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LAYER POTENTIAL ANALYSIS OF A NEUMANN PROBLEM FOR THE BRINKMAN SYSTEM

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Abstract. In this paper we obtain the existence and uniqueness result (up to a constant pressure) in some Sobolev spaces for a Neumann problem associated to the Brinkman system on Lipschitz domains in compact boundaryless Riemannian manifolds. In order to obtain the desired result, we use an indirect boundary integral formulation based on the potential theory for the Brinkman system.

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Key words. Brinkman system, existence and uniqueness, Lipschitz domain, layer potential theory, boundary integral equations, Sobolev spaces.

1. INTRODUCTION

Recently, various boundary value problems for elliptic operators on smooth or even Lipschitz domains have been studied by using potential theory. The importance of this subject is already well known (see, e.g., [1], [5] [12], [13], [14]). Note that a valuable contribution in the study of the Dirichlet problem for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$ has been provided by Fabes, Kenig and Verchota in [4]. Well-posedness results for the main boundary value problems associated to the Stokes system on Lipschitz domains in Euclidean setting, with the boundary data in various function spaces, have been obtained by Mitrea and Wright [18]. Mitrea and Taylor [17] used the layer potential theory to study the Poisson problem for the Navier-Stokes equations on arbitrary Lipschitz domains in compact Riemannian manifolds and with boundary data in Sobolev or Besov spaces. They have also developed the layer potential theory for elliptic operators on Lipschitz domains in compact Riemannian manifolds and studied related boundary value problems on such domains (see e.g., [15]). The Poisson problem for the Stokes system on C^1 or, more generally, on Lipschitz domains in a smooth compact Riemannian manifold and with data in Sobolev or Besov spaces has been studied by Dindoš and Mitrea in [2], by using a layer potential approach. In addition, they treated the Poisson problem for the stationary, nonlinear Navier-Stokes equations on Riemannian manifolds. Recently, Kohr, Pintea and Wendland [6]-[11] used layer potential methods to study boundary value problems (including transmission problems) for pseudodifferential Brinkman operators on

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Lipschitz domains in compact Riemannian manifolds, with the given boundary data in L^p, Sobolev or Besov spaces. Note that the pseudodifferential Brinkman operator is an extension of the Brinkman operator from the Euclidean setting to the case of the compact Riemanian manifolds and has been introduced in [7].

The purpose of this paper is to show the existence and uniqueness (up to a constant pressure) for a Neumann problem associated to the Brinkman system on Lipschitz domains in compact Riemannian manifolds.

2. PRELIMINARIES

We consider a compact boundaryless manifold (M, g) of dimension $m \geq 2$ equipped with a smooth Riemannian metric tensor $g = \sum_{j,k=1}^{m} g_{jk} dx_j \otimes dx_k =: g_{jk} dx_j \otimes dx_k$, and let (g^{jk}) be the inverse of (g_{jk}). Let us mention that the volume element on M is given by $\mathrm{dVol} = \sqrt{g} dx_1 \dots dx_m$, where $g := \det(g_{jk})$. The tangent and cotangent bundles are $TM = \bigcup_{p \in M} T_p M$ and $T^*M = \bigcup_{p \in M} T_p^* M$, respectively. By $\mathcal{X}(M)$ we denote the space of smooth vector fields on M, i.e., the space $C^\infty(M, TM)$ of smooth sections of TM. In a natural way we can identify T^*M with TM and $\Lambda^1 TM$ with $\mathcal{X}(M)$. Next, we define following inner product on $\Lambda^1 TM$:

$$(2.1) \qquad \langle dx_j, dx_k \rangle = g^{jk}, \quad \langle X, Y \rangle = X_j g^{jk} Y_k,$$

where the vector field $X = X^k \partial_k \in TM$ is identified with the one form $X_r dx_r = X^k g_{kr} dx_r$, $X_r = g_{kr} X^k$, and the notation $\langle \cdot, \cdot \rangle$ is used for the inner product. Consequently, the gradient operator grad : $C^\infty(M) \to \mathcal{X}(M)$ identifies the exterior derivative operator $d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM)$, $d = \partial_j dx_j$. On the other hand, $-\mathrm{div} : \mathcal{X}(M) \to C^\infty(M)$ is identified with the exterior co-derivative operator $\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M)$, $\delta = d^*$. Next, assume that $X \in \mathcal{X}(M)$. Then, the symmetric part of the tensor field $\nabla X : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M, TM \otimes TM)$, $(\nabla X)(Y, Z) = \langle \nabla_Y X, Z \rangle$, where by ∇ we denote the Levi-Civita connection on M, is the *deformation* of X, denoted by Def X. Thus,

$$(2.2) \qquad (\mathrm{Def}\ X)(Y, Z) = \frac{1}{2}\{\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle\}, \quad \forall\, Y, Z \in \mathcal{X}(M).$$

A *Killing field* is a vector field $X \in \mathcal{X}(M)$, which satisfies the equation Def $X = 0$ on M. All along this paper, we assume that *the manifold M has no nontrivial Killing fields* (for more details such manifolds see [17]). Note that the Killing fields in $\mathbb{R}^n$ are the usual rigid body motion fields.

