx'\right) &\leq c\varphi_1(d\left(x, x'\right), d\left(x, RSx\right), d\left(x', RSx'\right), \rho\left(Sx, Sx'\right)), \\ \rho\left(SRy, SRy'\right) &\leq c\varphi_2(\rho\left(y, y'\right), \rho\left(y, SRy\right), \rho\left(y', SRy'\right), d\left(Ry, Ry'\right)) \end{aligned}$$

for all $x, x' \in X, y, y' \in Y$ *and* $\varphi_1, \varphi_2 \in \Phi_4$*, then* RS *has a unique fixed point* $\alpha \in X$ *and* SR *has a unique fixed point* $\beta \in Y$*. Moreover,* $S\alpha = \beta$ *and* $R\beta = \alpha$*.*

For different expressions of φ_1 and φ_2 in Corollary 2 we get different theorems:

If $\varphi_1 = \varphi_2 = \varphi$, where $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, we obtain an extension of Fisher's Theorem [4] to quasi-metric spaces; and in the cases $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3\}$, $\varphi(t_1, t_2, t_3, t_4) = \frac{t_2+t_3+t_4}{3}$ and

$$\varphi(t_1, t_2, t_3, t_4) = \frac{at_1 + bt_2 + ct_3 + ct_4}{a + b + c + d}$$

we obtain, respectively, the extensions of Bianchini's Theorem [2], Kannan's Theorem [6] and Reich's Theorem [10] from one metric space to two quasi-metric spaces.

In the same way, we are concluded for the cases $m = 3, 4, \ldots$

Corollary 3. *From Theorem 1, in the case* $m = 3$ *and* $\varphi_1, \varphi_2, \varphi_3 \in \Phi_5$*, we obtain a generalization and extension of Jain's et al. Theorem* [5] *from three metric spaces to three quasi-metric spaces.*

For different expressions of φ_1, φ_2 and φ_3 we have different corollaries, for example the corollaries which extend the Theorems of Kannan [6], Bianchini [2] and Reich [10], from one metric space to three quasi-metric spaces.

At last, we would like to emphasize the fact that all the above corollaries hold not only for 1, 2 and 3 quasi-metric spaces, but also for an arbitrary number of metric spaces.

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