Let us consider the second-order partial differential operator

$$(2.3) \qquad L : \mathcal{X}(M) \to \mathcal{X}(M), \quad L := 2\mathrm{Def}^*\mathrm{Def} = -\triangle + \mathrm{d}\delta - 2\mathrm{Ric},$$

where Def^* is the adjoint of Def, $\triangle := -(\mathrm{d}\delta + \delta d)$ is the Hodge Laplacian and Ric is the Ricci tensor. Note that L is the natural operator for the Stokes system on an arbitrary Riemannian manifold (cf. [3]). Next, by $OPS^{\ell}_{\rm cl}$ one denotes the class of classical pseudodifferential operators of order ℓ (for details the interested reader can consult e.g. [19], [20]). Let $P \in OPS^0_{\rm cl}(\Lambda^1TM, \Lambda^1TM)$ be a self-adjoint and non-negative operator with respect to the $L^2(M, \Lambda^1TM)$ - inner product $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle Pu, w\rangle = \langle u, Pw\rangle, \quad \langle Pu, u\rangle \geq 0 \ \text{ for all } \ u, w \in L^2(M, \Lambda^1TM). \tag{2.4}$$

Then the pseudodifferential Brinkman operator on M is given by (see [7])

$$B_P := \begin{pmatrix} L+P & d \\ \delta & 0 \end{pmatrix} : C^\infty(M, \Lambda^1TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1TM) \times C^\infty(M), \tag{2.5}$$

and the Stokes operator is defined as

$$B_0 := \begin{pmatrix} L & d \\ \delta & 0 \end{pmatrix} : C^\infty(M, \Lambda^1TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1TM) \times C^\infty(M).$$

All along this paper we consider the pseudodifferential operator P in the form $P = \lambda^2\mathbb{I}$, where $\lambda \neq 0$ is a constant. Thus, the operator (2.5) takes the form

$$B_\lambda := \begin{pmatrix} L+\lambda^2\mathbb{I} & d \\ \delta & 0 \end{pmatrix} : C^\infty(M, \Lambda^1TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1TM) \times C^\infty(M). \tag{2.6}$$

3. SOBOLEV SPACES ON LIPSCHITZ DOMAINS IN M

Let $\Omega_+ := \Omega \subset M$ be a Lipschitz domain and assume that $\Omega_- := M \setminus \overline{\Omega}$ is connected. Next, we denote by $\gamma_\pm(x) := \{y \in \Omega_\pm : |x-y| < (1+\kappa)\text{dist}(y, \partial\Omega)\}$, $x \in \partial\Omega$, the non-tangential approach regions lying in Ω_+ and Ω_-, respectively, for fixed $\kappa = \kappa(\partial\Omega) > 0$. Let $\text{Tr}^\pm$ be the non-tangential boundary trace operators on $\partial\Omega$, $(\text{Tr}^\pm u)(x) := \lim\limits_{\gamma_\pm(x)\ni y\to x} u(y)$, $x \in \partial\Omega$ (see e.g. [15]). For $s \geq 0$, consider the Sobolev spaces of functions

$$H^s(\Omega_\pm) := \{f|_{\Omega_\pm} : f \in H^s(M)\}, \quad \tilde{H}^s(\Omega_\pm) := \{f \in H^s(M) : \text{supp} f \subseteq \overline{\Omega}_\pm\},$$

and denote by $H^{-s}(\Omega_\pm)$ the dual of the space $\tilde{H}^s(\Omega_\pm)$ with respect to the $L^2(\Omega_\pm)$-duality, i.e., $H^{-s}(\Omega_\pm) = \big(\tilde{H}^s(\Omega_\pm)\big)^*$.

In addition, consider the Sobolev spaces of one forms

$$H^s(\Omega_\pm, \Lambda^1TM|_{\Omega_\pm}) := H^s(\Omega_\pm) \otimes \Lambda^1TM|_{\Omega_\pm},$$
$$\tilde{H}^s(\Omega_\pm, \Lambda^1TM|_{\Omega_\pm}) := \tilde{H}^s(\Omega_\pm) \otimes \Lambda^1TM|_{\Omega_\pm}.$$

Also, $H^{-s}(\Omega_\pm, \Lambda^1TM) := (\tilde{H}^s(\Omega_\pm, \Lambda^1TM))^*$. Next, assume that $\beta \in \big(-\frac{1}{2}, \frac{1}{2}\big)$, and consider the spaces

$$\begin{aligned} &\tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1TM) = \{\mathbf{f} \in H^{-1+\beta}(M, \Lambda^1TM) : \text{supp } \mathbf{f} \subseteq \overline{\Omega}_\pm\}, \\ &H^{1+\beta}(\Omega_\pm, \mathcal{L}_\lambda) = \big\{(\mathbf{u}, \pi, \mathbf{f}) : \mathbf{u} \in H^{1+\beta}(\Omega_\pm, \Lambda^1TM), \pi \in H^\beta(\Omega_\pm), \\ &\mathbf{f} \in \tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1TM) \text{ such that } \mathcal{L}_\lambda(\mathbf{u}, \pi) = \mathbf{f}|_{\Omega_\pm}, \ \delta\mathbf{u} = 0 \text{ in } \Omega_\pm\big\}, \end{aligned} \tag{3.1}$$

where $\mathcal{L}_\lambda(\mathbf{u}, \pi) := L\mathbf{u} + \lambda^2\mathbf{u} + d\pi$.

Let us mention the following useful result (see e.g. [1, 2, 17]):

LEMMA 3.1. *For every $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, the restriction operator to the boundary, $C^\infty(\overline{\Omega}_\pm, \Lambda^1 TM) \to C^0(\overline{\Omega}_\pm, \Lambda^1 TM)$, $u \mapsto u|_{\partial\Omega_\pm}$, extends to a linear and bounded operator $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1 TM) \to H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1 TM)$, which is onto, having a bounded right inverse $\mathcal{Z}^\pm : H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1 TM) \to H^s(\Omega_\pm, \Lambda^1 TM)$. For $s > \frac{3}{2}$, $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1 TM) \to H^1(\partial\Omega_\pm, \Lambda^1 TM)$ is also bounded.*

The conormal derivative operator for the Brinkman system on Lipschitz domains in Riemannian manifolds has been introduced in [9, Lemma 2.2], as an extension to the notion of conormal derivative operator for the Stokes system on Euclidean setting, for $s \in [0,1]$ and some $X \subseteq M$, $\langle \cdot, \cdot \rangle_X := {}_{H^s(X,\Lambda^1 TM)}\langle \cdot, \cdot \rangle_{(H^s(X,\Lambda^1 TM))^*}$ denotes the pairing between two dual Sobolev spaces $H^s(X, \Lambda^1 TM)$ and $\big(H^s(X, \Lambda^1 TM)\big)^*$, due to Mitrea and Wright [18, Theorem 10.10] (see also [2, 6, 7, 17]):

LEMMA 3.2. *For any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, the conormal derivative operator*

$$\partial_\nu^\pm : H^{1+\beta}(\Omega_\pm, \mathcal{L}_P) \to H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \tag{3.2}$$

$$\begin{aligned} \pm\langle \partial_\nu^\pm(\mathbf{u},\pi,\mathbf{f}), \Phi\rangle_{\partial\Omega} &:= 2\int_{\Omega_\pm} \langle \mathrm{Def}\ \mathbf{u}, \mathrm{Def}\ (\mathcal{Z}^\pm\Phi)\rangle \mathrm{dVol} + \int_{\Omega_\pm} \langle P\mathbf{u}, \mathcal{Z}^\pm\Phi\rangle \mathrm{dVol} \\ &+ \int_{\Omega_\pm} \langle \pi, \delta(\mathcal{Z}^\pm\Psi)\rangle \mathrm{dVol} - \langle \mathbf{f}, \mathcal{Z}^\pm\Phi\rangle_{\Omega_\pm},\ \forall\ \Phi \in H^{\frac{1}{2}-\beta}(\partial\Omega, \Lambda^1 TM), \end{aligned} \tag{3.3}$$

is well defined and bounded. Also, the Green formula

$$\begin{aligned} \pm\langle \partial_\nu^\pm(\mathbf{u},\pi,\mathbf{f}), \mathrm{Tr}^\pm\ \mathbf{v}\rangle_{\partial\Omega} - 2\int_{\Omega_\pm} \langle \mathrm{Def}\ \mathbf{u}, \mathrm{Def}\ \mathbf{v}\rangle \mathrm{dVol} - \int_{\Omega_\pm} \langle P\mathbf{u}, \mathbf{v}\rangle \mathrm{dVol} \\ = \int_{\Omega_\pm} \langle \pi, \delta\mathbf{v}\rangle \mathrm{dVol} - \langle \mathbf{f}, \mathbf{v}\rangle_{\Omega_\pm} \end{aligned} \tag{3.4}$$

holds for all $(\mathbf{u}, \pi, \mathbf{f}) \in H^{1+\beta}(\Omega_\pm, \mathcal{L}_P)$ and $\mathbf{v} \in H^{1-\beta}(\Omega_\pm, \Lambda^1 TM)$.

4. THE INVERTIBILITY OF THE BRINKMAN OPERATOR

The Brinkman operator (2.6) is elliptic in the sense of Agmon-Douglis-Nirenberg (see [6]) and extends to a Fredholm operator with index zero

$$B_\lambda : H^1(M, \Lambda^1 TM) \times L^2(M) \to H^{-1}(M, \Lambda^1 TM) \times L^2(M).$$

The kernel of this operator is the set $\{0\} \times \mathbb{R}$, and its range is $H^{-1}(M, \Lambda^1 TM) \times L^2_*(M)$, where $L^2_*(M) := \{q \in L^2(M) : \langle q, 1\rangle = 0\}$. In addition, the restriction of the Brinkman operator to $H^1(M, \Lambda^1 TM) \times L^2_*(M)$, denoted by B^0_λ, is invertible (for more details see [6]).

Next, let us refer to the second order differential operator

$$L_\lambda = 2\mathrm{Def}^*\mathrm{Def} + \lambda^2\mathbb{I} : H^1(M, \Lambda^1 TM) \to H^{-1}(M, \Lambda^1 TM), \tag{4.1}$$

which is invertible as a Fredholm operator with index zero and injective (due to the assumption of non-existence of non-trivial Killing fields on M) (for details, see [6, Lemma 5.8]). Then one obtains the following result (see also [6, 7]):

Lemma 4.1. *The following operators are invertible*

$$\Upsilon_\lambda : L^2_*(M) \to L^2_*(M), \quad \Upsilon_\lambda := \delta L_\lambda^{-1} d, \tag{4.2}$$

$$B^0_\lambda : H^1(M, \Lambda^1 TM) \times L^2_*(M) \to H^{-1}(M, \Lambda^1 TM) \times L^2_*(M), \tag{4.3}$$

and the inverse of $B^0_\lambda : H^{-1}(M, \Lambda^1 TM) \times L^2_*(M) \to H^1(M, \Lambda^1 TM) \times L^2_*(M)$ *is the operator given by*

$$(B^0_\lambda)^{-1} := \begin{pmatrix} \mathfrak{A}_\lambda & \mathfrak{B}_\lambda \\ \mathfrak{C}_\lambda & \mathfrak{D}_\lambda \end{pmatrix}, \tag{4.4}$$

where the pseudodifferential operators $\mathfrak{A}_\lambda,\ \mathfrak{B}_\lambda,\ \mathfrak{C}_\lambda,\ \mathfrak{D}_\lambda$ *are defined as*

$$\mathfrak{A}_\lambda := L_\lambda^{-1} - L_\lambda^{-1} d\, \Upsilon_\lambda^{-1}\, \delta L_\lambda^{-1}, \quad \mathfrak{B}_\lambda := L_\lambda^{-1} d \Upsilon_\lambda^{-1}, \tag{4.5}$$

$$\mathfrak{C}_\lambda := \Upsilon_\lambda^{-1} \delta L_\lambda^{-1}, \quad \mathfrak{D}_\lambda := -\,\Upsilon_\lambda^{-1}\,. \tag{4.6}$$

Note that the matrix operator

$$(B^0_0)^{-1} := \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{B}_0 \\ \mathfrak{C}_0 & \mathfrak{D}_0 \end{pmatrix} \tag{4.7}$$

is the inverse of the operator B^0_0, which corresponds to the Stokes system.

5. THE FUNDAMENTAL SOLUTION FOR THE BRINKMAN OPERATOR

In view of Lemma 4.1, one obtains the following relations M:

$$L_\lambda \mathfrak{A}_\lambda + \mathrm{d}\mathfrak{C}_\lambda = \mathbb{I}, \quad \delta \mathfrak{A}_\lambda = 0, \tag{5.1}$$

where $\mathbb{I}$ is the identity operator on $H^{-1}(M, \Lambda^1 TM)$. Let us denote by $\mathcal{G}_\lambda(x,y)$ and $\Pi_\lambda(x,y)$ the Schwartz kernels of the operators $\mathfrak{A}_\lambda$ and $\mathfrak{C}_\lambda$, respectively. In addition, let $\mathcal{G}(x,y)$ and $\Pi(x,y)$ be the Schwartz kernels of Φ_0 and Ψ_0. By using (5.1) one then obtains the following equations on M:

$$(L_x + \lambda_x)\mathcal{G}_\lambda(x,y) + d_x \Pi_\lambda(x,y) = \mathrm{Dirac}_y(x), \quad \delta_x \mathcal{G}_\lambda(x,y) = 0, \tag{5.2}$$

where Dirac_y denotes the Dirac distribution with mass at y. Hence the pair $(\mathcal{G}_\lambda(x,y), \Pi_\lambda(x,y))$ is the fundamental solution of the Brinkman system on M.

5.1. Layer potential operator for the Brinkman system.

In this section we present the main properties of layer potential operators for the Brinkman system on Lipschitz domains in compact Riemannian manifolds.

For $s \in [0,1]$, $\mathbf{f} \in H^{s-1}(\partial\Omega, \Lambda^1 TM)$ and $\mathbf{h} \in H^s(\partial\Omega, \Lambda^1 TM)$, the *single-layer potential* $\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}$ is the one form given on $M \setminus \partial\Omega$ by

$$(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f})(x) := \langle \mathcal{G}_\lambda(x,\cdot), \mathbf{f} \rangle_{\partial\Omega}, \quad x \in M \setminus \partial\Omega. \tag{5.3}$$

In addition, the corresponding pressure potential has the expression

$$\mathcal{P}^s_{\lambda;\partial\Omega}\mathbf{f} := \langle \Pi_\lambda(x,\cdot), \mathbf{f} \rangle_{\partial\Omega}, \quad x \in M \setminus \partial\Omega. \tag{5.4}$$

Similarly, the *double-layer potential* is defined at any point $x \in M \setminus \partial\Omega$ by

$$(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \Big\langle -2\big[(\mathrm{Def}_y\ \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x)\nu_{\partial\Omega}(y), \mathbf{h}(y)\Big\rangle \mathrm{d}\sigma(y), \tag{5.5}$$

and the corresponding pressure potential

$$(\mathcal{P}_{\lambda;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \langle -2[(\mathrm{Def}_y\ \Pi_\lambda(x,\cdot))\nu_{\partial\Omega}](y) - E_\lambda(x,y)\nu_{\partial\Omega}(y), \mathbf{h}(y)\rangle \mathrm{d}\sigma(y), \tag{5.6}$$

where $E_\lambda(x,y)$ is the Schwartz kernel of $(-\mathfrak{D}_\lambda)^\top \in OPS^0_{\mathrm{cl}}(\mathbb{R},\mathbb{R})$, and

$$(L_\lambda)_x(\Pi_\lambda)^\top(y,x) = d_x E_\lambda(x,y)$$

(see [6]). These layer potentials satisfy in $M \subset \partial\Omega$ the Brinkman system:

$$\delta(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = 0,\ (L + \lambda^2\mathbb{I})\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f} + \mathrm{d}Q_{\lambda;\partial\Omega}\mathbf{f} = 0, \tag{5.7}$$

$$\delta\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} = 0,\ (L + \lambda^2\mathbb{I})\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} + \mathrm{d}\mathcal{P}_{\lambda;\partial\Omega}\mathbf{h} = 0. \tag{5.8}$$

Now, consider the (principal value) boundary version of $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ a.e. $x \in \partial\Omega$ by (see e.g. [2])

$$(\mathbf{K}_{\lambda;\partial\Omega}\mathbf{h})(x) := \mathrm{p.v.}\int_{\partial\Omega} \Big\langle -2\big[(\mathrm{Def}_y\ \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x) \otimes \nu_{\partial\Omega}(y), \mathbf{h}(y)\Big\rangle \mathrm{d}\sigma_y, \tag{5.9}$$

where *p.v.* means the principal value of a singular integral. Thus, one has

$$(\mathbf{K}_{\lambda;\partial\Omega}\mathbf{h})(x) = \lim_{\epsilon\to 0}\int_{\{y\in\partial\Omega: r(x,y)>\varepsilon\}} \Big\langle -2\big[(\mathrm{Def}_y\ \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x) \otimes \nu_{\partial\Omega}(y), \mathbf{h}(y)\Big\rangle \mathrm{d}\sigma_y, \tag{5.10}$$

where $r(x,y)$ means the geodesic distance between the points x and y in M. In addition, one has the following jump relations a.e. on $\partial\Omega$ (see e.g. [2, 6])

(5.11)

$$\mathrm{Tr}^\pm(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}) = \Big(\pm\tfrac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\Big)\mathbf{h},$$

$$\partial_\nu^\pm(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}, \mathcal{P}_{\lambda;\partial\Omega}\mathbf{h}) := \mathbf{D}^\pm_{\lambda;\partial\Omega}\mathbf{h},\quad \mathbf{D}^+_{\lambda;\partial\Omega}\mathbf{h} - \mathbf{D}^-_{\lambda;\partial\Omega}\mathbf{h} \in \mathbb{R}\nu_{\partial\Omega}$$

$$\mathrm{Tr}^+(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathrm{Tr}^-(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) := \mathcal{V}_{\lambda;\partial\Omega}\mathbf{f},$$

$$\partial_\nu{}^\pm(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}, Q_{\lambda;\partial\Omega}\mathbf{f}) = \mp\tfrac{1}{2}\mathbf{f} + \mathbf{K}^*_{\lambda;\partial\Omega}\mathbf{f},$$

where

$$(\mathbf{K}^*_{\lambda;\partial\Omega}\mathbf{f})(x) := \mathrm{p.v.}\int_{\partial\Omega} \Big\langle -2[\mathrm{Def}_x\mathcal{G}_\lambda(\cdot,y)\nu](x) + \Pi_\lambda(x,y)\otimes\nu(x), \mathbf{f}(y)\Big\rangle_y \mathrm{d}\sigma(y) \quad \text{a.e. } x \in \partial\Omega,$$

is the formal transpose of $\mathbf{K}_{\lambda;\partial\Omega}$.

The next results extend to the Brinkman system the results of M. Mitrea and M. Taylor [17] and M. Dindoš and M. Mitrea [2] obtained in the case $\lambda = 0$, i.e., for the Stokes system (see [6]).

Theorem 5.1. *Let $\Omega \subset M$ be a Lipschitz domain. Then the following results hold:*

(i) *For any $s \in [0,1]$ and $\mathbf{f} \in H^{-s}(\partial\Omega, \Lambda^1 TM)$, one has*

$$\mathrm{Tr}^+(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathrm{Tr}^-(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathcal{V}_{\lambda;\partial\Omega}\mathbf{f}. \tag{5.12}$$

If $f \in H^{-s}(\partial\Omega, \Lambda^1 TM)$, $s \in (0,1)$ then the property (5.12) *holds as well.*

(ii) *If $s \in [0,1]$ and $\lambda \in [0, \frac{1}{2}]$, the following operators are linear and bounded:*

$$\begin{aligned}
&\mathcal{V}_{\lambda,\partial\Omega} : H^{-s}(\partial\Omega, \Lambda^1 TM) \to H^{1-s}(\partial\Omega, \Lambda^1 TM),\\
&\mathbf{K}_{\lambda;\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s}(\partial\Omega, \Lambda^1 TM)\\
&\mathbf{K}^*_{\lambda;\partial\Omega} : H^{s-1}(\partial\Omega, \Lambda^1 TM) \to H^{s-1}(\partial\Omega, \Lambda^1 TM)\\
&\mathbf{D}^{\pm}_{\lambda;\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s-1}(\partial\Omega, \Lambda^1 TM).
\end{aligned} \tag{5.13}$$

Theorem 5.2. *If $\Omega \subset M$ is a Lipschitz domain, then for any $s \in [0,1]$ the kernel of the operator $\mathcal{V}_{\lambda,\partial\Omega} : H^{-s}(\partial\Omega, \Lambda^1 TM) \to H^{1-s}(\partial\Omega, \Lambda^1 TM)$ is given by*

$$\mathrm{Ker}(\mathcal{V}_{\lambda,\partial\Omega}; H^{-s}(\partial\Omega, \Lambda^1 TM)) = \mathbb{R}\nu,\ \mathbb{R}\nu := \{c\nu : c \in \mathbb{R}\}. \tag{5.14}$$

In addition, one has the property

$$\mathbf{V}_{\lambda,\partial\Omega}\nu = 0 \text{ on } M. \tag{5.15}$$

In the case $\lambda = 0$ one gets the result by Mitrea and Taylor [17, Lemma 6.1].

6. INVERTIBILITY RESULTS FOR RELATED LAYER POTENTIAL OPERATORS

The Fredholm and invertibility results below have been recently obtained in [8, Lemma 5.3, Lemma 5.4].

Theorem 6.1. *Let $\Omega \subset M$ be a Lipschitz domain and let $\lambda \neq 0$, $\mu \in [0,1)$ be given constants. Then for any $s \in (0,1)$ the following statements hold:*

(i) *The operators*

$$\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s}(\partial\Omega, \Lambda^1 TM) \tag{6.1}$$

are Fredholm with index zero.

(ii) *The operators*

$$\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) \to H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) \tag{6.2}$$

are isomorphisms, where

$$H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) := \{\Phi \in H^{s}(\partial\Omega, \Lambda^1 TM) : \langle \Phi, \nu\rangle_{\partial\Omega} = 0\}. \tag{6.3}$$

7. NEUMANN PROBLEM FOR THE BRINKMAN SYSTEM ON LIPSCHITZ DOMAINS IN COMPACT RIEMANNIAN MANIFOLDS

Let $\Omega \subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, $\dim(M) \geq 2$, $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$. For $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, consider the Neumann problem for the Brinkman system:

$$(7.1) \qquad \begin{cases} (L + \lambda^2 \mathbb{I})\mathbf{u} + \mathrm{d}\pi = 0, \ \delta\mathbf{u} = 0 \text{ in } \Omega \\ [\partial_\nu^+(\mathbf{u}, \pi)] = [G] \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu, \end{cases}$$

where $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$. Note that the condition $[\partial_\nu^+(\mathbf{u}, \pi)] = [G]$ is equivalent with $\partial_\nu^+(\mathbf{u}, \pi) - G \in \mathbb{R}\nu$ on $\partial\Omega$.

Uniqueness result for the Neumann problem (7.1) The following result yields the uniqueness of solutions of the Neuman problem (7.1).

THEOREM 7.1. *Let $\Omega \subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, dim$(M) \geq 2$, and let $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ and $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ be given. Then the boundary value problem of Neumann type* (7.1) *has at most one solution $(\mathbf{u}, \pi) \in H^{1+\beta}(\Omega, \Lambda^1 TM) \times H^\beta(\Omega)$ (up to a constant pressure).*

Proof. Let us consider the homogenous problem:

$$(7.2) \qquad \begin{cases} (L + \lambda^2 \mathbb{I})\mathbf{u_0} + \mathrm{d}\pi_0 = 0, \ \delta\mathbf{u_0} = 0 \text{ in } \Omega \\ [\partial_\nu^+(\mathbf{u_0}, \pi_0)] = [0] \text{ on } \partial\Omega. \end{cases}$$

Therefore, there exists a constant $c_0 \in \mathbb{R}$ such that $\partial_{\nu_\Gamma}^+(\mathbf{u_0}, \pi_0) = c_0\nu$.

Since $(\mathbf{u_0}, \pi_0) \in H^{1+\beta}(\Omega, \Lambda^1 TM) \times H^\beta(\Omega)$ satisfies the Brinkman system, one has the layer potential representation (see e.g., [2]):

$$(7.3) \qquad \mathbf{u}_0 = \mathbf{W}_{\lambda,\partial\Omega}(\mathrm{Tr}^+\mathbf{u}_0) - \mathbf{V}_{\lambda,\partial\Omega}(\partial_\nu^+(\mathbf{u}_0, \pi_0)) \text{ in } \Omega,$$

where, in view of (5.15), the single-layer potential vanishes, as $\mathbf{V}_{\lambda,\partial\Omega}(\nu) = 0$ on M. Therefore, (7.3) becomes

$$(7.4) \qquad \mathbf{u}_0 = \mathbf{W}_{\lambda,\partial\Omega}(\mathrm{Tr}^+\mathbf{u}_0) \text{ in } \Omega,$$

Next, going non-tangentially to the boundary in (7.4) we get the equation $\mathrm{Tr}^+\mathbf{u}_0 = \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathrm{Tr}^+\mathbf{u}_0$ a.e. on $\partial\Omega$, i.e.,

$$(7.5) \qquad \left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathrm{Tr}^+\mathbf{u}_0 = 0 \text{ a.e. on } \partial\Omega.$$

Since the operator $-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega} : H_\nu^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H_\nu^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ is invertible (see e.g. [6, 2]) and $\mathrm{Tr}^+\mathbf{u}_0 \in H_\nu^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$, it follows that $\mathrm{Tr}^+\mathbf{u}_0 = 0$ on $\partial\Omega$. Consequently, the pair $(\mathbf{u}_0, \pi_0) \in H^{1+\beta}(\Omega, \Lambda^1 TM) \times H^\beta(\Omega)$ is a solution of the following Dirichlet problem for the Stokes system:

$$(7.6) \qquad \begin{cases} (L + \lambda^2 \mathbb{I})\mathbf{u_0} + \mathrm{d}\pi_0 = 0, \ \delta\mathbf{u_0} = 0 \text{ in } \Omega \\ \mathrm{Tr}^+\mathbf{u}_0 = 0 \text{ a.e. on } \partial\Omega. \end{cases}$$

In view of the uniqueness result (up to a constant pressure) for this problem (see e.g. [6, 10]), we get:

$$\mathbf{u}_0 = 0,\ \pi_0 = c_0 \in \mathbb{R} \text{ in } \Omega \tag{7.7}$$

as desired. □

8. LAYER POTENTIAL FORMULATION OF THE PROBLEM

Next, we show the existence of a solution $(\mathbf{u}, \pi) \in H^{1+\beta}(\Omega, \Lambda^1 TM) \times H^{\beta}(\Omega)$ of the Neumann problem (7.1), by means of the layer potential theory. To this aim, we use the invertibility property of the operators

$$\pm\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega} : H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM),$$

for any $\beta \in (-\frac{1}{2}, \frac{1}{2})$, as follows from Theorem 6.1. In addition, in view of the property (5.7) and the divergence theorem, we find that $\langle \mathcal{V}_{\lambda,\partial\Omega} F, \nu \rangle_{\partial\Omega} = 0$, i.e., $\mathcal{V}_{\lambda,\partial\Omega} F \in H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ for any $F \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$. Taking into account these properties, we consider the following layer potentials:

$$\mathbf{u} = \mathbf{W}_{\lambda,\partial\Omega}\mathbf{h}, \pi = Q^d_{\lambda,\partial\Omega}\mathbf{h} \text{ in } \Omega, \tag{8.1}$$

with the density $\mathbf{h} \in H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ in the form

$$\mathbf{h} := \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)^{-1} \left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)^{-1} \mathcal{V}_{\lambda,\partial\Omega} G. \tag{8.2}$$

Let us now show that the pair $(\mathbf{u}, \pi)$ given by (8.1) is a solution of the Neumann problem (7.1). Indeed, by (5.8) this pair satisfies the Brinkman system

$$\left(L + \lambda^2 \mathbb{I}\right)\mathbf{u} + \mathrm{d}\pi = 0,\ \delta\mathbf{u} = 0 \text{ in } \Omega. \tag{8.3}$$

It remains to show that the Neumann condition in (7.1) is also satisfied by the layer potentials (8.1). To this aim, note that $\mathbf{u}$ admits the layer potential representation (see [2], [8]):

$$\mathbf{u} = \mathbf{W}_{\lambda,\partial\Omega}\left(\mathrm{Tr}^+\mathbf{u}\right) - \mathbf{V}_{\lambda,\partial\Omega}\left(\partial_{\nu}^{+}(\mathbf{u}, \pi)\right) \quad \text{on } \partial\Omega. \tag{8.4}$$

Going non-tangentially to the boundary in (8.4) and using the relations (5.11), one then finds that

$$\mathrm{Tr}^+\mathbf{u} = \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathrm{Tr}^+\mathbf{u} - \mathcal{V}_{\lambda,\partial\Omega}\left(\partial_{\nu}^{+}(\mathbf{u}, \pi)\right) \text{ on } \partial\Omega,$$

i.e.,

$$\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathrm{Tr}^+\mathbf{u} = \mathcal{V}_{\lambda,\partial\Omega}\left(\partial_{\nu}^{+}(\mathbf{u}, \pi)\right) \text{ a.e. on } \partial\Omega. \tag{8.5}$$

In addition, in view of the fact that $\mathbf{u} = \mathbf{W}_{\lambda,\partial\Omega}\mathbf{h}$ in Ω, one has $\mathrm{Tr}^+\mathbf{u} = \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathbf{h}$ on $\partial\Omega$, and hence the equation (8.5) becomes

$$\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda,\partial\Omega}\right)\mathbf{h} = \mathcal{V}_{\lambda,\partial\Omega}\left(\partial_{\nu}^{+}(\mathbf{u}, \pi)\right) \text{ a.e. on } \partial\Omega. \tag{8.6}$$

Now, by using the expression (8.2) of the one form $\mathbf{h}$, (8.6) takes the form

$$\mathcal{V}_{\lambda,\partial\Omega}\left(\partial_\nu^+(\mathbf{u},\pi)\right) = \mathcal{V}_{\lambda,\partial\Omega} G \text{ a.e. on } \partial\Omega$$

i.e.,

$$\mathcal{V}_{\lambda,\partial\Omega}\left(\partial_\nu^+(\mathbf{u},\pi) - G\right) = 0 \text{ a.e. on } \partial\Omega. \tag{8.7}$$

Finally, by using the property (5.15), we conclude that

$$\partial_\nu^+(\mathbf{u},\pi) \in \mathbb{R}\nu \text{ a.e. on } \partial\Omega,$$

i.e., $[\partial_\nu^+(\mathbf{u},\pi)] = [G]$. Consequently, the pair $(\mathbf{u},\pi)$ given by (8.1) is a solution of the Neumann problem (7.1), in the space $H^{1+\beta}(\Omega,\Lambda^1 TM)\times H^\beta(\Omega)$. In view of Theorem 7.1, this is the unique solution (up to a constant pressure) of the Neumann problem (7.1). The boundedness properties of the layer potentials (8.1) and those of the operators in (8.2) imply that there exists a constant $C > 0$ such that this solution satisfies the estimate

$$\|\mathbf{u}\|_{H^{1+\beta}(\Omega,\Lambda^1 TM)} + \|\pi\|_{H^\beta(\Omega)} \leq C\|[G]\|_{H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1 TM)/\mathbb{R}\nu}. \tag{8.8}$$

By the above arguments, we obtain:

Theorem 8.1. *Let $\Omega \subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, $dim(M) \geq 2$, and let $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1 TM)$ and $\beta \in \left(-\frac{1}{2},\frac{1}{2}\right)$ be given. Then the layer potentials* (8.1) *with the density $\mathbf{h} \in H_\nu^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1 TM)$ given by* (8.2) *determine the unique solution $(\mathbf{u},\pi) \in H^{1+\beta}(\Omega,\Lambda^1 TM) \times H^\beta(\Omega)$ (up to a constant pressure) of the boundary value problem of Neumann type* (7.1)*, which satisfies the estimate* (8.8).

Remark 8.2. If $\Omega \subset M$ is a C^1 domains then for any $\mu \in [0,1)$ and $p \in (1,\infty)$ the following operators are isomorphisms:

$$\mp\tfrac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : L^p_{s;\nu}(\partial\Omega,\Lambda^1 TM) \to L^p_{s;\nu}(\partial\Omega,\Lambda^1 TM),\ s = 0,1, \tag{8.9}$$

where $L^p_{s;\nu}(\partial\Omega,\Lambda^1 TM) := \{\Phi \in L^p_s(\partial\Omega,\Lambda^1 TM) : \langle\Phi,\nu\rangle_{\partial\Omega} = 0\}$. The Fredholm and zero index property of these operators follows from [2, Proposition 3.5] and the compactness of the complementary layer potential operators for the Stokes and Brinkman systems (see [10]). In addition, the injectivity property of these operators is provided by similar arguments to those for the injectivity of the operators in Lemma 6.1.

By using similar arguments to those in the proof of Theorem 8.1, one obtains the well-posedness of the boundary value problem of Neumann type (7.1) whenever $G \in L^p_{s-1}(\partial\Omega,\Lambda^1 TM)$, $s \in (0,1)$ and $p \in (1,\infty)$. Hence, we get:

Theorem 8.3. *Let $\Omega \subset M$ be a C^1 domain on a compact boundaryless Riemannian manifold M, $dim(M) \geq 2$. Then for any $G \in L^p_{s-1}(\partial\Omega,\Lambda^1 TM)$, with $s \in (0,1)$ and $p \in (1,\infty)$, the layer potentials* (8.1) *having the density $h \in L^p_{s,\nu}(\partial\Omega,\Lambda^1 TM)$ given by* (8.2) *determine the unique solution $(u,\pi) \in$*

$L^p_{s+\frac{1}{p}}(\partial\Omega, \Lambda^1 TM) \times L^p_{s+\frac{1}{p}-1}(\Omega)$ *(up to a constant pressure) of the boundary value problem of Neumann type* (7.1)*, which satisfies the estimate*

$$(8.10) \qquad ||\mathbf{u}||_{L^p_{s+\frac{1}{p}}(\partial\Omega,\Lambda^1 TM)} + ||\pi||_{L^p_{s+\frac{1}{p}-1}(\Omega)} \leq C||[G]||_{L^p_{s-1}(\partial\Omega,\Lambda^1 TM)/\mathbb{R}/\nu},$$

with some constant $C > 0$.

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